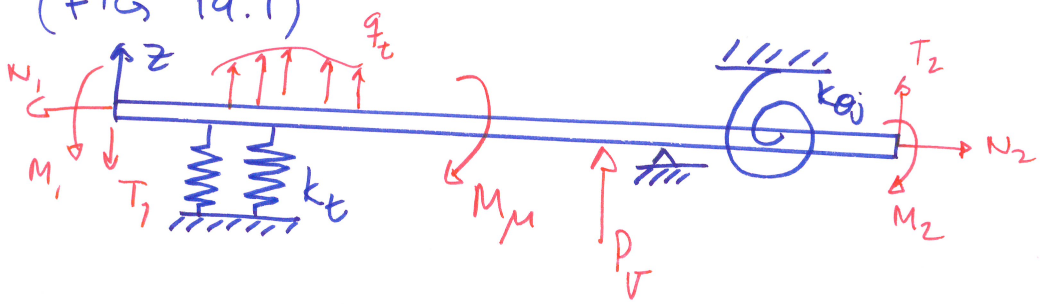
ÖVN 14: Energimetoder

2.4.105a → 2.4.105b är hemupg!
2.4.107
2.4.108

formelsamlingen kap 19, sid 279

(FIG 19.1)



Den potentiella energin som funktion av w

(19.1) → $$U(w) = W(w) + W_r(w) - \left[Tw - Mw'\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} q_t w \, dx - \sum p_v w_v + \sum_{\mu \neq 1,2} M_\mu w_\mu + \frac{1}{2}\int_{x_1}^{x_2} N w'^2 dx$$

Elastisk energi lagrad i balken:

(19.2) $$W(w) = \frac{1}{2}\int_{x_1}^{x_2} EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

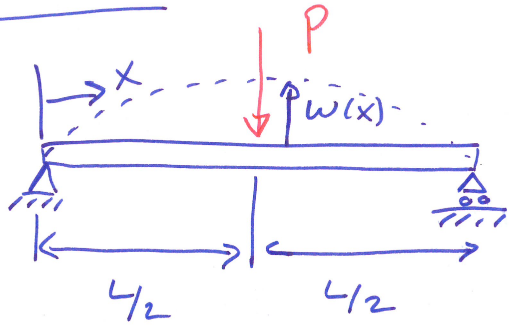
Elastisk energi i stöden

(19.3) $$W_r = \frac{1}{2}\int_{x_1}^{x_2} k_t w^2 dx + \frac{1}{2}\sum_j k_{\theta j} w_j'^2 + \ldots + \ldots$$

Användning.

1. Gör en ansats för utböjningen Ex: $w(x) = Ax$
2. Minimera potentiella energin, $\frac{\partial U(w)}{\partial A} = 0$

2.4.105



Bestäm nedböjningen under lasten, vinkeländring vid vänstra stödet och maximala momentet i balken exakt och/eller approximativt.

Beror på ansats!

1. Potentiella energin

$$(19.1) \rightarrow \quad U(w) = W(w) - Pw\left(\tfrac{L}{2}\right) \qquad (1)$$

$$(19.2) \rightarrow \quad W(w) = \frac{1}{2}\int_{x_1}^{x_2} EI\left(\frac{\partial^2 w}{\partial x^2}\right) dx \qquad (2)$$

2. Randvillkor

$w(0) = 0, \quad w(L) = 0$

3. Ansats

$$\begin{cases} w(0) = 0 \\ w(L) = 0 \end{cases}$$

- Måste uppfylla kinematiska randvillkor
- Bra om den även uppfyller övriga rv. för T & M vid ändarna.

$-EIw''' = T, \quad -EIw'' = M$

$$w(x) = Ax(L-x)$$

4. Energins minima

$$\frac{\partial w}{\partial x} = AL - 2Ax, \qquad \frac{\partial^2 w}{\partial x^2} = -2A.$$

$$\rightarrow W(w) = \frac{1}{2}\int_0^L EI(-2A)^2\, dx = 2EIA^2L$$

Insatt i (1)

$$\rightarrow U(w) = 2EIA^2L - P\,\overbrace{A\left(\frac{L}{2}\right)\left(L-\frac{L}{2}\right)}^{w(\frac{L}{2})}$$

$$= 2EIA^2L = 2EIA^2L + \frac{1}{4}PAL^2$$

Minimera

$$\frac{\partial U}{\partial A} = 0 \Rightarrow 4EIAL + \frac{1}{4}PL^2 = 0$$

$$\rightarrow A = \frac{-PL}{16EI}.$$

5. SVAREN

$$\therefore\; w(x) = Ax(L-x) = -\frac{PLx}{16EI}(L-x)$$

$$\delta = -w\left(\frac{L}{2}\right) = \frac{PL^3}{64EI}$$

$$\theta = -\left.\frac{\partial w}{\partial x}\right|_{x=0} = -\left[\frac{PL^2}{16EI} - \frac{2PLx}{16EI}\right]_0^L = \frac{PL^2}{16EI}$$

$$M(x) = -EI\frac{\partial^2 w}{\partial x} = \frac{PL}{8} \sim \text{konstant!}$$

$$\rightarrow M_{max} = \frac{PL}{8}$$

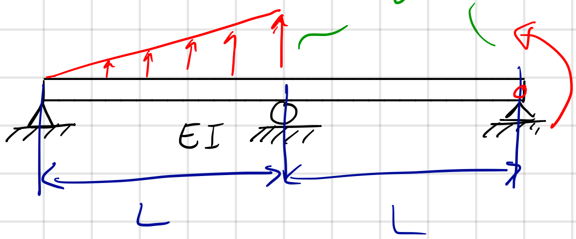
Exakt lösning, Elementarfall 31.2.1

$$\delta = \frac{PL^3}{48EI}, \quad \theta = \frac{PL^2}{16EI}, \quad M_{max} = \frac{PL}{4}.$$

Att välja ansats:

DS 2 -2016

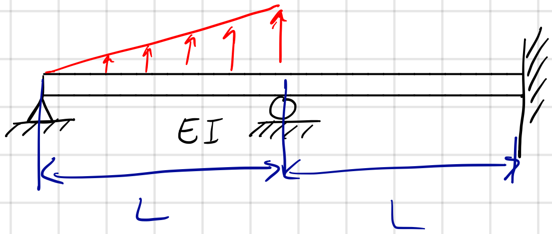
Spelar lasten någon roll....



Vi ser att

$$w(0)=0,\quad w(L)=0,\quad w(2L)=0 \;\longrightarrow\; w(x) = a\,x(x-L)(x-2L)$$

DS2 -2017

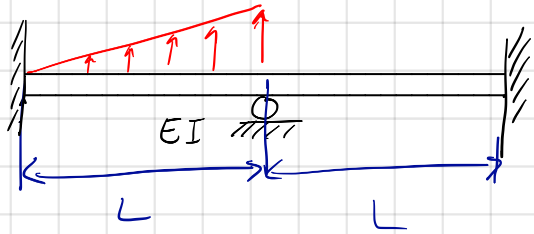


Ansats:

$$\longrightarrow\; w(x) = a\,x(x-L)(x-2L)^2$$

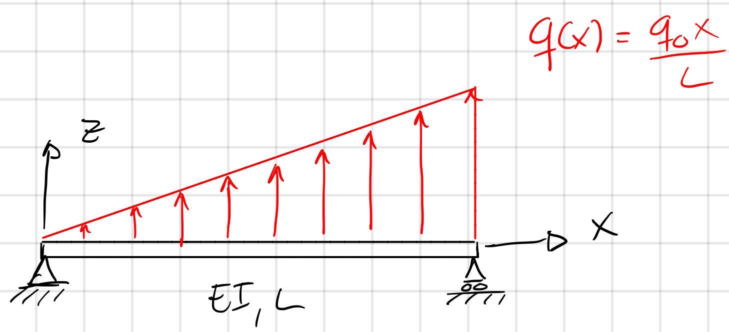
$$w'(2L)=0 \quad !$$

Framtida DS?



$$\longrightarrow\; w(x) = a\,x^2(x-L)(x-2L)^2$$

2.4.107



Sökt: maximala utböjningen

① Potentiella energin

$$U(w) = W(w) - \int_{x_1}^{x_2} q_t \, w \, dx \qquad (1)$$

Där

$$W(w) = \frac{1}{2} \int_{x_1}^{x_2} EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \qquad (2)$$

② Ansats

RV:

$w(0) = 0$
$w(L) = 0$

noll då $x=0$ noll då $x=L$

$$\rightarrow \quad w(x) = x(L-x)(A+Bx) \qquad (*)$$
$$= ALx + Bx^2L - Ax^2 - Bx^3$$

③ Minimera potentiella energin.

$$\begin{cases} \dfrac{\partial w}{\partial x} = AL + 2BxL - 2Ax - 3Bx^2 \\[2ex] \dfrac{\partial^2 w}{\partial x^2} = 2BL - 2A - 6Bx \end{cases}$$

$(*)$ Insatt i (2)

$$W(w) = \frac{1}{2}\int_0^L EI\left(2BL - 2A - 6Bx\right)^2 dx = \frac{EI}{2}\left[\frac{-1}{6B \cdot 3}\left(2BL - 2A - 6Bx\right)^3\right]_0^L$$

$$= \ldots = 2EIL\left(A^2 + ABL + B^2L^2\right) \qquad (3)$$

Och (x) i termen för kraftens energi.

$$\int_0^L q(x)\,w(x)\,dx = \int_0^L \frac{q_0 x}{L} \cdot x(L-x)(A+Bx)\,dx$$

$$= \dots = \frac{q_0}{60}\left[L^3(5A+3BL)\right] \qquad (3)$$

$$\rightarrow W(w) = \underbrace{2EIL\left(A^2 + ABL + B^2L^2\right)}_{(2)} - \frac{q_0}{60}L^3(5A+3BL) \quad (4)$$

$$\begin{cases} \dfrac{\partial U}{\partial A} = 2EIL(2A+BL) - \dfrac{q_0L^3}{12}, & \dfrac{\partial U}{\partial A} = 0 \\ \dfrac{\partial U}{\partial B} = 2EIL(AL+2BL^2) - \dfrac{q_0BL^4}{20}, & \dfrac{\partial U}{\partial B} = 0 \end{cases}$$

TVÅ EKVATIONER, TVÅ OBEKANTA, LÖS!

$$\rightarrow \begin{cases} A = \dfrac{7}{360} \cdot \dfrac{q_0L^2}{EI} \\ B = \dfrac{1}{360} \cdot \dfrac{q_0L}{EI} \end{cases}$$

4 Utböjningen.

$$\rightarrow w(x) = \frac{x}{360} \cdot \frac{q_0L}{EI}(L-x)(7L+x)$$

5 MAXIMALA UTBÖJNINGEN.

$$\frac{\partial w}{\partial x} = 0$$

Kan ej vara 0!

$$\rightarrow 0 = \frac{q_0L}{360EI}\left(7L^2 - 12Lx - 3x^2\right)$$

Måste vara noll!

$$\rightarrow x^2 + 4Lx - \frac{7L^2}{3} = 0 \Rightarrow x = -2L \pm \sqrt{4L^2 + \frac{7L^2}{3}} = -2L \pm 2{,}517L$$

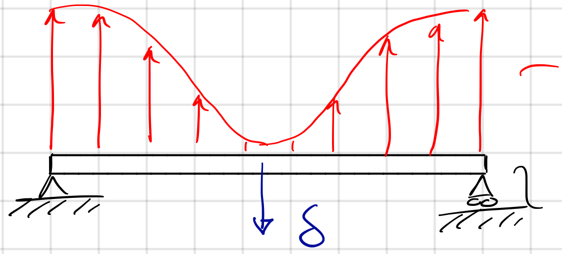
$\rightarrow \quad x_1 = 0{,}517\ L$

$(x_2 = -4{,}517\,L)$

$$w_{max} = w(0{,}517L) = 1{,}88\ \frac{q_0 L^4}{360EI}.$$

$$\boxed{\text{Svar: } w_{max} = 1.88 \cdot \frac{q_0 L^4}{360EI}.}$$

2.4.108



$q(x) = q_0\left(1 + \cos\left(\frac{2\pi x}{L}\right)\right)$

<u>Sökt</u>: (Approximativ) utböjning mitt på balken, δ, genom att använda ansatsen

$$w(x) = A\sin\left(\frac{\pi x}{L}\right) + B\sin\left(\frac{3\pi x}{L}\right)$$

Uppfyller RV !

$$w(0) = A\cdot\sin(0) + B\sin(0) = 0$$
$$w(L) = A\cdot\sin(\pi) + B\sin(3\pi) = 0$$

1. Potentiella energin

$(19,1) \to \quad U(w) = W(w) - \int_0^L q(x)w(x)\,dx$

$(19,2) \to \quad W(w) = \frac{EI}{2}\int_0^L \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$

2. Energins minima

$$\begin{cases} \frac{\partial w}{\partial x} = \frac{A\pi}{L}\cos\left(\frac{\pi x}{L}\right) + \frac{3B\pi}{L}\cdot\cos\left(\frac{3\pi x}{L}\right) \\ \frac{\partial^2 w}{\partial x^2} = -\frac{A\pi^2}{L^2}\sin\left(\frac{\pi x}{L}\right) - \frac{9B\pi^2}{L^2}\sin\left(\frac{3\pi x}{L}\right) \end{cases}$$

$$\to W(w) = \frac{EI}{2}\int_0^L \left(-\frac{A\pi^2}{L^2}\sin\left(\frac{\pi x}{L}\right) - \frac{9B\pi^2}{L^2}\sin\left(\frac{3\pi x}{L}\right)\right)^2 dx =$$

$$= \ldots = \frac{\pi^4 EI}{4L^3}(A^2 + 81B)$$

<u>And</u>

$$\int_0^L \underbrace{q_0\left(1+\cos\left(\frac{2\pi x}{L}\right)\right)}_{q_t} \underbrace{A\sin\left(\frac{\pi x}{L}\right) + B\sin\left(\frac{3\pi x}{L}\right)}_{w(x)}\,dx =$$

$$= \ldots = \frac{4q_0L}{15\pi}(5A + 7B)$$

$$\therefore \quad U(w) = \frac{\pi^4 EI}{4L^3}\left(A^2 + 81B^2\right) - \frac{4 q_0 L}{15\pi}\left(5A + 7B\right)$$

Hitta minimum map A & B.

$$\begin{cases} \dfrac{\partial U}{\partial A} = \dfrac{A\pi^4 EI}{2L^3} - \dfrac{20 q_0 L A}{15\pi}, & \dfrac{\partial U}{\partial A} = 0 \\[2ex] \dfrac{\partial U}{\partial B} = \dfrac{B \cdot 81\pi^4 EI}{2L^3} - \dfrac{28 q_0 L}{15\pi}, & \dfrac{\partial U}{\partial B} = 0 \end{cases}$$

Lös ekuationssystemet

$$\rightarrow \quad A = \frac{8 q_0 L^4}{3\pi^5 EI}, \quad B = \frac{56 q_0 L^4}{1215\pi^5 EI}$$

TILLBAKA I ANSATSEN

$$\rightarrow \quad w(x) = \frac{q_0 L^4}{\pi^5 EI}\left[\frac{8}{3}\sin\left(\frac{\pi x}{L}\right) + \frac{56}{1215}\sin\left(\frac{3\pi x}{L}\right)\right]$$

$$\delta = \left\| w\left(\frac{L}{2}\right)\right\| = \frac{q_0 L^4}{\pi^5 EI}\left[\frac{8}{3}\sin\left(\frac{\pi}{2}\right) + \frac{56}{1215}\sin\left(\frac{\pi}{2}\right)\right]$$

$$= 0.008563 \cdot \frac{q_0 L^4}{EI}$$

3. Exakt lösning.

Elastiska linjen (finns inget elementarfall som matchar $q(x)$!)

$$(6.20) \rightarrow \quad EI w^4(x) = q(x) = q_0\left(1 + \cos\left(\frac{2\pi x}{L}\right)\right)$$

Integrera.

$$EI \cdot \frac{\partial^3 w}{\partial x^3} = q_0 x + \frac{L}{2\pi} \sin\left(\frac{2\pi x}{L}\right) + C$$

$$EI \frac{\partial^2 w}{\partial x} = \frac{q_0 x^2}{2} - \frac{L^2}{4\pi^2} \cos\left(\frac{2\pi x}{L}\right) + Cx + D$$

$$EI \frac{\partial w}{\partial x} = \frac{q_0 x^3}{6} - \frac{L^3}{8\pi^3} \sin\left(\frac{2\pi x}{L}\right) + \frac{Cx^2}{2} + Dx + E$$

$$EIw = \frac{q_0 x^4}{24} + \frac{L^3}{16\pi^4} \cos\left(\frac{2\pi x}{L}\right) + \frac{Cx^3}{6} + \frac{Dx^2}{2} + Ex + D$$

Randvillkor

$$\underbrace{w(0) = w(L)}_{\text{Ingen utböjning vid stöden}} = \underbrace{\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L}}_{\text{Inget moment vid stöden.}} = 0$$

Använd RV för att lösa ut konstanterna

$$\rightarrow w(x) = \frac{q_0 L^4}{EI}\left[\frac{1}{16\pi^4}\cos\left(\frac{2\pi x}{L}\right) + \frac{1}{24}\left(\frac{x}{L}\right)^4 - \frac{1}{12}\left(\frac{x}{L}\right)^3 + \frac{1}{8\pi^2}\left(\frac{x}{L}\right)^2 + \left(\frac{1}{24} - \frac{1}{8\pi^2}\right)\frac{x}{L} - \frac{1}{16\pi^4}\right]$$

$$w\left(\frac{L}{2}\right) = 0.008571 \cdot \frac{q_0 L^4}{EI}$$

Skillnad :

$$e = \frac{w\left(\frac{L}{2}\right)_{exakt} - w\left(\frac{L}{2}\right)_{Energi}}{w\left(\frac{L}{2}\right)_{exakt}} \simeq 0{,}0009 \quad !$$