Analysis of Rate-Optimized Throughput for Large-Scale MIMO-(H)ARQ Schemes

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Presented at GlobeCom 2014-12-09

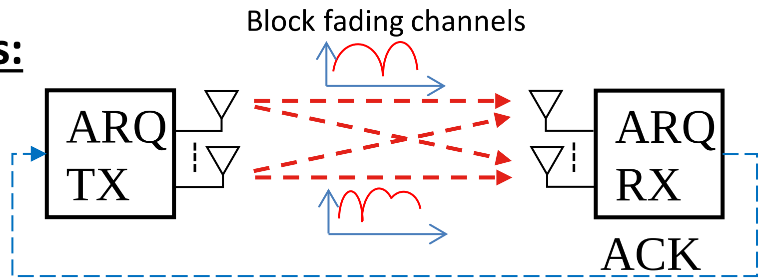
Outline

- Problem
  - Scenario, Objective & Motivation
- Solution
  - Analysis & Results
- Conclusions

Problem Scenario: MIMO-(H)ARQ

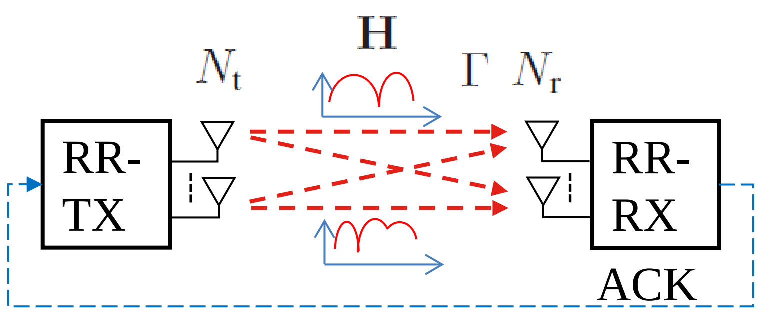
Three scenarios:

- MIMO-ARQ



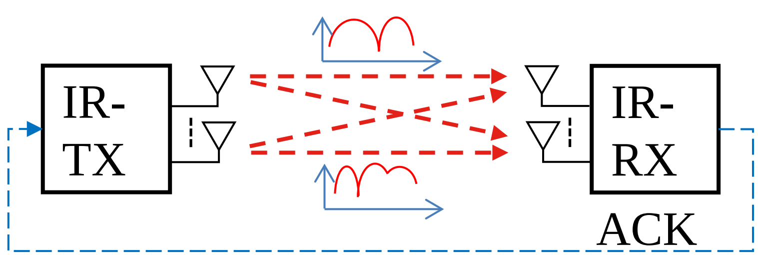
- Retransmit: Repeat packet
- Combine packets: No

- MIMO-RR -HARQ



- Retransmit: Repeat packet
- Combine packets: Yes

- MIMO-IR -HARQ



- Retransmit: Incremental blocks
- Combine blocks: Yes

N.B. Automatic Repeat reQuest (ARQ), Hybrid-ARQ (HARQ)

Problem objectives & motivation

Our objectives

- Analytically characterize throughput $T \triangleq \frac{R}{S}$
- Optimize throughput
  - Opt. rate point R^*
  - Opt. throughput value T^*

Why is this hard?

- Exact MIMO analysis often intractable
- (H)ARQ makes it more complicated
  - SISO-IR-HARQ problem unsolved

The contributions are:

- **Proposing the use of RMT for MIMO-(H)ARQ analysis.**

- **Deriving closed-form throughput expressions (ARQ, RR, IR).**

- **Deriving very tight closed-form bounds for the optimal rate points and optimal throughput values, (ARQ, RR, IR).**

N.B. Random Matrix Theory (RMT) is a tool to study large matrices as they tend to infinite sizes. Example: the (asymptotic) eigenvalue-distribution.

Analysis: MIMO-ARQ Throughput

RMT and MIMO

- MIMO channel capacity

$$C = \ln\det\left(\mathbf{I}_{N_\mathrm{r}} + \Gamma N_\mathrm{t}^{-1}\mathbf{H}\mathbf{H}^H\right)$$

- RMT – Asymptotically Gaussian [Kamath]

$$(C - N_\mathrm{t} I_c)/S_c \to \mathcal{N}(0,1)$$

$$N_\mathrm{t} \to \infty, N_\mathrm{t}/N_\mathrm{r} = c$$

with parameters

$$I_c = c\ln(1+\Gamma(1-m_c)) + \ln(1+\Gamma(c-m_c)) - m_c$$

$$S_c^2 = -\ln\left(1 - m_c^2/c\right)$$

Marchenko-Pastur law:

$$m_c = \frac{1}{2}\left(1 + c + 1/\Gamma - \sqrt{(1+c+1/\Gamma)^2 - 4c}\right)$$

For simplicity, we will assume

$$N = N_\mathrm{t} = N_\mathrm{r},\ c = 1$$

M. Kamath, B. Hughes, and Y. Xinying, "Gaussian approximations for the capacity of MIMO Rayleigh fading channels," in *Asilomar Conf. on Signals, Systems and Computers*, Nov. 2002.

MIMO-ARQ throughput

- Decoding probability

$$P = \mathbb{P}\{C > R\}$$

- Throughput

$$T = RP$$

- Asymptotic decoding probability

$$P \simeq Q\left((R - NI)/S\right)$$

$$Q(x) = \int_x^\infty \mathrm{e}^{-t^2/2}/\sqrt{2\pi}\,dt$$

- Asymptotic throughput

$$\tilde{T} \simeq RQ\left((R - NI)/S\right)$$

Analysis: MIMO-HARQ Throughput

IR-HARQ throughput

- MIMO IR-HARQ capacity

$$C_k^{\mathrm{IR}} = \sum_{j=1}^{k} \ln\det\left(\mathbf{I}_{N_\mathrm{r}} + \frac{\Gamma}{N_\mathrm{t}}\mathbf{H}_j\mathbf{H}_j^H\right)$$

- Decoding failure probability

$$Q_k^{\mathrm{IR}} = \mathbb{P}\left\{C_k^{\mathrm{IR}} < R\right\}$$

- Throughput

$$T^{\mathrm{HARQ}} = \frac{R}{1+\sum_{k=1}^{\infty} Q_k^{\mathrm{HARQ}}}$$

- Asymptotic decoding failure probability

$$\tilde{Q}_k^{\mathrm{IR}} \simeq Q\left(-(R-kNI)/\sqrt{k}S\right)$$

- Asymptotic throughput

$$\tilde{T}^{\mathrm{IR}} \simeq \frac{R}{1+\sum_{k=1}^{\infty} Q\left(-\frac{R-kNI}{\sqrt{k}S}\right)}$$

RR-HARQ throughput

- MIMO RR-HARQ capacity

$$C_k^{\mathrm{RR}} = \ln\det\left(\mathbf{I}_{N_r} + \frac{\Gamma}{N_t}\sum_{j=1}^{k}\mathbf{H}_j\mathbf{H}_j^H\right)$$

$$= \ln\det\left(\mathbf{I}_{N_\mathrm{t}} + \Gamma N_\mathrm{t}^{-1}\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\right)$$

$$\tilde{\mathbf{H}} = [\mathbf{H}_1;\mathbf{H}_2;\ldots\mathbf{H}_k]$$

- Decoding failure prob. & throughput

  See IR-HARQ

- Asymptotic decoding failure probability

$$\tilde{Q}_k^{\mathrm{RR}} \simeq Q\left(-(R-NI_k)/S_k\right)$$

- Asymptotic throughput

$$\tilde{T}^{\mathrm{RR}} \simeq \frac{R}{1+\sum_{k=1}^{\infty} Q\left(-\frac{R-NI_k}{S_k}\right)}$$

"Exact" Simulation vs. Approx. RMT Analysis

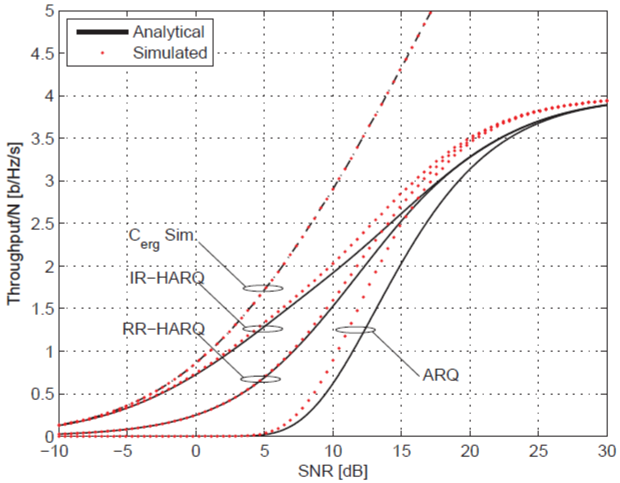


Figure 1. $\tilde{T}/N$ vs. Γ for MIMO-ARQ with $R/N = 4$, $N = 1$

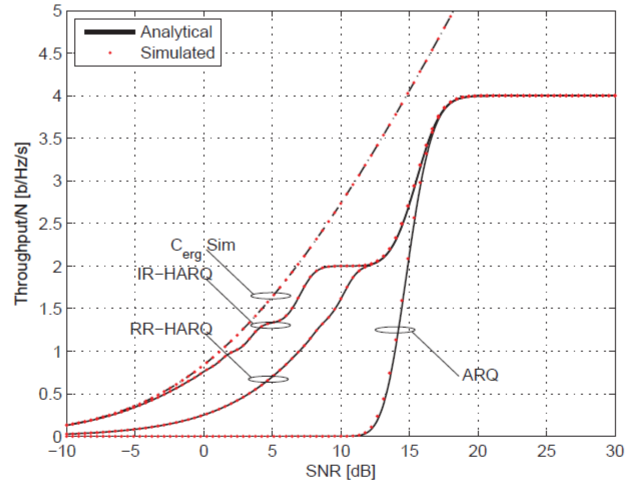


Figure 3. $\tilde{T}/N$ vs. Γ for MIMO-ARQ with $R/N = 4$, $N = 4$

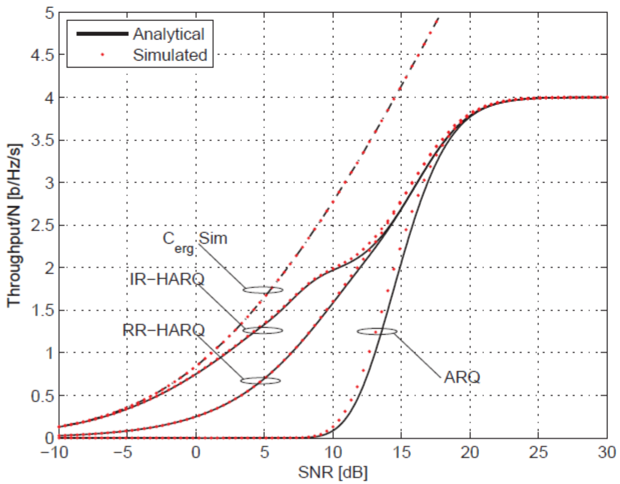


Figure 2. $\tilde{T}/N$ vs. Γ for MIMO-ARQ with $R/N = 4$, $N = 2$

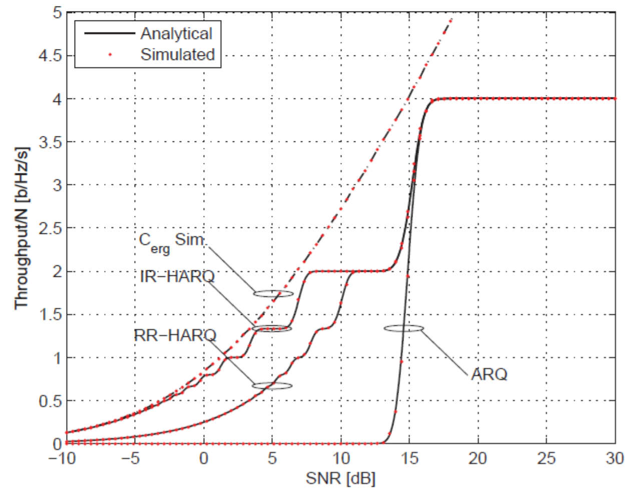


Figure 4. $\tilde{T}/N$ vs. Γ for MIMO-ARQ with $R/N = 4$, $N = 8$

Analysis: MIMO-ARQ Optimization

MIMO-ARQ throughput optimization

- Substitution

$x \triangleq \frac{NI-R}{S}$

$T = (NI - Sx)(1 - Q(x))$

- Optimization

$$\frac{d\ln(\tilde{T})}{dx} = \frac{-S}{NI - Sx^*} + \frac{Q'(x^*)}{1 - Q(x^*)} = 0$$

- Optimality condition

$$\sqrt{2\pi}e^{x^2/2}(1 - Q(x)) + x = NI/S$$

- Define

$f(x) \triangleq \sqrt{2\pi}e^{x^2/2}(1 - Q(x)) + x.$

Cont…

- Solution

$x^* = f^{-1}(NI/S)$

$\tilde{R}^* = NI - Sx^*$

- Problem!

$f^{-1}(x)$ not easily computed!

- Craft good bounds!

$f_l(x) \leqslant f(x) \leqslant f_u(x)$, with

$$f_u(x) = \sqrt{2\pi}e^{x^2/2} - \frac{x}{1+x^2} + x,$$

$$f_l(x) = \sqrt{2\pi}e^{x^2/2} - \frac{1}{x} + x,$$

Used well-known Q-function bounds

$$\frac{x}{1+x^2}e^{-x^2/2} \leqslant \sqrt{2\pi}Q(x) \leqslant \frac{1}{x}e^{-x^2/2}$$

Analysis: MIMO-ARQ Optimization

Cont…

- Craft invertible bounds to the bounds!

$$f_{\mathrm{ll}}(x) \underset{x_1 \geqslant 1}{\leqslant} f_{\mathrm{l}}(x) \leqslant f(x) \leqslant f_{\mathrm{u}}(x) \leqslant f_{\mathrm{uu}}(x)$$

$$f_{\mathrm{uu}}(x) = \sqrt{2\pi} \mathrm{e}^{x^2/2} + \frac{x^2}{2},$$

$$f_{\mathrm{ll}}(x) = \sqrt{2\pi} \mathrm{e}^{x^2/2}.$$

- Solution with the $f_{\mathrm{uu}}(x)$ bound

$$x_{\mathrm{uu}} = \sqrt{2 \ln \left(\frac{W_0 \left(\sqrt{2\pi} \exp\left(NI/S \right) \right)}{\sqrt{2\pi}} \right)}$$

$$R_{\mathrm{uu}} = NI - S x_{\mathrm{uu}}$$

- Solution with the $f_{\mathrm{ll}}(x)$ bound

$$x_{\mathrm{ll}} = \sqrt{2 \ln \left(\frac{NI/S}{\sqrt{2\pi}} \right)}.$$

$$R_{\mathrm{ll}} = NI - S x_{\mathrm{ll}}$$

Cont…

- Bounds on optimal rate points

$$R_{\mathrm{ll}} \leqslant \tilde{R}^* \leqslant R_{\mathrm{uu}}$$

- Diminishing optimal rate gap

$$\lim_{N \to \infty} \Delta R \triangleq \lim_{N \to \infty} R_{\mathrm{uu}} - R_{\mathrm{ll}} = 0$$

- Optimal throughput with bounds

$$T_{\mathrm{uu}} = R_{\mathrm{uu}}(1 - Q(x_{\mathrm{uu}})),$$
$$T_{\mathrm{ll}} = R_{\mathrm{ll}}\ (1 - Q(x_{\mathrm{ll}})).$$

- Lower bounds on optimal throughput

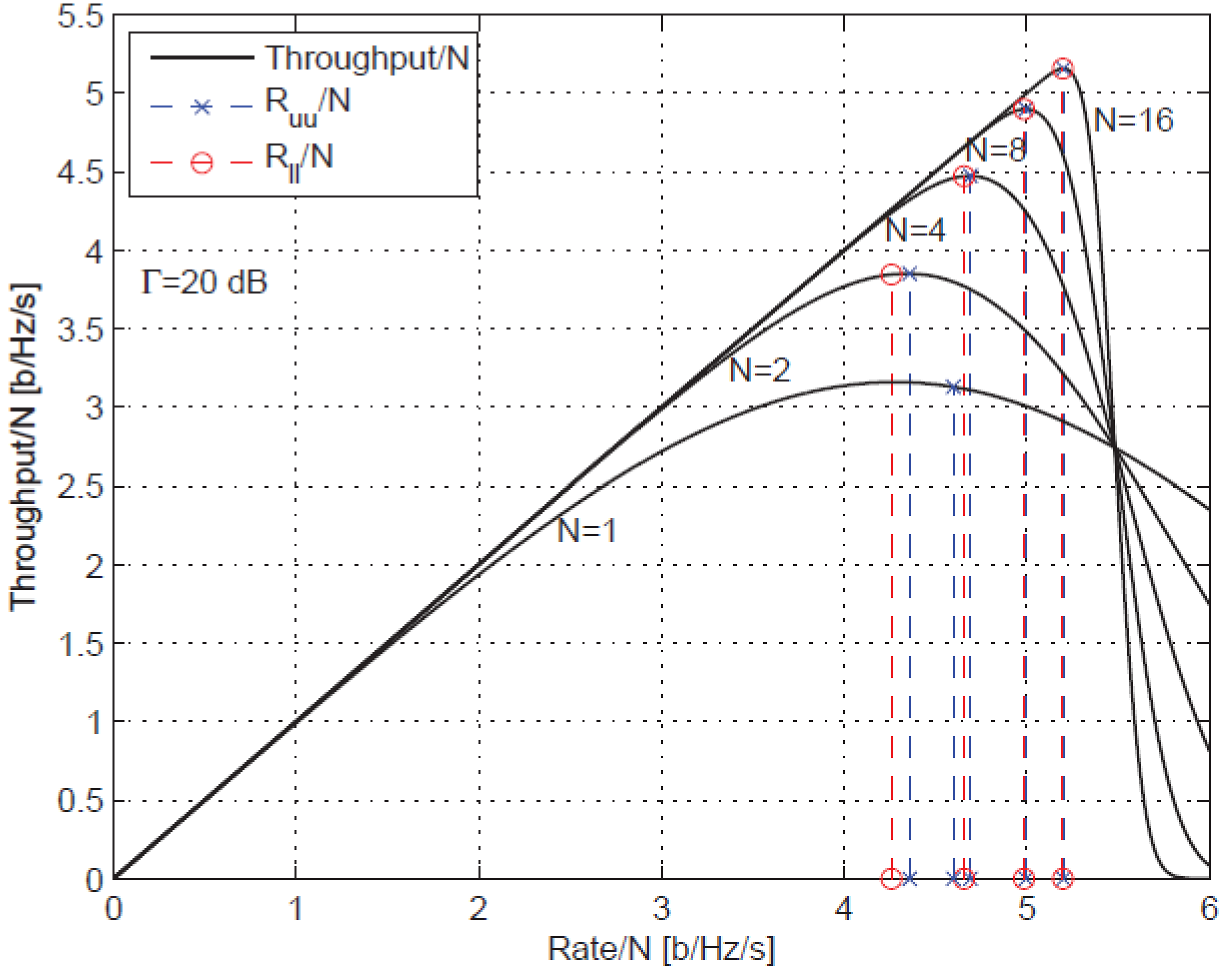
$$T_{\mathrm{ll}} \leqslant \tilde{T}^* \text{, and } T_{\mathrm{uu}} \leqslant \tilde{T}^*$$

- Diminishing optimal throughput gap

$$\lim_{N \to \infty} \Delta T_{\mathrm{uu}} \triangleq \lim_{N \to \infty} T_{\mathrm{uu}} - \tilde{T}^* = 0,$$
$$\lim_{N \to \infty} \Delta T_{\mathrm{ll}} \triangleq \lim_{N \to \infty} \tilde{T}^* - T_{\mathrm{ll}} = 0.$$

Analysis: MIMO-ARQ Optimization

Optimal rate bounds



Analysis: MIMO-HARQ Optimization

IR-HARQ optimal throughput

- Solution with the $f_{\text{uu}}(x_K)$ bound

$$R_{K,\text{uu}}^{\text{IR}} = KNI - \sqrt{K}Sx_{K,\text{uu}}, \text{ with}$$
$$x_{K,\text{uu}} = \sqrt{2\ln\left(\frac{W_0(K\sqrt{2\pi}\exp(\frac{\sqrt{K}NI}{S} - \sqrt{\frac{\pi}{2}}))}{K\sqrt{2\pi}}\right)}$$

- Solution with the $f_{\text{ll}}(x_K)$ bound

$$R_{K,\text{ll}}^{\text{IR}} = KNI - \sqrt{K}Sx_{K,\text{ll}}, \text{ with}$$
$$x_{K,\text{ll}} = \sqrt{2\ln\left(\frac{NI/S}{\sqrt{2\pi K}} - \frac{1}{2K}\right)}.$$

RR-HARQ optimal throughput

- Solution with the $f_{\text{uu}}(x_K)$ bound

$$R_{K,\text{uu}}^{\text{RR}} = NI_K - S_K \cdot x_{K,\text{uu}}, \text{ with}$$
$$x_{K,\text{uu}} = \sqrt{2\ln\left(\frac{W_0(K\sqrt{2\pi}\exp(\frac{NI_K}{S_K} - \sqrt{\frac{\pi}{2}}))}{K\sqrt{2\pi}}\right)},$$

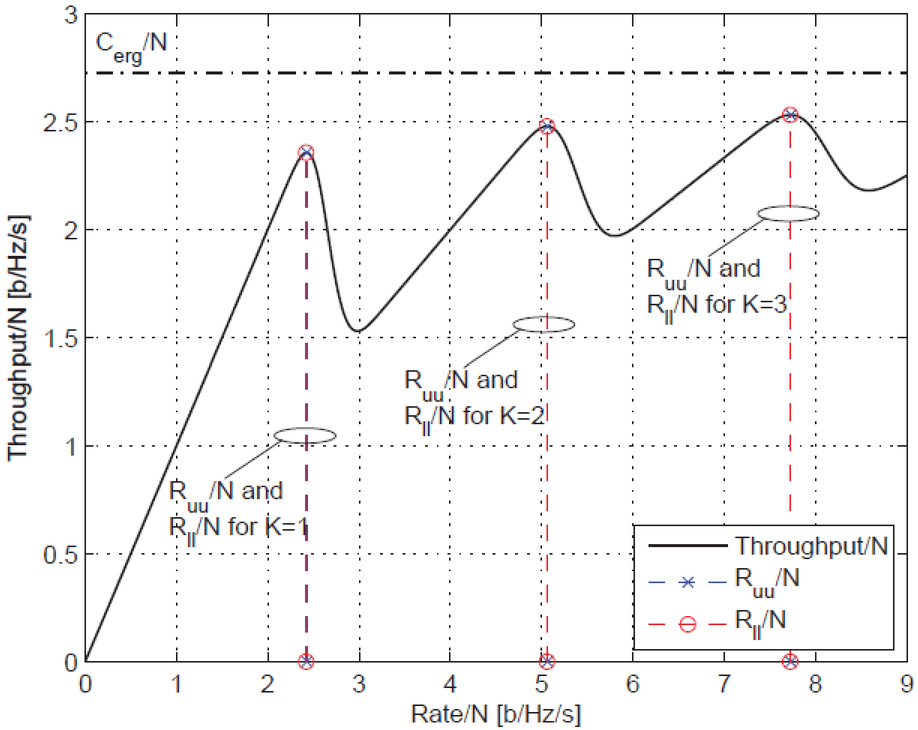
- Solution with the $f_{\text{ll}}(x_K)$ bound

$$R_{K,\text{ll}}^{\text{RR}} = NI_K - S_K \cdot x_{K,\text{ll}}, \text{ with}$$
$$x_{K,\text{ll}} = \sqrt{2\ln\left(\frac{NI_K/S_K}{K\sqrt{2\pi}} - \frac{1}{2K}\right)}.$$

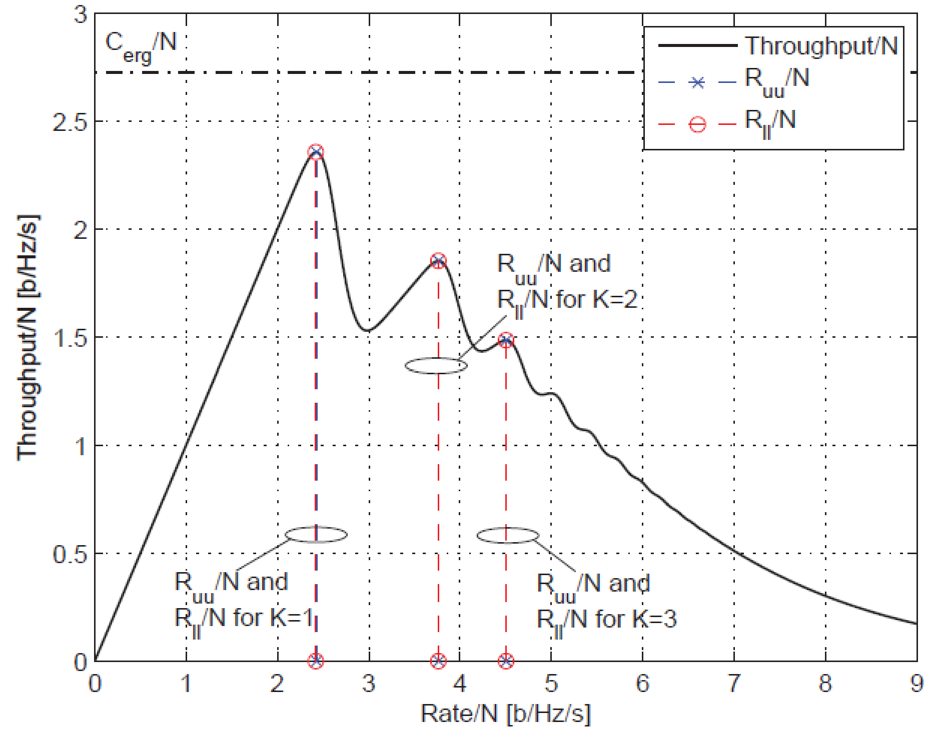
Analysis: MIMO-HARQ Optimization

Optimal rate bounds

IR-HARQ



RR-HARQ



Summary & Conclusions

- RMT is an excellent technique for studying MIMO-(H)ARQ. Great match already for N≥4.

- Max ARQ, RR and IR throughput, similar for N≥4.

- Future systems may aim for rate-adapted MIMO-ARQ, instead of the more complex MIMO-RR/IR-HARQ.

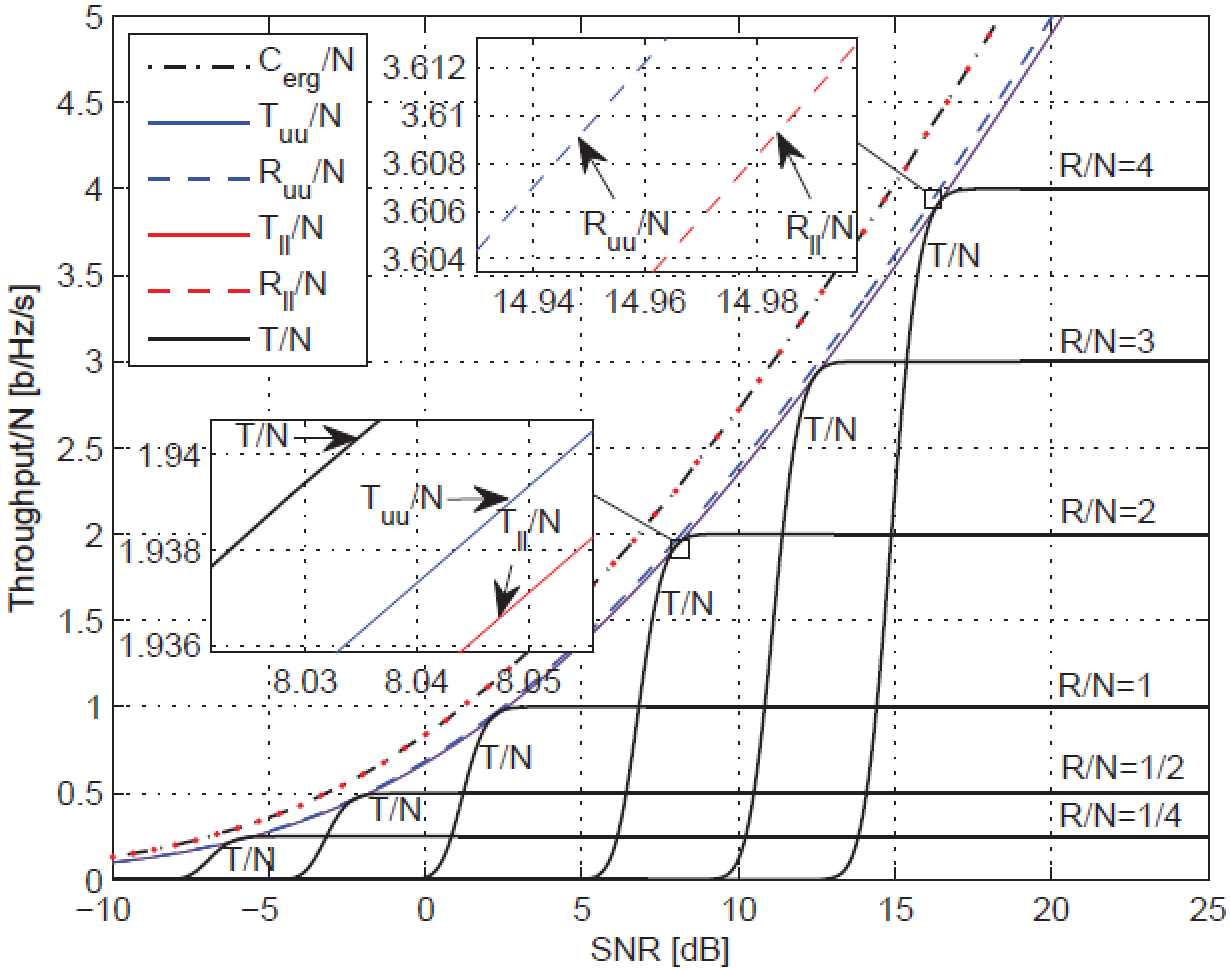
Related Papers

- Please also consider the following related papers:
  - P. Larsson, L. K. Rasmussen, M. Skoglund, "**Throughput Analysis of ARQ Schemes in Gaussian Block Fading Channels**," *IEEE Trans. Commun.*, July 2014.
  - P. Larsson, L. K. Rasmussen, M. Skoglund, **"Throughput Analysis of ARQ in Interference with Nakagami-m Block Fading Channels,"** in *Proc. of ICC*, Sydney, June 2014.
  - P. Larsson, L. K. Rasmussen, M. Skoglund, "**Multi-layer Network Coded ARQ for multiple unicast flows,**" in *Proc. of SWE-CTW*, Lund, Oct. 2012.

Supplementary slides

Analysis: MIMO-ARQ Optimization

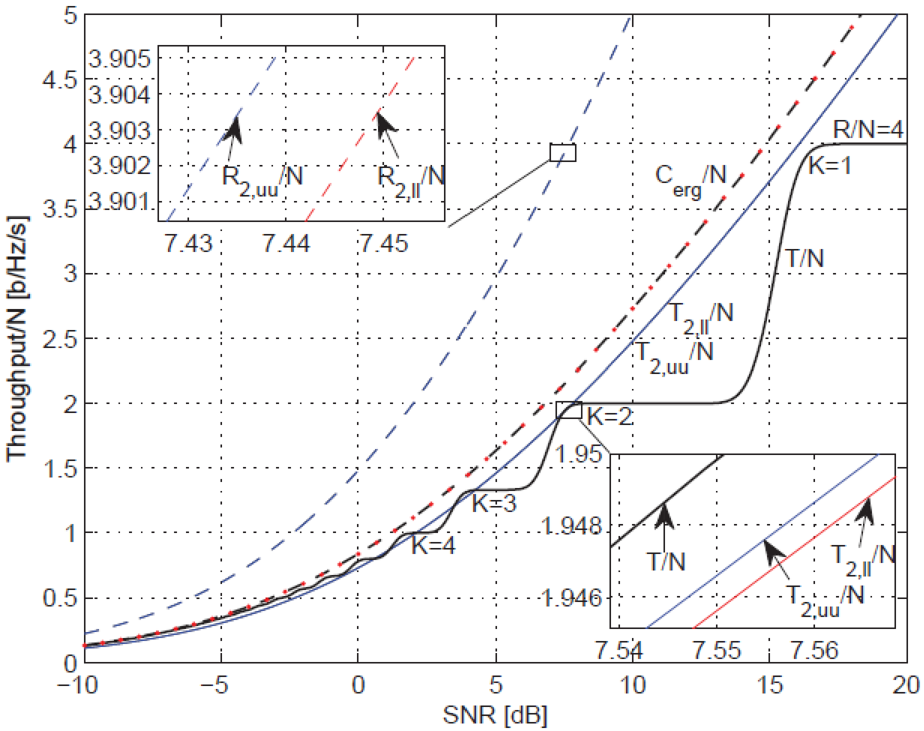
Optimal rate and throughput bounds



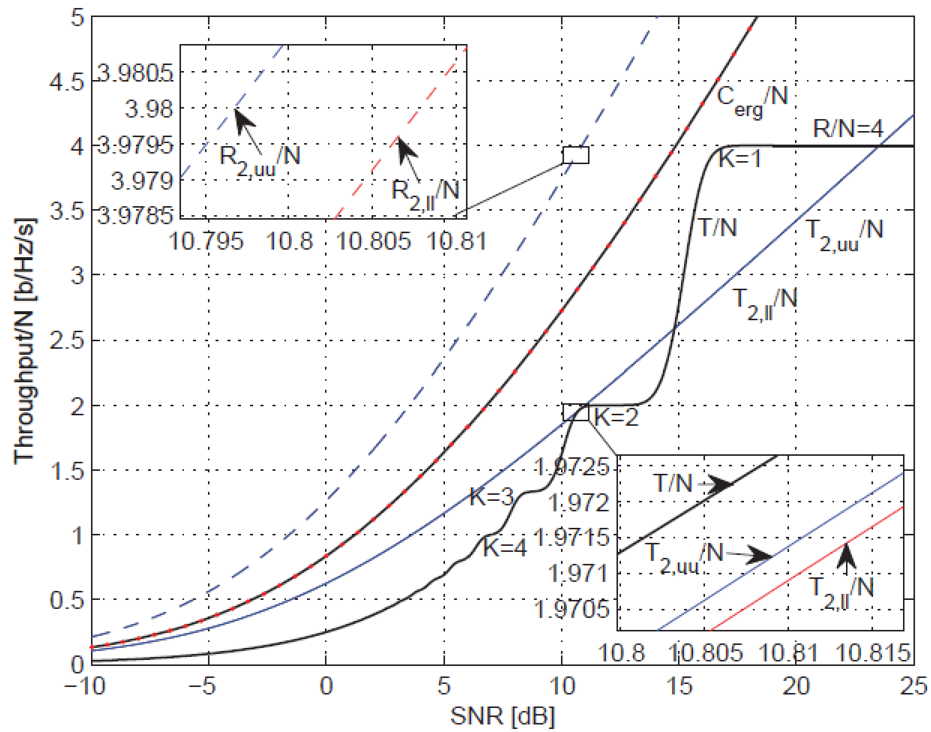
Analysis: MIMO-ARQ Optimization

Optimal rate and throughput bounds

IR-HARQ



RR-HARQ



Simulation vs. RMT-based analysis

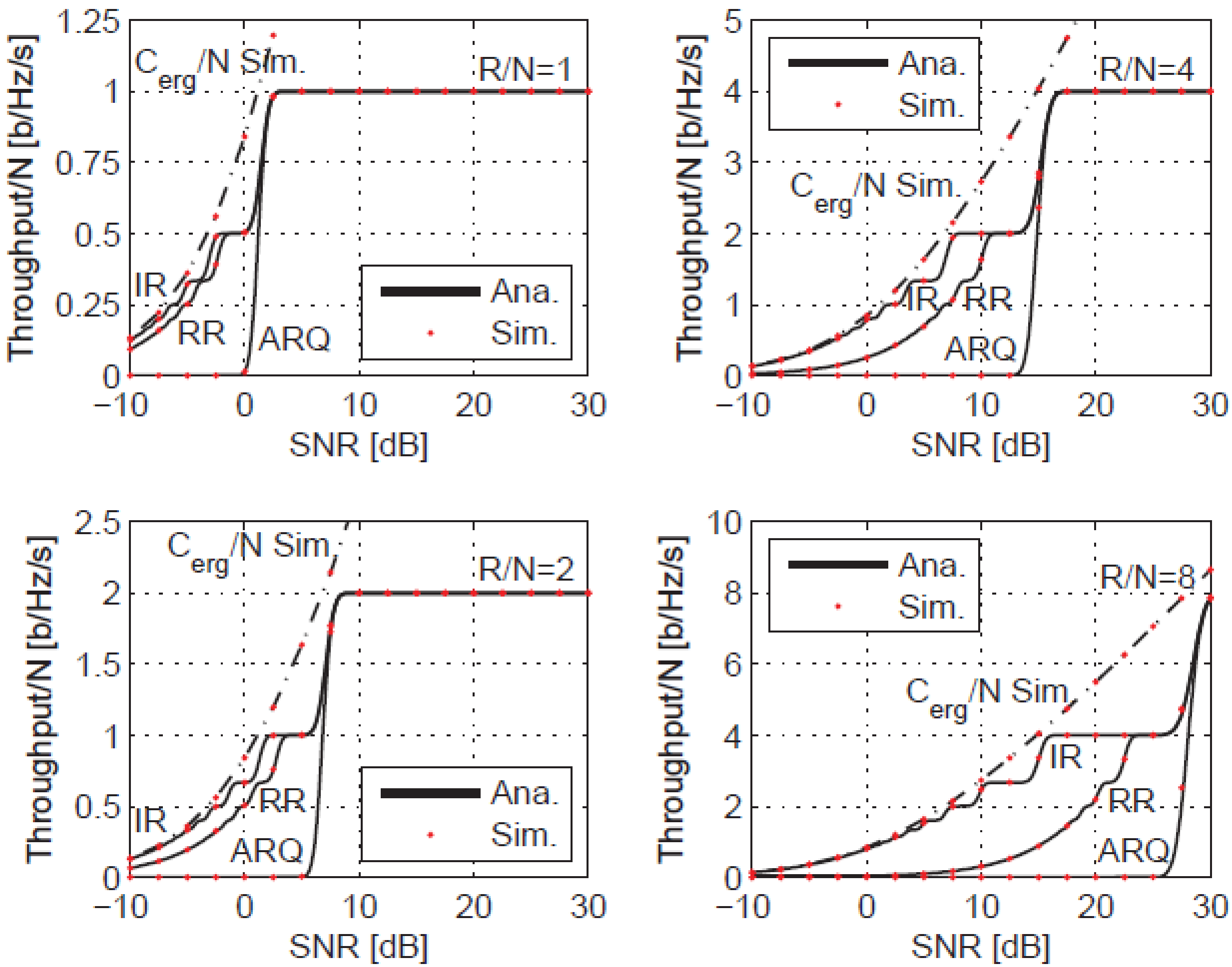


Figure 5. $\tilde{T}/N$ vs. Γ for MIMO-ARQ with $R/N = 2^{\{0,1,2,3\}}$, $N = 8$.

Diminishing optimal rate gap