CR compactification for asymptotically locally complex hyperbolic almost Hermitian manifolds

[arXiv:2307.04062]

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Outline
of the presentation

$(S^n, [\hat{g}])$ the conformal round sphere

$(\mathbb{H}^{n+1}, g)$ the hyperbolic space

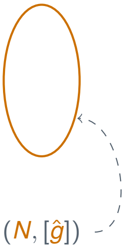
$(\overline{B^{n+1}(0;1)}, \overline{g})$ the unit ball

$\rho(x) = \frac{1-|x|^2}{2}$ is such that $\overline{g} = \rho^2 g$ restricts as $\hat{g}$ on S^n

$$\mathrm{Isom}(\mathbb{H}^{n+1}) = \mathrm{Conf}(S^n)$$

Conformally compact manifolds

$(N, [\hat{g}])$ conformal manifold



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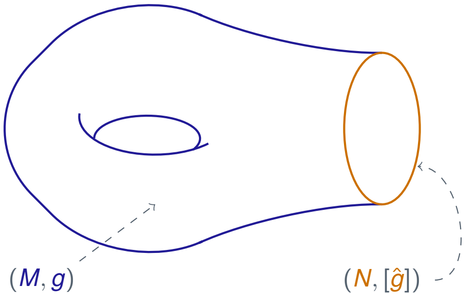

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(M, g) Riemannian manifold

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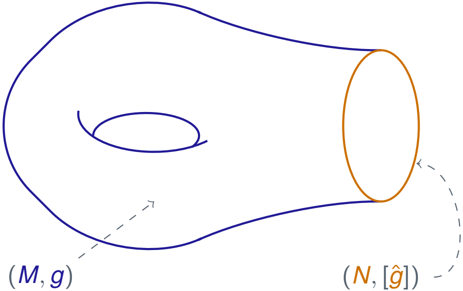

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(M, g) Riemannian manifold
$\overline{M} = M \cup N$ compact manifold with boundary
$\partial \overline{M} = N$
$\rho \colon \overline{M} \to [0, \infty)$ defining function for N

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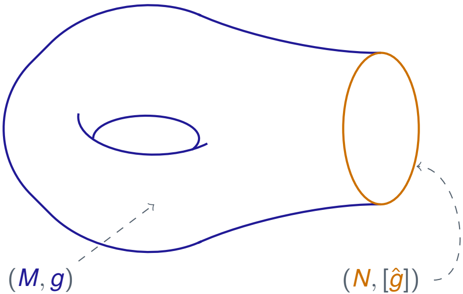
(equivalently, $\overline{g} = \rho^2 g$ restricts to $\hat{g}$ on N)



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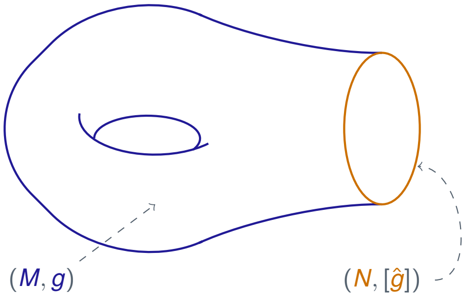

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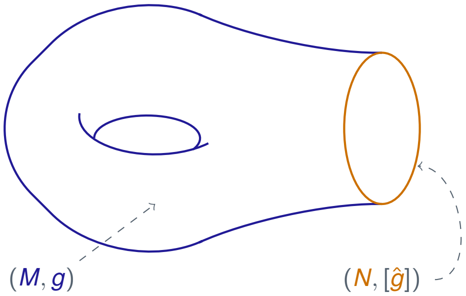
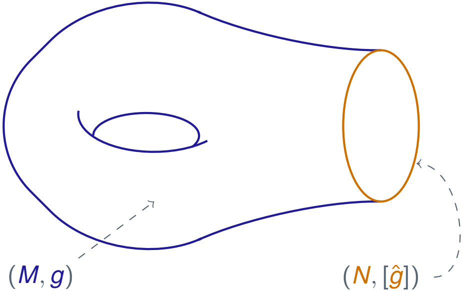
Riemannian invariants $\Longrightarrow$ conformal invariants



Sectional curvature

$$\sec(g) = -|d\rho|_g^2 + o(1)$$

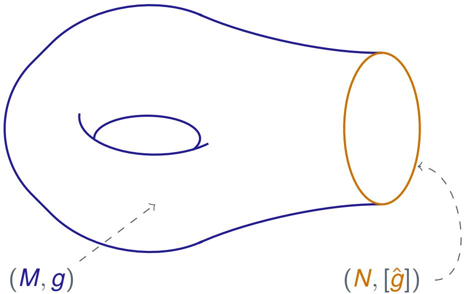


Sectional curvature

$$\sec(g) = -|d\rho|^2_{\bar{g}} + o(1)$$

(M^{n+1}, g) is called

- **asymptotically hyperbolic** if $|d\rho|^2_{\bar{g}} = 1$ on N
- **Poincaré-Einstein** if moreover $\mathrm{Ric}(g) = -ng$



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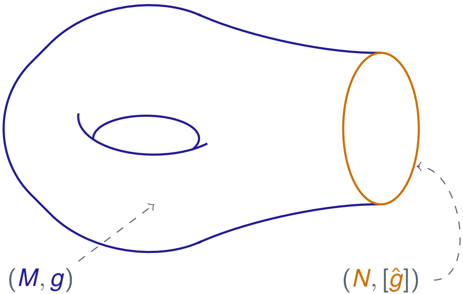

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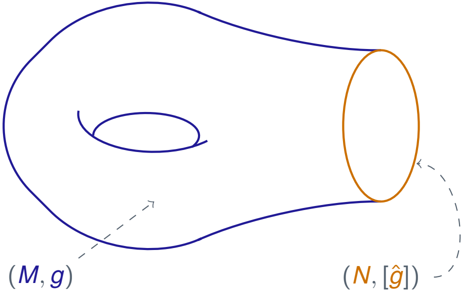

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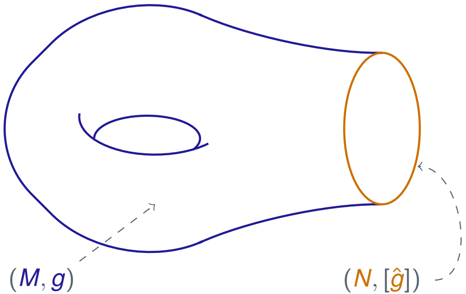

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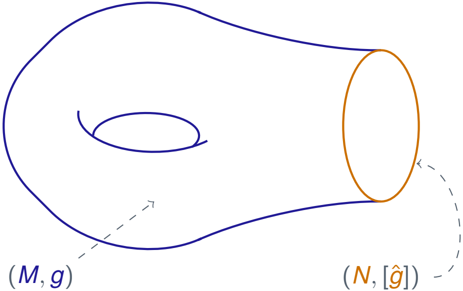
Bahuaud-Gicquaud-Lee-Marsh 10's: **yes** if



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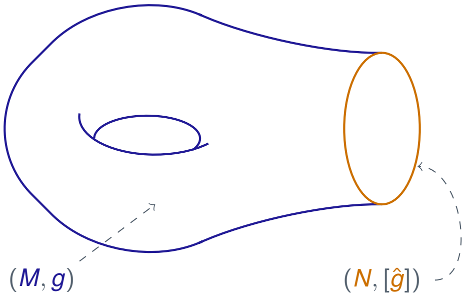

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$R^0(X, Y)Z = g(Y, Z)X - g(X, Z)Y$
$r = d_g(\cdot, o)$

Complex hyperbolic geometry

$(\mathbb{CH}^{n+1}, g, J)$

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Quarter-pinched
$-1 \leqslant \sec(P) \leqslant -\frac{1}{4}$

- $\sec(P) = -1 \iff P = JP$
- $\sec(P) = -\frac{1}{4} \iff P \perp JP$

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Einstein
$\mathrm{Ric}(g) = -\left(\frac{n}{2}+1\right) g$

CR manifold
(N, H, J_H) with

- H hyperplane distribution
- J_H formally integrable almost complex structure on H

Strictly pseudoconvex if

- H contact
- $\exists\theta$ calibrating H with $d\theta|_{\ker\theta}(\cdot, J_H\cdot) > 0$

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$$\begin{aligned} g_{\mathbb{CH}^{n+1}} &= dr^2 + 4\sinh^2(r)\theta_{\mathbb{S}^{2n+1}} \otimes \theta_{\mathbb{S}^{2n+1}} + 4\sinh^2(r/2)d\theta_{\mathbb{S}^{2n+1}}(\cdot, i\cdot) \\ &= dr^2 + e^{2r}\theta_{\mathbb{S}^{2n+1}} \otimes \theta_{\mathbb{S}^{2n+1}} + e^r d\theta_{\mathbb{S}^{2n+1}}(\cdot, i\cdot) + \dots \end{aligned}$$

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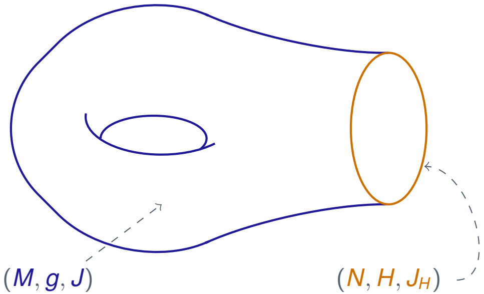
Intrinsic geometry of $(\mathbb{CH}^{n+1}, g, J) \implies$ CR geometry of S^{2n+1}



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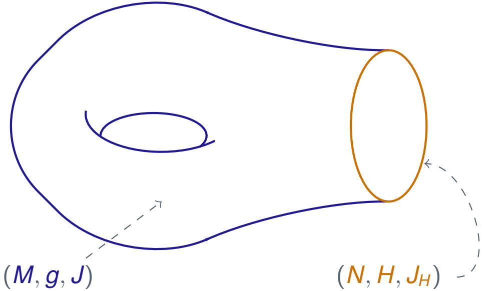
Asymptotically complex hyperbolic manifolds



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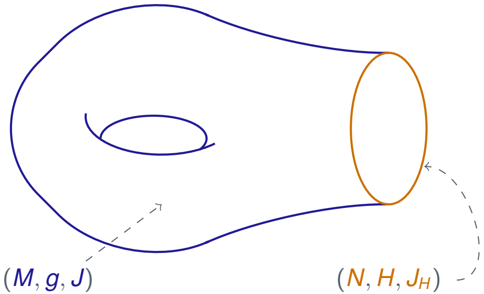

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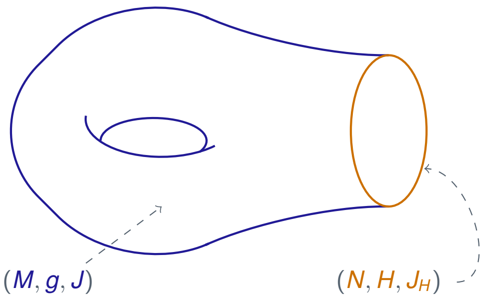
(almost) Hermitian / Kähler invariants $\Longrightarrow$ CR invariants



- $\sec^h(g) = -1 + o(1)$
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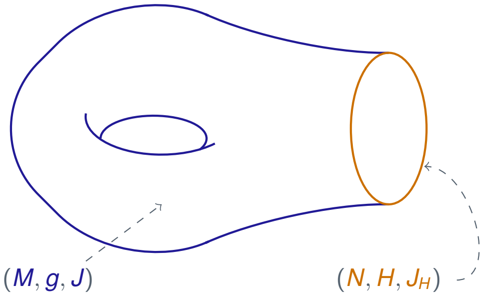

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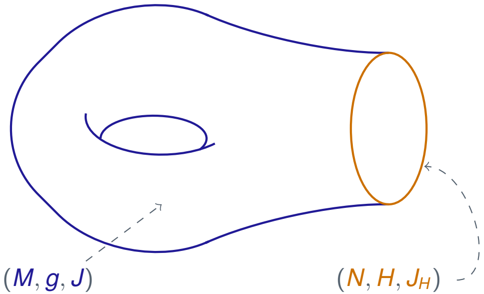

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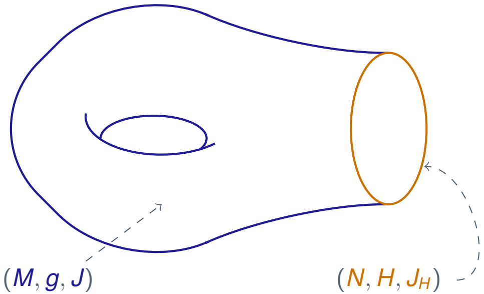

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Theorem [P. '22]

Let (M, g, J) be a complete, non-compact, Kähler manifold of real dimension $\geqslant 4$, with a convex core. Assume that

$$|R - R^0|_g, \qquad |\nabla R|_g \qquad = O(e^{-ar}), \quad a > 3/2$$

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$$|R - R^0|_g, \quad |\nabla J|_g, \quad |\nabla R|_g, \quad |\nabla^2 J|_g = O(e^{-ar}), \quad a > 1$$

Then (M, J) is the interior of a compact almost complex manifold $(\overline{M}, \overline{J})$ with boundary such that

- $\overline{J}$ is continuous up to the boundary
- $H_0 = \left(T\partial\overline{M}\right) \cap \left(\overline{J}T\partial\overline{M}\right)$ is a contact distribution of class $\mathcal{C}^1$
- $J_0 = \overline{J}|_{H_0}$ is of class $\mathcal{C}^1$, formally integrable
- $(\partial\overline{M}, H_0, J_0)$ is CR strictly pseudoconvex

Moreover, g is asymptotically complex hyperbolic: there exist ρ defining function, η^0 a $\mathcal{C}^1$ contact form on $\partial\overline{M}$, and a $\mathcal{C}^1$ Carnot metric $\gamma = d\eta^0|_{H_0 \times H_0}(\cdot, J_0 \cdot)$ such that

$$g = \frac{d\rho \otimes d\rho + \eta^0 \otimes \eta^0 + \rho\gamma + o(\rho^{3/2})}{\rho^2}$$

Bland 80's:
Compactification for some open Kähler manifolds

- assumptions not totally geometric
- $|R - R^0|_g = O(e^{-4r})$ *a posteriori*
- $\dim M = 4$, g Einstein: only applies to spherical boundaries (consequence of Biquard-Herzlich '05)

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Bracci-Gaussier-Zimmer '18:
$\Omega \subset \mathbb{C}^n$ bounded domain, $\partial\Omega$ of class $\mathcal{C}^{2,\alpha}$
$\partial\Omega$ strictly pseudoconvex if and only if
Ω carries a complete Kähler metric with

$$-1 - \varepsilon \leqslant \sec^h \leqslant -1 + \varepsilon$$

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P. '22-'23

- $a > 3/2 \implies \mathcal{C}^1$ everywhere
- g Kähler $\implies$ assumptions on J are superfluous

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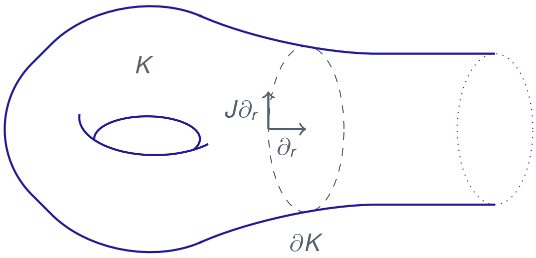


Convex core

- K compact, convex, $\mathrm{codim} = 0$
- ∂K smooth, orientable
- $\sec(M \setminus K) < 0$

Normal exponential map

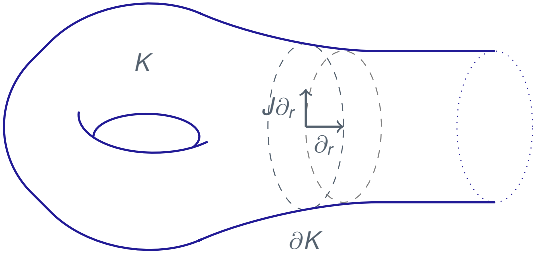
$$\mathcal{E}\colon (0,\infty) \times \partial K \xrightarrow{\sim} \overline{M \setminus K}$$

$$\mathcal{E}(r,p) = \mathcal{E}_r(p) := \exp_p(r\nu(p))$$

Intrinsic data

- Radial vector field ∂_r
- $J\partial_r$
- $\{\partial_r, J\partial_r\}^\perp$

Outline of the proof



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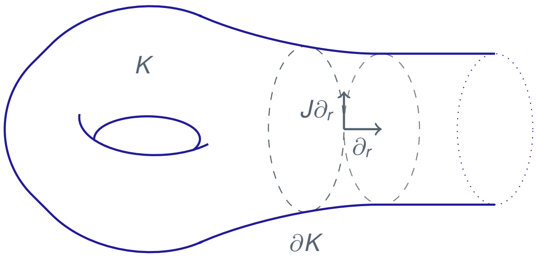

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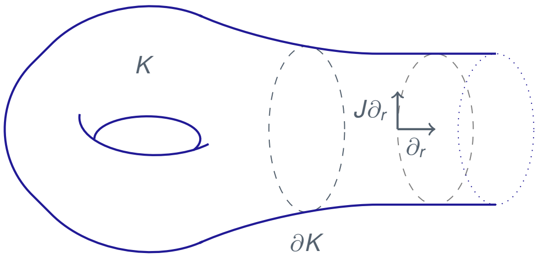

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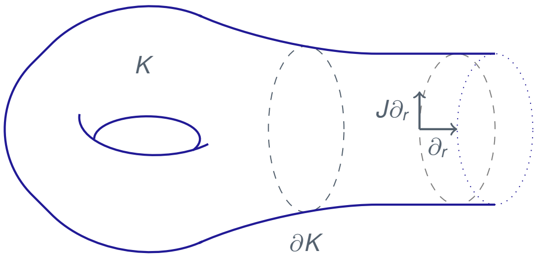

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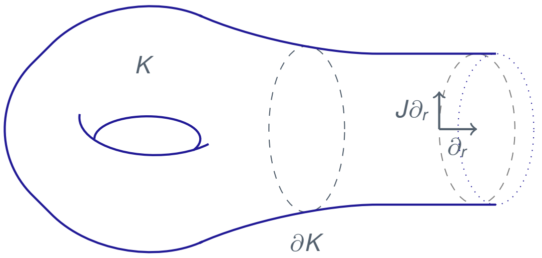

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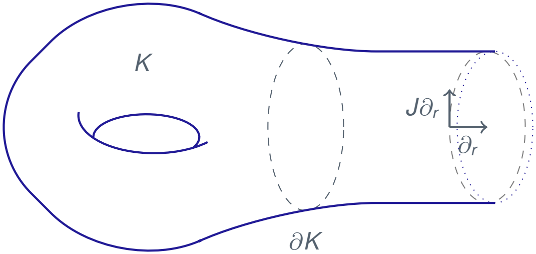

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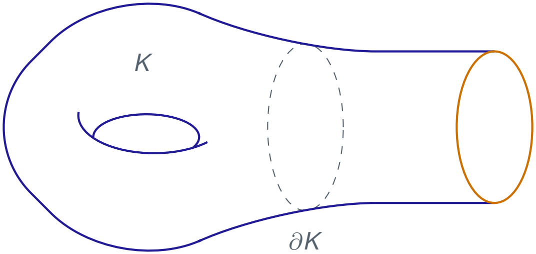

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Admissible frame: $\{\partial_r, J\partial_r, E_1, \ldots, E_{2n}\}$ orthonormal frame with $\nabla_{\partial_r} E_j = 0$

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in the Kähler case

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Associated coframes

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$$\varphi_r = \mathcal{E}_r^*(\pi^\perp \circ J \circ \pi^\perp)$$

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The metric reads

$$g = dr \otimes dr + e^{2r}\eta_r^0 \otimes \eta_r^0 + e^r \gamma_r$$

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Relations

- ${\varphi_r}^2 = -\mathrm{Id} + \eta_r^0 \otimes \xi_0^r$ with $\xi_0^r = \mathcal{E}_r^*(e^r J\partial_r)$
- $\eta_r^0 \circ \varphi_r = 0$
- $\gamma_r(\varphi_r \cdot, \varphi_r \cdot) = \gamma_r$
- $d\eta_r^0(\cdot, \varphi_r \cdot) = \gamma_r + O(e^{(1-a)r})$

Convergence of the coframes

- In $\mathcal{C}^0$ topology

$$\begin{cases} \partial_r^2 \eta_r^0 + 2\partial_r \eta_r^0 &= \sum_{k=0}^{2n} O\left(e^{(1/2-a)r}\right) \eta_r^k \\ \partial_r^2 \eta_r^j + \partial_r \eta_r^j &= \sum_{k=0}^{2n} O\left(e^{(1/2-a)r}\right) \eta_r^k \end{cases}$$

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$$\begin{cases} \partial_r^2 \left(e^r \mathcal{L}_u(\eta_r^0(v))\right) - e^r \mathcal{L}_u \left(\eta_r^0(v)\right) &= O\left(e^{(2-a)r}\right) \\ \partial_r^2 \left(e^{r/2} \mathcal{L}_u(\eta_r^j(v))\right) - e^{r/2} \mathcal{L}_u \left(\eta_r^j(v)\right) &= O\left(e^{(3/2-a)r}\right) \end{cases}$$

Convergence of φ_r

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$$\partial_r \varphi_r = \varphi_r \circ S_r - S_r \circ \varphi_r = O\left(e^{(3/2-a)}\right)$$

where $S_r = \mathcal{E}_r^*(S)$, $S = \nabla \partial_r$

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$$\varphi_r = \sum_{j=1}^{n} \eta_r^{2j-1} \otimes \xi_r^{2j} - \eta_r^{2j} \otimes \xi_{2j-1}^r$$

where $\{\xi_0^r, \ldots, \xi_{2n}^r\}$ is dual to $\{\eta_r^0, \ldots, \eta_r^{2n}\}$

Consequence: ${\varphi_r}^2 = -\mathrm{Id} + \eta_r^0 \otimes \xi_0^r \implies J_0 = \varphi|_{\ker \eta^0}$ is an almost complex structure

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Consequence: ${\varphi_r}^2 = -\mathrm{Id} + \eta_r^0 \otimes \xi_0^r \implies J_0 = \varphi|_{\ker \eta^0}$ is an almost complex structure
The integrability of J_0 is more complicated

Outline of the proof
in the non-Kähler case

$\nabla J \neq 0$
But $|\nabla J|_g = O(e^{-ar}) \implies \nabla_{\partial_r} J \partial_r = O(e^{-ar})$

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Key fact

There exists a unique vector field E_0 on $\overline{M \setminus K}$ such that

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- $\beta_r(v) - \beta_0(v) = \int_0^r g(v^{\parallel}, \nabla_{\partial_r} J\partial_r) = \int_0^r O(|v|_g |\nabla J|_g)$, hence $\implies \beta_r(v) \to \beta(v)$

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- $|\beta - \beta_r|_{g|_{\partial K}} \leqslant \int_r^\infty |\nabla J|_g$, hence β is continuous, nowhere zero

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Define e_0 the unique vector field with $\beta(e_0) = 1$, $e_0 \perp \ker \beta$
and E_0 its radial parallel transport on $\overline{M \setminus K}$

Outline of the proof
in the non-Kähler case

One can now initiate the same strategy as in the Kähler case, but

- substitute $J\partial_r$ by E_0 "almost everywhere"
  caution: the resulting admissible frame $\{\partial_r, E_0, \ldots, E_{2n}\}$ is not smooth
- large amount of extra estimates
- extra error terms in all equations must be understood

Tack så mycket!

KTH
VETENSKAP
OCH KONST