


§ 4: Relation to integrable hierarchies

Lecture 12 (B4)
2021/05/07

1° A Lax pair for P_{IV} (3×3 matrices)

$$x \partial_x \vec{u} = A(x, t) \vec{u}, \quad \partial_t \vec{u} = B(x, t) \vec{u}$$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad u_i = u_i(x, t)$$

compatibility condition (Frobenius integrability, zero curvature)

$$[x \partial_x - A(x), \partial_t - B(x)] = 0$$

$$A(x) = A(x; t)$$

$$B(x) = B(x; t)$$

$$\text{i.e.} \quad \partial_t A(x) - x \partial_x B(x) + [A(x), B(x)] = 0$$

We specify $A(x), B(x)$ as follows:

nonlinear for the equations for the matrix elements

$$A(x) = - \begin{bmatrix} \varepsilon_1 & f_1 & 1 \\ x & \varepsilon_2 & f_2 \\ x f_0 & x & \varepsilon_3 \end{bmatrix}, \quad B(x) = \begin{bmatrix} v_1 & -1 & 0 \\ 0 & v_2 & -1 \\ -x & 0 & v_3 \end{bmatrix}$$

$$\partial_t A(x) - x \partial_x B(x) + [A(x), B(x)]$$

$$= \begin{bmatrix} -\varepsilon_1' & \varepsilon_1 - \varepsilon_2 + (v_1 - v_2) f_1 - f_1' & f_1 - f_2 + v_1 - v_3 \\ (f_2 - f_0 + (v_2 - v_1)) x & -\varepsilon_2' & \varepsilon_2 - \varepsilon_3 + (v_2 - v_3) f_2 - f_2' \\ (1 - \varepsilon_1 + \varepsilon_3 + (v_3 - v_1) f_0 - f_0') x & (f_0 - f_1 + (v_3 - v_2)) x & -\varepsilon_3' \end{bmatrix}$$

$$' = \partial_t$$

$$\left\{ \begin{array}{l} \varepsilon_1' = 0 \\ \varepsilon_2' = 0 \\ \varepsilon_3' = 0 \end{array} \right\} \left\{ \begin{array}{l} f_0' = f_0(v_3 - v_1) + 1 + \varepsilon_3 - \varepsilon_1 \\ f_1' = f_1'(v_1 - v_2) + \varepsilon_1 - \varepsilon_2 \\ f_2' = f_2(v_2 - v_3) + \varepsilon_2 - \varepsilon_3 \end{array} \right\} \left\{ \begin{array}{l} f_1 - f_2 = v_3 - v_1 \\ f_2 - f_0 = v_1 - v_2 \\ f_0 - f_1 = v_2 - v_3 \end{array} \right.$$

$$\left\{ \begin{array}{l} \alpha_0 = 1 - \varepsilon_1 + \varepsilon_3 \\ \alpha_1 = \varepsilon_1 - \varepsilon_2 \\ \alpha_2 = \varepsilon_2 - \varepsilon_3 \end{array} \right\} \left\{ \begin{array}{l} f_0' = f_0(f_1 - f_2) + \alpha_0 \\ f_1' = f_1'(f_2 - f_0) + \alpha_1 \\ f_2' = f_2'(f_0 - f_1) + \alpha_2 \end{array} \right. \quad (S_{IV})$$

$$\text{cf. } \left\{ \begin{array}{l} v_1 = h_1 - h_0 \\ v_2 = h_2 - h_1 \\ v_3 = h_0 - h_2 \end{array} \right.$$

$$-A(x) = \begin{bmatrix} \varepsilon_1 & f_1 & 1 \\ x & \varepsilon_2 & f_2 \\ x f_0 & x & \varepsilon_3 \end{bmatrix}$$

$$B(x) = \text{diag}(v_1, v_2, v_3) I_3$$

$$-\text{diag}(1, 1, 1) \Lambda(x)$$

$$= \begin{bmatrix} \varepsilon_1 & f_1 & 1 \\ 0 & \varepsilon_2 & f_2 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ f_0 & 1 & 0 \end{bmatrix} x$$

$$= \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3) I_3 + \text{diag}(f_1, f_2, f_0) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & 0 & 0 \end{bmatrix} = \Lambda(x)$$

$$+ \text{diag}(1, 1, 1) \begin{bmatrix} 0 & 0 & 1 \\ x & 0 & 0 \\ 0 & x & 0 \end{bmatrix} = \Lambda(x)^2$$

2° Relation to the n -reduced KP hierarchy

① Painlevé system of type (n, m) ^{height}

$$x \partial_x \vec{u} = A(x) \vec{u}, \quad \partial_{t_k} \vec{u} = B_k(x) \vec{u} \quad (k=1, \dots, m)$$

$$\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \quad t = (t_1, \dots, t_m) \quad A(x), B(x) : n \times n \text{ matrices}$$

$$A(x) = \sum_{r=0}^m \text{diag}(a^{(r)}) \Lambda(x)^r, \quad B_k(x) = \sum_{r=0}^{k-1} \text{diag}(b_k^{(r)}) \Lambda(x)^r + \underline{\Lambda(x)^k} \quad (k=1, \dots, m)$$

$$\Lambda(x) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & \\ x & & & 0 \end{bmatrix} = \sum_{i=1}^{n-1} E_{i, i+1} + x E_{n, 1} \quad \text{cyclic element}$$

compatibility conditions

$$(A) \quad [x\partial_x - A(x), \partial_{t_k} - B_k(x)] = \partial_{t_k} A(x) - x\partial_x B_k(x) + [A(x), B_k(x)] = 0$$

($k=1, \dots, m$)

$$(B) \quad [\partial_{t_k} - B_k(x), \partial_{t_l} - B_l(x)] = \partial_{t_l} B_k(x) - \partial_{t_k} B_l(x) + [B_k(x), B_l(x)] = 0$$

($k, l=1, \dots, m$)

(B) --- Zakharov-Shabat equations
for n -reduced modified KP hierarchy

(A) --- Similarity reduction

(n, m)

$(2, 3)$

$P_{II} \leftarrow$ modified KdV (2-reduced)

$(3, 2)$

$P_{IV} \leftarrow$ modified Boussinesq (3-reduced)

$(4, 2)$

$P_V \leftarrow$ modified ? (4-reduced)

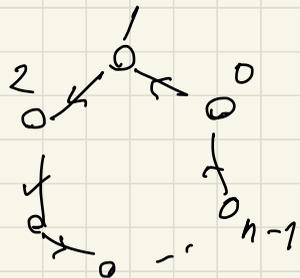
$(n, 2)$

higher order Painlevé equation of type $A_{n-1}^{(1)}$
(Noumi-Yamada system)

$n \geq 3$

$$\tilde{W}(A_{n-1}^{(1)}) = \langle s_0, s_1, \dots, s_{n-1}, \eta \rangle$$

$$S_i(f_j) = f_j + \frac{\alpha_i}{f_i} u_{ij}$$



① n -reduced modified KP hierarchy

$$[\partial_{t_k} - B_k(x), \partial_{t_l} - B_l(x)] = 0 \quad (k, l = 1, 2, \dots, m) \quad (m = \infty)$$

Zakharov-Shabat equations

In the theory of modified KP hierarchy, we usually assume the existence of the "wave functions" of the form

$$\bar{\Psi}(x; t) = W(x; t) \circ \sum_{r \geq 1} \text{tr} \Lambda(x)^r$$

$$\bar{\Psi}(x) = \bar{\Psi}(x; t)$$

formal
solution

$$W(x; t) = I_n + \sum_{r=1}^{\infty} \text{diag}(w_r) \Lambda(x)^{-r}$$

$$\partial_{t_k} \bar{\Psi}(x; t) = B_k(x; t) \bar{\Psi}(x; t) \quad (k = 1, 2, \dots, m)$$

Remark Equations to be satisfied by $W(x) = \bar{W}(x; t)$

$$(*) \quad \partial_{t_k} W(x) = B_k(x) W(x) - W(x) \Lambda(x)^k \quad (k=1, 2, \dots)$$

Sato equations

Suppose that $W(x)$ is obtained through the Riemann-Hilbert-Birkhoff decomposition

$$C(x) = W(x; t) \Sigma(x; t), \quad \Sigma(x) = \sum_{r \geq 0} \text{diag}(Z_r) \Lambda(x)^r$$

positive part of the Heisenberg alg. \leftarrow constant w.r.t t (t_1, t_2, \dots)

Then $W(x)$ solves $(*)$.

$N_- B_+ \subset G = GL(n; \mathbb{C}(\!(x)\!))$
big $\hookrightarrow \mathcal{U} \subset G/B_+$ flag manifold

In this setting, we consider

$$L(x) = W(x) \Lambda(x) W(x)^{-1}$$

$$M(x) = W(x) e^{\sum_{r=2}^{\infty} \text{tr} \Lambda(x)^r} (n \times \partial_x + \text{diag}(\rho)) e^{-\sum_{r=2}^{\infty} \text{tr} \Lambda(x)^r} W(x)^{-1}$$

Lax operator
 $\rho = (n-1, n-2, \dots, 0)$

$$\left\{ \begin{array}{l} [M(x), L(x)] = L(x) \\ \partial_{\text{tr}} L(x) = [B_n(x), L(x)] \\ \partial_{\text{tr}} M(x) = [B_n(x), M(x)] \end{array} \right.$$

$$B_n(x) = (L(x)^n)_{\geq 0}$$

Lax equations

non negative part
in the expansion by
 $\Lambda(x)^r$

$$L(x) = \Lambda(x) + \sum_{r=2}^{\infty} \text{diag}(A_r) \Lambda(x)^{1-r}$$

$$M(x) = \underbrace{n x \partial_x + \text{diag}(p) - \sum_{r \geq 1} r t_r \Lambda(x)^r}_{M(x)_{\geq 0}} + \underbrace{\sum_{r=1}^{\infty} \text{diag}(m_r) \Lambda(x)^{-r}}_{M(x)_{< 0}}$$

Similarity reduction

$$M(x) = M(x)_{\geq 0} \iff M(x)_{< 0} = 0$$

$$\iff \sum_{k \geq 1} k t_k \partial_{t_k} (\omega_r) = -r \omega_r \quad (r=1, 2, \dots)$$

$$W(x) = \sum_{r \geq 0} \text{diag}(\omega_r) \Lambda(x)^{-r}$$

Set

$$A(x) = -\text{diag}\left(\frac{p}{n}\right) + \sum_{k \geq 1} \frac{k}{n} t_k B_k(x) \quad \text{so that}$$

$$\frac{1}{n} M(x)_{\geq 0} = x \partial_x - A(x).$$

If this similarity condition is satisfied, then

$$\bar{\Phi}(x;t) = W(x;t) e^{\sum_{k \geq 1} t_k \Lambda(x)^k} \cdot X^{-P/n}$$

satisfies

↳ fundamental system of formal solutions

$$\left\{ \begin{array}{l} x \partial_x \bar{\Phi}(x;t) = A(x;t) \bar{\Phi}(x;t) \\ \partial_{t_k} \bar{\Phi}(x;t) = B_k(x;t) \bar{\Phi}(x;t) \quad (k=1,2,\dots) \end{array} \right\}$$

solving the associated Painlevé system.

§5: PVI and Umemura polynomials

1^o Hamiltonian system

2^o Lax pair (Monodromy preserving deformation)

3^o Affine Weyl group symmetry

4^o Typical classical solutions

5^o Umemura polynomials