The Matrix Exponential Distribution – A Tool for Wireless System Performance Analysis

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Objectives

- Introduce the matrix exponential (ME)-distribution as a new fading channel model.
- Present a unified, tractable, and powerful performance evaluation framework for wireless system with fading channels.
- Exemplify the ME-distribution (and its generalizations) in related areas.

Evolution

The Matrix Exponential Distribution –
A Tool for Wireless System Performance Analysis

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arXiv preprint arXiv:1612.06809 (2016).

P. Larsson, J. Gross, H. Al-Zubaidy, L. K. Rasmussen, and M. Skoglund, “Effective capacity of retransmission schemes: A recurrence relation approach,” *IEEE Transactions on Communications*, vol. 64, no. 11, pp. 4817–4835, Nov 2016.

P. Larsson, L. K. Rasmussen, and M. Skoglund, “Throughput analysis of hybrid-ARQ – A matrix exponential distribution approach,” *IEEE Transactions on Communications*, vol. 64, no. 1, pp. 416–428, Jan 2016.

P. Larsson, L. K. Rasmussen, and M. Skoglund, “Throughput analysis of ARQ schemes in Gaussian block fading channels,” *IEEE Transactions on Communications*, vol. 62, no. 7, pp. 2569–2588, July 2014.

Outline

- Preliminaries
- Performance Analysis Framework
- New ME-distr. Properties
- Applications
- Other Uses of ME-distr.
  - (ME-distributed discrete-time signals)
  - (ME-distribution generalizations)

PRELIMINARIES

Preliminaries: Matrix Exponential

- ME-series definition
$$e^{t\mathbf{X}} \triangleq \sum_{k=0}^{\infty} \frac{(t\mathbf{X})^k}{k!},$$
- ME-limit definition
$$e^{t\mathbf{X}} = \lim_{k\to\infty} \left(1 + \frac{t\mathbf{X}}{k}\right)^k.$$
- Derivative
$$\frac{\mathrm{d}}{\mathrm{d}t} e^{t\mathbf{X}} = \mathbf{X} e^{t\mathbf{X}}.$$
- Integral
$$\int_a^b e^{t\mathbf{X}}\,\mathrm{d}t = \mathbf{X}^{-1}\left(e^{t\mathbf{X}} - \mathbf{I}\right)\Big|_a^b,$$

[33] C. Moler and C. V. Loan, "Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later," *SIAM Review*, vol. 45, no. 1, pp. 3–49, 2003.
[34] N. J. Higham, *Functions of matrices – Theory and computation*. SIAM, 2008.

Matrix exponential func. in Matlab: *expm(.)*
Exponential func. in Matlab: *exp(.)*

Preliminaries:
Matrix Exponential Distribution (1)

- cdf

$$F_T(t) = 1 + \mathbf{x}e^{t\mathbf{Y}}\mathbf{Y}^{-1}\mathbf{z}, t \geq 0,$$

The ME-distribution is dense on t>=0

- pdf

$$f_T(t) = \mathbf{x}e^{t\mathbf{Y}}\mathbf{z}, t \geq 0,$$

Sum of products of Exp., Trig., and Poly.

- Moments

$$\mathbb{E}\{T^k\} = (-1)^{k+1}k!\mathbf{x}\mathbf{Y}^{-(k+1)}\mathbf{z}.$$

- Laplace transform

$$F(s) = \mathbf{x}(s\mathbf{I} - \mathbf{Y})^{-1}\mathbf{z}.$$

[35] M. W. Fackrell, "Characterization of matrix-exponential distributions," Ph.D. dissertation, The University of Adelaide, Faculty of Engineering, Computer and Mathematical Sciences, 2003.
[36] S. Asmussen and C. A. Ocinneide, *Matrix-Exponential Distributions*. John Wiley & Sons, Inc., 2004. [Online]. Available: http://dx.doi.org/10.1002/0471667196.ess1092.pub2
[37] J. E. Ruiz-Castro, "Matrix-exponential distributions: Closure properties," *International Journal of Advanced Statistics and Probability*, vol. 1, no. 2, pp. 44–52, 2013.
[38] M. F. Neuts, *Matrix-geometric solutions in stochastic models : An algorithmic approach*, ser. Johns Hopkins series in the mathematical sciences. Baltimore: Johns Hopkins University Press, 1981.

Preliminaries: Matrix Exponential Distribution (2)

- Companion form

$$\mathbf{Y} \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -y_1 & -y_2 & -y_3 & \cdots & -y_{d-1} & -y_d \end{bmatrix}, \quad (9)$$

$$\mathbf{x} \triangleq [x_1 \ x_2 \ \ldots \ x_{d-1} \ x_d] \in \mathbb{R}^{1\times d}, \quad (10)$$

$$\mathbf{y} \triangleq [y_1 \ y_2 \ \ldots \ y_{d-1} \ y_d] \in \mathbb{R}^{1\times d}, \quad (11)$$

$$\mathbf{z} \triangleq [0 \ 0 \ \ldots \ 0 \ 1]^{\mathrm{T}} \in \mathbb{R}^{d\times 1}, \quad (12)$$

- Rational LT

$$F(s) = \frac{x(s)}{y(s)}, \quad (13)$$

$$x(s) \triangleq x_d s^{d-1} + x_{d-1} s^{d-2} + \ldots + x_2 s^1 + x_1, \quad (14)$$

$$y(s) \triangleq s^d + y_d s^{d-1} + y_{d-1} s^{d-2} + \ldots + y_2 s^1 + y_1. \quad (15)$$

[29] N. G. Bean, M. Fackrell, and P. Taylor, "Characterization of matrix-exponential distributions," *Stochastic Models*, vol. 24, no. 3, pp. 339–363, 2008. [Online]. Available: http://dx.doi.org/10.1080/15326340802232186

[30] S. Asmussen and M. Bladt, *Renewal theory and queueing algorithms for matrix-exponential distributions*. Marcel Dekker Incorporated, 1996, pp. 313–341.

[36] S. Asmussen and C. A. Ocinneide, *Matrix-Exponential Distributions*. John Wiley & Sons, Inc., 2004. [Online]. Available: http://dx.doi.org/10.1002/0471667196.ess1092.pub2

Preliminaries: Matrix Exponential Distribution (3)

- Convolution

Proposition 2.1: (Convolution of two ME-distributed r.v.s. [37, Proposition 3.1]) Let the r.v.s. $T_j, j = \{1,2\}$ have pdfs $f_T^{(j)}(t) = \mathbf{x}_j \mathrm{e}^{t\mathbf{Y}_j}\mathbf{z}_j$. Then, $T = T_1 + T_2$ has the pdf

$$f_T(t) = \mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}, \tag{16}$$

where

$$\mathbf{x} = \begin{bmatrix}\mathbf{x}_1 & \mathbf{0}\end{bmatrix}, \tag{17}$$

$$\mathbf{Y} = \begin{bmatrix}\mathbf{Y}_1 & \mathbf{z}_1\mathbf{x}_2 \\ \mathbf{0} & \mathbf{Y}_2\end{bmatrix}, \tag{18}$$

$$\mathbf{z} = \begin{bmatrix}\mathbf{0} \\ \mathbf{z}_2\end{bmatrix}. \tag{19}$$

Proof:

$$\begin{aligned}
&\mathbf{x}_1\mathrm{e}^{t\mathbf{Y}_1}\mathbf{z_1} * \mathbf{x}_2\mathrm{e}^{t\mathbf{Y}_2}\mathbf{z_2} \\
&= \mathcal{L}_t^{-1}\left\{\mathbf{x}_1(s\mathbf{I}-\mathbf{Y}_1)^{-1}\mathbf{z_1}\cdot\mathbf{x}_2(s\mathbf{I}-\mathbf{Y}_2)^{-1}\mathbf{z_2}\right\} \\
&= \mathcal{L}_t^{-1}\left\{\begin{bmatrix}\mathbf{x}_1 & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{Y}_1 - s\mathbf{I} & \mathbf{z}_1\mathbf{x}_2 \\ 0 & \mathbf{Y}_2 - s\mathbf{I}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0} \\ \mathbf{z}_2\end{bmatrix}\right\} \\
&= \begin{bmatrix}\mathbf{x}_1 & \mathbf{0}\end{bmatrix}\mathrm{e}^{t\begin{bmatrix}\mathbf{Y}_1 & \mathbf{z}_1\mathbf{x}_2 \\ \mathbf{0} & \mathbf{Y}_2\end{bmatrix}}\begin{bmatrix}\mathbf{0} \\ \mathbf{z}_2\end{bmatrix}.
\end{aligned}$$

[37] J. E. Ruiz-Castro, "Matrix-exponential distributions: Closure properties," *International Journal of Advanced Statistics and Probability*, vol. 1, no. 2, pp. 44–52, 2013.
[38] M. F. Neuts, *Matrix-geometric solutions in stochastic models : An algorithmic approach*, ser. Johns Hopkins series in the mathematical sciences. Baltimore: Johns Hopkins University Press, 1981.

Preliminaries: Matrix Exponential Distribution (4)

- Maximum

Proposition 2.2: (Maximum of two ME-distributed r.v.s. [37, Proposition 3.5], [29]). Consider the ME-distributions $F_T^{(j)}(t) = 1 + \mathbf{x}_j e^{t\mathbf{Y}_j}\mathbf{Y}_j^{-1}\mathbf{z}_j, t \geq 0, j \in \{1,2\}$. Then, $T = \max\{T_1, T_2\}$ has the ME-distribution

$$F_T(t) = 1 + \mathbf{x}e^{t\mathbf{Y}}\mathbf{Y}^{-1}\mathbf{z}, \tag{20}$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \otimes \mathbf{x}_2 & \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}, \tag{21}$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \oplus \mathbf{Y}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \end{bmatrix}, \tag{22}$$

$$\mathbf{z} = \begin{bmatrix} (\mathbf{Y}_1^{-1} \oplus \mathbf{Y}_2^{-1})(\mathbf{z}_1 \otimes \mathbf{z}_2) \\ \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}. \tag{23}$$

- Minimum

Proposition 2.3: (Minimum of two ME-distributed r.v.s [37, Proposition 3.6], [29]). Consider the ME-distributions $F_T^{(j)}(t) = 1 + \mathbf{x}_j e^{t\mathbf{Y}_j}\mathbf{Y}_j^{-1}\mathbf{z}_j, t \geq 0, j \in \{1,2\}$. Then, $T = \min\{T_1, T_2\}$ has the ME-distribution

$$F_T(t) = 1 + \mathbf{x}e^{t\mathbf{Y}}\mathbf{Y}^{-1}\mathbf{z}, \tag{24}$$

where

$$\mathbf{x} = \mathbf{x}_1 \otimes \mathbf{x}_2, \tag{25}$$

$$\mathbf{Y} = \mathbf{Y}_1 \oplus \mathbf{Y}_2, \tag{26}$$

$$\mathbf{z} = -(\mathbf{Y}_1^{-1} \oplus \mathbf{Y}_2^{-1})(\mathbf{z}_1 \otimes \mathbf{z}_2). \tag{27}$$

[37] J. E. Ruiz-Castro, "Matrix-exponential distributions: Closure properties," *International Journal of Advanced Statistics and Probability*, vol. 1, no. 2, pp. 44–52, 2013.
[38] M. F. Neuts, *Matrix-geometric solutions in stochastic models : An algorithmic approach*, ser. Johns Hopkins series in the mathematical sciences. Baltimore: Johns Hopkins University Press, 1981.

PERFORMANCE ANALYSIS FRAMEWORK

ME-distributed fading (1)

Bivariate pdf $f_h(h) =$	Amplitude pdf $f_{\lvert h\rvert}(\lvert h\rvert) =$	SNR pdf $f_G(g) =$	LT of SNR pdf $F(s) =$	SNR cdf $F_G(g) =$
Bivariate Gaussian distr. $\frac{1}{\pi\Omega}\mathrm{e}^{-(h_\mathrm{r}^2+h_\mathrm{i}^2)/\Omega}$	Rayleigh distr. $\frac{2\lvert h\rvert}{\Omega}\mathrm{e}^{-\lvert h\rvert^2/\Omega}$	Exponentially distr. $\frac{1}{S}\mathrm{e}^{-g/S}$	$\frac{1}{1+sS}$	$1-\mathrm{e}^{-g/S}$
–	Nakagami-m distr. $\frac{2m^m\lvert h\rvert^{2m-1}}{\Gamma(m)\Omega^m}\mathrm{e}^{-m\lvert h\rvert^2/\Omega}$	Gamma distr. $\frac{m^m g^{m-1}}{\Gamma(m)S^m}\mathrm{e}^{-mg/S}$	$\left(\frac{1}{1+sS/m}\right)^m$	$\frac{1}{\Gamma(m)}\gamma(m, mg/S)$
(Unnamed distr.) $\frac{1}{\pi}\mathbf{p}_{\lvert h\rvert}\mathrm{e}^{(h_\mathrm{r}^2+h_\mathrm{i}^2)\mathbf{Q}_{\lvert h\rvert}}\mathbf{r}$	(Unnamed distr.) $2\lvert h\rvert\mathbf{p}_{\lvert h\rvert}\mathrm{e}^{\lvert h\rvert^2\mathbf{Q}_{\lvert h\rvert}}\mathbf{r}$	ME-distr. $\mathbf{p}\mathrm{e}^{g\mathbf{Q}}\mathbf{r}$	$\frac{p(s)}{q(s)}$	$1+\mathbf{p}\mathrm{e}^{g\mathbf{Q}}\mathbf{Q}^{-1}\mathbf{r}$

Table I

COMPARISON OF PDFS (AND CDFS) FOR UNPROCESSED FADING WIRELESS CHANNELS SNRS. THE FOLLOWING NOTION IS USED: THE INSTANTANEOUS SNR IS $g \triangleq \lvert h\rvert^2 P/\sigma^2$, WHERE $\lvert h\rvert$ IS THE CHANNEL AMPLITUDE GAIN, P IS THE RECEIVED POWER, σ^2 IS THE RECEIVER NOISE POWER. THE MEAN SNR IS $S \triangleq \mathbb{E}\{g\}$. THE COMPLEX AMPLITUDE GAIN IS $h \triangleq h_\mathrm{R} + ih_\mathrm{I}$, AND $\Omega \triangleq \mathbb{E}\{\lvert h\rvert\}$.

- Gamma- and exponential-distributions are special cases of the ME-distribution class.
- ME-distribution is dense on positive *g*.

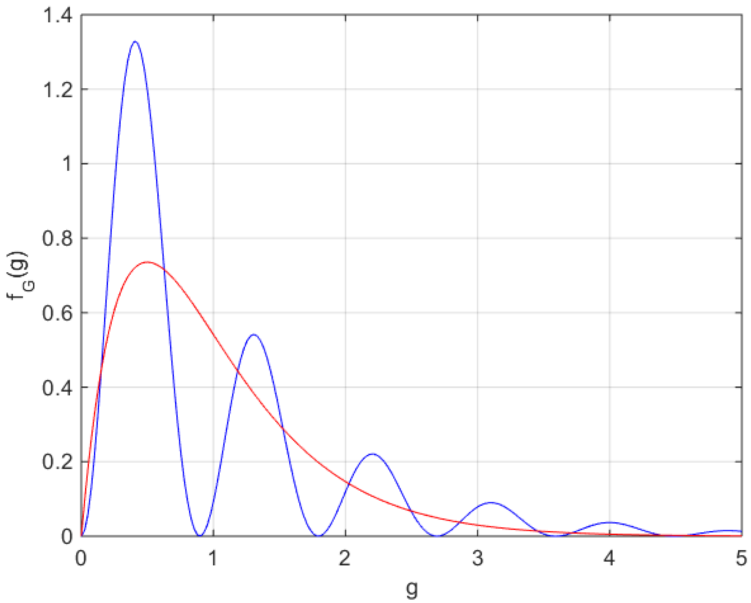
ME-distributed fading (2)

- Examples of ME-pdfs
  - Gamma pdf

$$f_G(g) = \frac{m^m g^{m-1}}{\Gamma(m) S^m} \mathrm{e}^{-mg/S}$$
$$F(s) = 1/(1 + sS/m^{\mathrm{N}})^{m^{\mathrm{N}}}$$

  - Oscillating pdf

$$f_T(t) = (1 + 7^{-2})\,(1 - \cos(7t))\,\mathrm{e}^{-t}$$
$$F(s) = 50/(s^3 + 3s^2 + 52s + 50)$$



ME-distributed fading (3)

- pdfs can also be approximated with ME-pdfs
  – In general a hard problem [35]
- Suggestions of approximation methods
  – Least squares in pdf domain [1]
  – Truncated continued fraction in LT domain [1]
  – Pade' approximation in the LT domain [1]
- ME-distribution (or ME-pdf) may be directly fitted to measured fading channel gains [2]

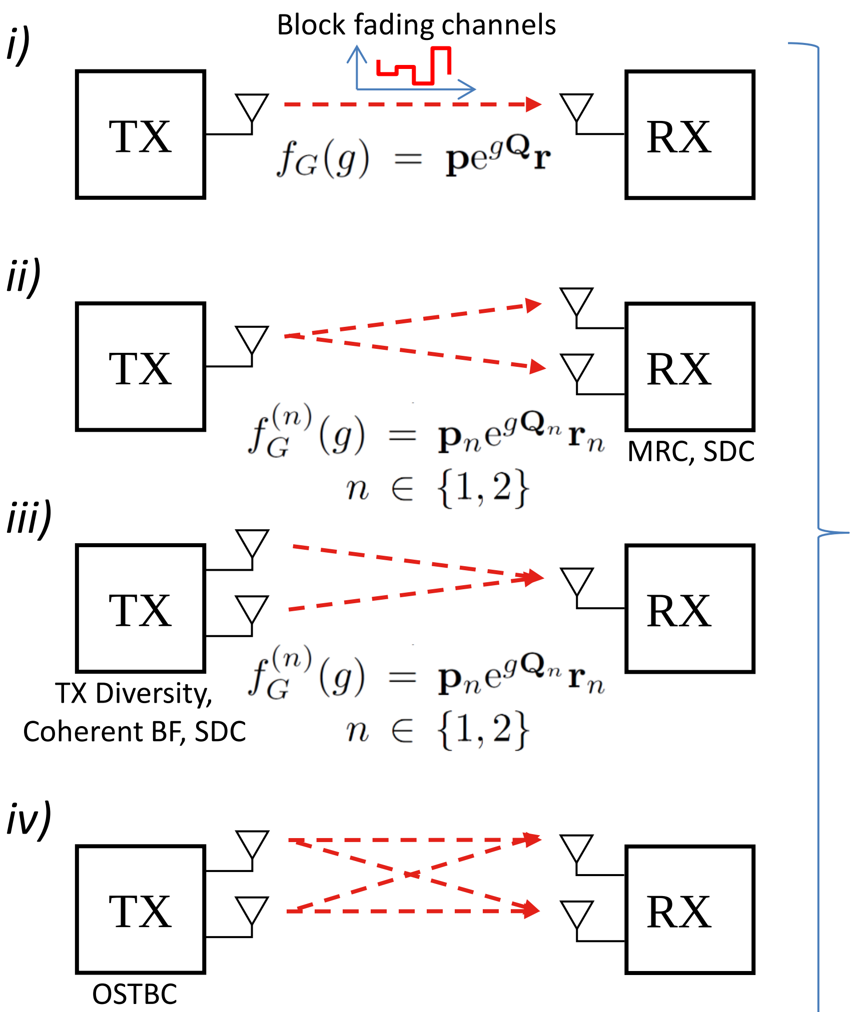
[1] P. Larsson, L. K. Rasmussen, and M. Skoglund, "Throughput analysis of hybrid-ARQ – A matrix exponential distribution approach," *IEEE Transactions on Communications*, vol. 64, no. 1, pp. 416–428, Jan 2016.

[2] P. Larsson, J. Gross, H. Al-Zubaidy, L. K. Rasmussen, and M. Skoglund, "Effective capacity of retransmission schemes: A recurrence relation approach," *IEEE Transactions on Communications*, vol. 64, no. 11, pp. 4817–4835, Nov 2016.

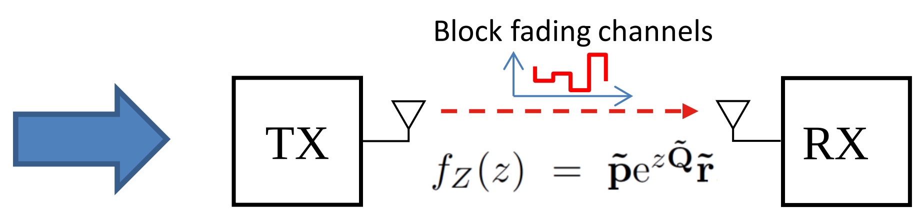
[35] M. W. Fackrell, "Characterization of matrix-exponential distributions," Ph.D. dissertation, The University of Adelaide, Faculty of Engineering, Computer and Mathematical Sciences, 2003.

Effective SNR Processing (1)

SNR processing examples



Effective (SNR) channel model



Translate unprocessed SNR(s) and ”complex” systems to an equivalent (but simpler) system which is charachterized by an effective SNR.

Effective SNR Processing (2)

Example 5.1: (MRC of two non-identical independent ME-distributed r.v.s) Consider two ME-distributed r.v.s z_u, with pdfs $f_G^{(n)}(g) = \mathbf{p}_n \mathrm{e}^{g\mathbf{Q}_n}\mathbf{r}_n$, $n \in \{1,2\}$. Then, the effective SNR, $Z = G_1 + G_2$, is also ME-distributed with pdf $f_Z(z) = \tilde{\mathbf{p}}\mathrm{e}^{z\tilde{\mathbf{Q}}}\tilde{\mathbf{r}}$, and parameters

$$\tilde{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{P}_2 \\ 0 & \mathbf{Q}_2 \end{bmatrix}, \tag{55}$$

$$\tilde{\mathbf{p}} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{0} \end{bmatrix}, \tag{56}$$

$$\tilde{\mathbf{r}} = \begin{bmatrix} \mathbf{0} & \mathbf{r}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \tag{57}$$

where $\mathbf{Q}_1 = \mathbf{S} - \mathbf{r}_1\mathbf{q}_1$, $\mathbf{Q}_2 = \mathbf{S} - \mathbf{r}_2\mathbf{q}_2$, and $\mathbf{P}_2 = \mathbf{r}_1\mathbf{p}_2$. The above follows from Corollary 4.1.

Example 5.3: (Effective channel of SDC and Rayleigh fading) The pdf of the SDC effective channel with (unit-mean) exponentially distributed fading SNRs has, as given by Ex. 3.5, LT $F(s) = N!/\prod_{n=1}^{N}(n+s)$, which gives

$$\tilde{\mathbf{Q}}_{\mathrm{um}} = \begin{bmatrix} -1 & 1 & \cdots & 0 \\ 0 & -2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -N \end{bmatrix}, \tag{58}$$

$$\tilde{\mathbf{p}}_{\mathrm{um}} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} N!, \tag{59}$$

$$\tilde{\mathbf{r}}_{\mathrm{um}} = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}^{\mathrm{T}}. \tag{60}$$

$$\mathbf{p}\mathrm{e}^{z\mathbf{Q}}\mathbf{r} = S^{-1}\mathbf{p}_{\mathrm{um}}\mathrm{e}^{zS^{-1}\mathbf{Q}_{\mathrm{um}}}\mathbf{r}_{\mathrm{um}}$$

Effective Channel Algebra

- SDC between branch 2+3 and then MRC with branch 1

$$Z = G_1 + \max(G_2, G_3)$$

- MRC of the two strongest branches

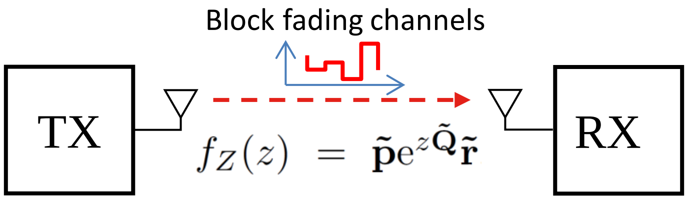
$$Z = G_1 + G_2 + G_3 - \min(G_1, G_2, G_3)$$

$= \max(G_1+G_2,\ G_1+G_3,\ G_2+G_3)$

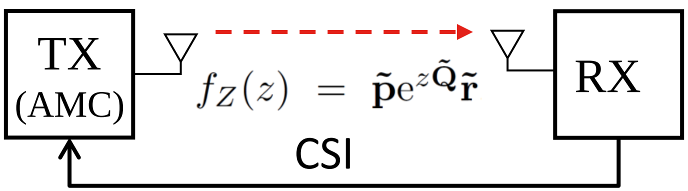
Perf.-evaluation and –metrics

Communication system model examples

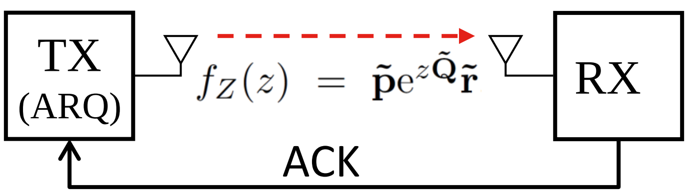
i)



ii)



iii)



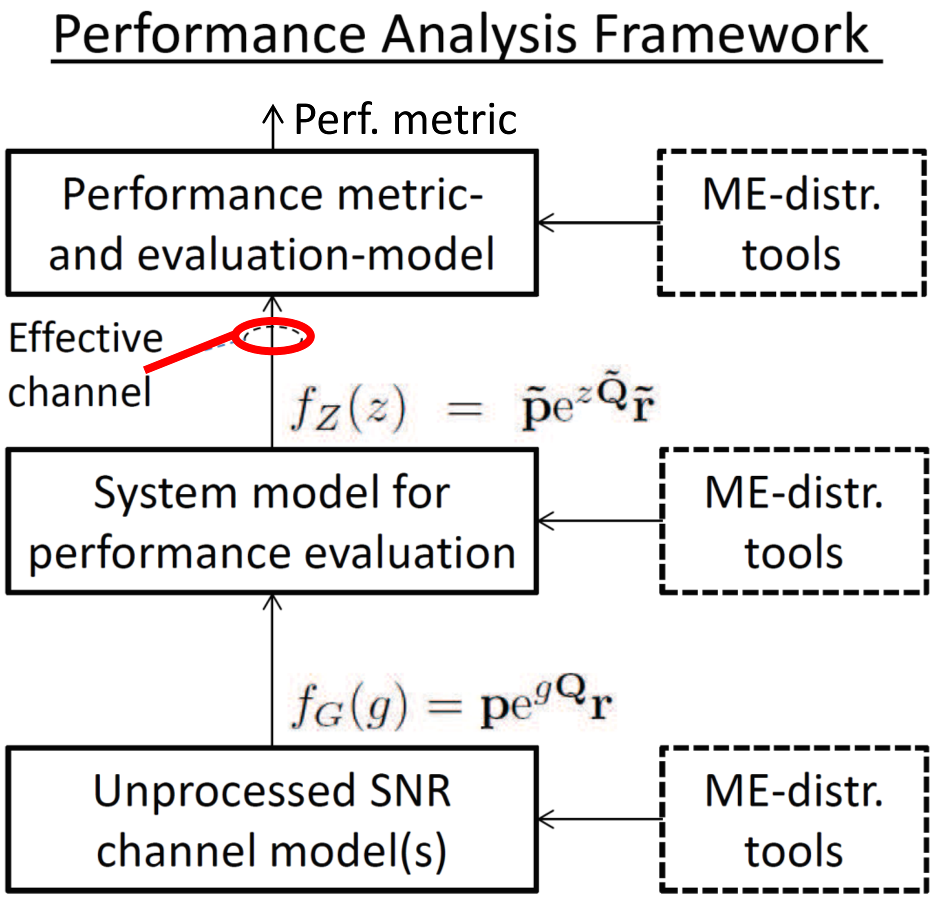
Performance metric examples

- Outage probability
- PEP/BER/SER (fast fading)
- …

- Ergodic capacity
- Effective capacity
- …

- Throughput
- Effective capacity
- …

Performance Analysis Framework



NEW ME-DISTRIBUTION PROPERTIES

New Integral of ME-pdf

Theorem 4.1: (Integration of ME-function on ME-pdf form). The integral of $f(t) = \mathbf{x}e^{t\mathbf{Y}}\mathbf{z}$, with intervals $(0, b)$ can be expressed as

$$\int_0^b \mathbf{x}e^{t\mathbf{Y}}\mathbf{z}\,\mathrm{d}t = \mathbf{E}_{1,d^{\mathrm{I}}}, \tag{41}$$

where

$$\mathbf{E} \triangleq e^{b\mathbf{Y}^{\mathrm{I}}}, \tag{42}$$

$$d^{\mathrm{I}} = d + 1, \tag{43}$$

$$\mathbf{Y}^{\mathrm{I}} = \begin{bmatrix} 0 & \mathbf{x} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix}. \tag{44}$$

Proof: Integration corresponds to convolution with a step function that has LT $1/s$. Using Proposition 2.1 gives

$$\int_0^b \mathfrak{L}_t^{-1}\left\{\frac{x(s)}{y(s)}\right\}\mathrm{d}t = \mathfrak{L}_b^{-1}\left\{\frac{1}{s}\frac{x(s)}{y(s)}\right\} = \mathbf{e}_1^{\mathrm{T}}e^{b\mathbf{Y}^{\mathrm{I}}}\mathbf{e}_{d^{\mathrm{I}}} = \mathbf{E}_{1,d^{\mathrm{I}}}.$$

■

Note: The standard integration approach, $\int_a^b e^{t\mathbf{X}}\,\mathrm{d}t = \mathbf{X}^{-1}\left(e^{t\mathbf{X}} - \mathbf{I}\right)|_a^b$, requires a non-singular matrix!

Maximum of Two ME-distr. r.v.s

Theorem 4.2: (Maximum of two ME-distributed r.v.s). Let $T_j, j \in \{1,2\}$ be ME-distributed r.v.s with pdf $f_T^{(j)}(t) = \mathbf{x}_j e^{t\mathbf{Y}_j}\mathbf{z}_j$, and degree d_j. Then, the CDF of the ME-distribution r.v. $T = \max(T_1, T_2)$ can be expressed as

$$F_T^{\max}(t) = \mathbf{E}_{1,d_1^{\mathrm{I}}+d_2^{\mathrm{I}}}, \tag{45}$$

where

$$\mathbf{E} \triangleq e^{t\left(\mathbf{Y}_1^{\mathrm{I}} \oplus \mathbf{Y}_2^{\mathrm{I}}\right)}, \tag{46}$$

$$\mathbf{Y}_j^{\mathrm{I}} = \begin{bmatrix} 0 & \mathbf{x}_j \\ \mathbf{0} & \mathbf{Y}_j \end{bmatrix}. \tag{47}$$

Proof:

$$\begin{aligned} F_T^{\max}(t) &= F_T^{(1)}(t) F_T^{(2)}(t) \\ &= \left(\int_0^{\mathrm{T}} \mathbf{x}_1 e^{u\mathbf{Y}_1}\mathbf{z}_1 \,\mathrm{d}u\right)\left(\int_0^{\mathrm{T}} \mathbf{x}_2 e^{u\mathbf{Y}_2}\mathbf{z}_2 \,\mathrm{d}u\right) \\ &\overset{(a)}{=} \left((\mathbf{e}_1^{(1)})^{\mathrm{t}} e^{t\mathbf{Y}_1^{\mathrm{I}}} \mathbf{e}_{d_1^{\mathrm{I}}}^{(1)}\right)\left((\mathbf{e}_1^{(2)})^{\mathrm{t}} e^{t\mathbf{Y}_2^{\mathrm{I}}} \mathbf{e}_{d_2^{\mathrm{I}}}^{(2)}\right) \\ &\overset{(b)}{=} \left(\mathbf{e}_1^{(1)} \otimes \mathbf{e}_1^{(2)}\right)^{\mathrm{T}} e^{t\left(\mathbf{Y}_1^{\mathrm{I}} \oplus \mathbf{Y}_2^{\mathrm{I}}\right)} \left(\mathbf{e}_{d_1^{\mathrm{I}}}^{(1)} \otimes \mathbf{e}_{d_2^{\mathrm{I}}}^{(2)}\right) \\ &= \mathbf{E}_{1,d_1^{\mathrm{I}}+d_2^{\mathrm{I}}},\ \mathbf{E} \triangleq e^{t\left(\mathbf{Y}_1^{\mathrm{I}} \oplus \mathbf{Y}_2^{\mathrm{I}}\right)}, \end{aligned}$$

- Alternative forms for max and min

$$F_T^{\max}(t) = \mathbf{E}_{1,d_1^{\mathrm{I}}}^{(1)} \mathbf{E}_{1,d_2^{\mathrm{I}}}^{(2)},$$

$$F_T^{\min}(t) = 1 - \left(1 - \mathbf{E}_{1,d_1^{\mathrm{I}}}^{(1)}\right)\left(1 - \mathbf{E}_{1,d_2^{\mathrm{I}}}^{(2)}\right)$$

$$\mathbf{E}^{(j)} \triangleq e^{t\mathbf{Y}_j^{\mathrm{I}}},$$

$$\mathbf{Y}_j^{\mathrm{I}} = \begin{bmatrix} 0 & \mathbf{x}_j \\ \mathbf{0} & \mathbf{Y}_j \end{bmatrix}.$$

Expectation of Function

Theorem 4.3: (Integral of ME-density- and Function-product). Let $g(t)$ be a function for which an integral representation $g(t) = \int_{a_u}^{b_u} g_1(u)\mathrm{e}^{-tg_2(u)}\,\mathrm{d}u$ exist. Then, the expectation of $g(t)$ is

$$\mathbb{E}\{g(t)\} = \int_0^\infty g(t)\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}\,\mathrm{d}t$$

$$= \int_{a_u}^{b_u} g_1(u)\mathbf{x}\left(g_2(u)\mathbf{I} - \mathbf{Y}\right)^{-1}\mathbf{z}\,\mathrm{d}u \tag{52}$$

$$= G_1 + \int_{a_u}^{b_u} g_1(u)\mathbf{x}\mathbf{Y}^{-1}\left(\mathbf{I} - \mathbf{Y}g_2(u)^{-1}\right)^{-1}\mathbf{z}\,\mathrm{d}u, \tag{53}$$

where $G_1 \triangleq \int_{a_u}^{b_u} g_1(u)\,\mathrm{d}u$.

Proof: The expectation is

$$\mathbb{E}\{g(t)\} = \int_0^\infty g(t)\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}\,\mathrm{d}t$$

$$= \int_0^\infty \left(\int_{a_u}^{b_u} g_1(u)\mathrm{e}^{-tg_2(u)}\,\mathrm{d}u\right)\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}\,\mathrm{d}t$$

$$= \int_{a_u}^{b_u} g_1(u)\left(\int_0^\infty \mathbf{x}\mathrm{e}^{t(\mathbf{Y}-g_2(u)\mathbf{I})}\mathbf{z}\,\mathrm{d}t\right)\mathrm{d}u$$

$$= \int_{a_u}^{b_u} g_1(u)\mathbf{x}\left(g_2(u)\mathbf{I} - \mathbf{Y}\right)^{-1}\mathbf{z}\,\mathrm{d}u$$

$$= -\int_{a_u}^{b_u} g_1(u)\mathbf{x}\mathbf{Y}^{-1}\left(\mathbf{I} - g_2(u)\mathbf{Y}^{-1}\right)^{-1}\mathbf{z}\,\mathrm{d}u$$

$$= -\int_{a_u}^{b_u} g_1(u)\mathbf{x}\mathbf{Y}^{-1}\left(\mathbf{I} - \left(\mathbf{I} - \mathbf{Y}g_2(u)^{-1}\right)^{-1}\mathbf{z}\right)\mathrm{d}u$$

$$= \int_{a_u}^{b_u} g_1(u)\,\mathrm{d}u + \int_{a_u}^{b_u} g_1(u)\mathbf{x}\mathbf{Y}^{-1}\left(\mathbf{I} - \mathbf{Y}g_2(u)^{-1}\right)^{-1}\mathbf{z}\,\mathrm{d}u$$

∎

APPLICATIONS

Applications

- We have a new cdf (pdf), the ME-distr.!
- Question: Where can we apply it?

We look in the bookshelf and in the litterature!

[3] S. G. Wilson, *Digital Modulation and Coding*, 1st ed. Delhi: Pearson Education, 1996.

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[19] D. Tse and P. Viswanath, *Fundamentals of Wireless Communications*. New York, NY, USA: Cambridge Univ. Press, 2004.

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[27] Y.-C. Ko, M. S. Alouini, and M. K. Simon, "Outage probability of diversity systems over generalized fading channels," *IEEE Transactions on Communications*, vol. 48, no. 11, pp. 1783–1787, Nov 2000.

[28] M. Jabi, L. Szczecinski, and M. Benjillali, "Accurate outage approximation of MRC receivers in arbitrarily fading channels," *IEEE Communications Letters*, vol. 16, no. 6, pp. 789–792, June 2012.

[23] M. K. Simon and M.-S. Alouini, *Digital communication over fading channels*, ser. Wiley series in telecommunications and signal processing. Hoboken, N.J. Wiley-Interscience, 2005. [Online]. Available: http://opac.inria.fr/record=b1102756

[24] A. Molisch, *Wireless Communications*. Wiley-IEEE Press, 2005.

Outage Probability

Theorem 5.4: (Outage probability for the ME-distributed effective channel) Let the effective channel pdf be $f_Z(z) = \tilde{\mathbf{p}} \mathrm{e}^{z\tilde{\mathbf{Q}}} \tilde{\mathbf{r}}$, $\tilde{\mathbf{p}} \in \mathbb{R}^{1\times\tilde{d}}$, $\tilde{\mathbf{Q}} \in \mathbb{R}^{\tilde{d}\times\tilde{d}}$, $\tilde{\mathbf{r}} = [0 \ldots 0\ 1]^{\mathrm{T}} \in \mathbb{R}^{\tilde{d}\times 1}$. Then, the outage probability, with decoding threshold Θ, is

$$Q_{\text{out}} = \mathbf{E}_{1,d^{\mathrm{I}}}, \tag{72}$$

where

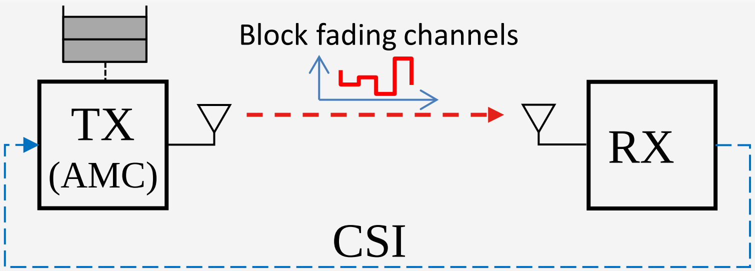
$$\mathbf{E} = \mathrm{e}^{\Theta \mathbf{Q}^{\mathrm{I}}}, \tag{73}$$

$$d^{\mathrm{I}} = \tilde{d} + 1, \tag{74}$$

$$\mathbf{Q}^{\mathrm{I}} = \begin{bmatrix} 0 & \tilde{\mathbf{p}} \\ \mathbf{0} & \tilde{\mathbf{Q}} \end{bmatrix}. \tag{75}$$

Proof: From Theorem 4.1, the outage probability can be directly computed as $Q_{\text{out}} = \mathbb{P}\{Z \leq \Theta\} = \int_0^{\Theta} \tilde{\mathbf{p}} \mathrm{e}^{z\tilde{\mathbf{Q}}} \tilde{\mathbf{r}}\, \mathrm{d}z = \mathbf{e}_1^{\mathrm{T}} \mathrm{e}^{\Theta \mathbf{Q}^{\mathrm{I}}} \mathbf{e}_{d^{\mathrm{I}}} = \mathbf{E}_{1,d^{\mathrm{I}}}$. ■

Adaptive Modulation and Coding (Rate-Adaptive Transmission)



- Transmission rate selected equal to the channel capacity

Effective Capacity (1)

- ”The EC identifies the maximum constant arrival rate that a given service process can support in orter to guarantee a desired statistical QoS specfied with the QoS exponent θ ”

- Definition (when the service rate varies independently)

$$C_{\text{eff}}^{\text{RA}} \triangleq -\frac{1}{\theta} \ln \left(\mathbb{E} \left\{ \mathrm{e}^{-\theta\zeta} \right\} \right)$$

[26] D. Wu and R. Negi, “Effective capacity: A wireless link model for support of quality of service,” *IEEE Transactions on Wireless Communications*, vol. 2, no. 4, pp. 630–643, July 2003.

Effective Capacity (2)

Theorem 5.1: (Effective capacity with ME-distributed service rate) Let the service rate ζ be iid, and have pdf $f_\zeta(\zeta) = \tilde{\mathbf{p}}e^{\zeta\tilde{\mathbf{Q}}}\tilde{\mathbf{r}}$, $\tilde{\mathbf{p}} \in \mathbb{R}^{1\times\tilde{d}}$, $\tilde{\mathbf{Q}} \in \mathbb{R}^{\tilde{d}\times\tilde{d}}$, $\tilde{\mathbf{r}} = [0\ldots0\ 1]^{\mathrm{T}} \in \mathbb{R}^{\tilde{d}\times 1}$. Then, the effective capacity for rate-adaptive transmission is

$$\begin{aligned} C_{\mathrm{eff}}^{\mathrm{RA}} &= -\frac{1}{\theta}\ln\left(\tilde{\mathbf{p}}(\theta\mathbf{I}-\tilde{\mathbf{Q}})^{-1}\tilde{\mathbf{r}}\right) \\ &= -\frac{1}{\theta}\ln\left(\frac{\tilde{p}(\theta)}{\tilde{q}(\theta)}\right). \end{aligned} \quad (61)$$

where θ is the effective capacity quality-of-service exponent.

Proof: When ζ is iid, the effective capacity is $C_{\mathrm{eff}}^{\mathrm{RA}} \triangleq -\frac{1}{\theta}\ln\left(\mathbb{E}\left\{e^{-\theta\zeta}\right\}\right) = -\frac{1}{\theta}\ln\left(\int_0^\infty e^{-\zeta\theta}\tilde{\mathbf{p}}e^{\zeta\tilde{\mathbf{Q}}}\tilde{\mathbf{r}}\,d\zeta\right)$, and we have $\int_0^\infty e^{-\zeta\theta}\tilde{\mathbf{p}}e^{\zeta\tilde{\mathbf{Q}}}\tilde{\mathbf{r}}\,d\zeta = \tilde{\mathbf{p}}(\theta\mathbf{I}-\tilde{\mathbf{Q}})^{-1}\tilde{\mathbf{r}} = \tilde{p}(\theta)/\tilde{q}(\theta)$. ■

Effective Capacity (3)

Theorem 5.2: (Effective capacity with the effective channel ME-distributed and the service rate equals the AWGN Shannon-capacity) Let the effective channel pdf be $f_Z(z) = \tilde{\mathbf{p}} e^{z\tilde{\mathbf{Q}}} \tilde{\mathbf{r}}$, $\tilde{\mathbf{p}} \in \mathbb{R}^{1\times\tilde{d}}$, $\tilde{\mathbf{Q}} \in \mathbb{R}^{\tilde{d}\times\tilde{d}}$, $\tilde{\mathbf{r}} = [0 \ldots 0\ 1]^{\mathrm{T}} \in \mathbb{R}^{\tilde{d}\times 1}$. Then,

the effective capacity, with CSI known at the transmitter and perfect rate adaptation, is

$$C_{\text{eff}}^{\text{RA}} \triangleq -\frac{1}{\theta} \ln \left(\int_0^\infty e^{-\theta \ln(1+z)} \tilde{\mathbf{p}} e^{z\tilde{\mathbf{Q}}} \tilde{\mathbf{r}}\, dz \right)$$

$$= -\frac{1}{\theta} \ln \left(\int_0^\infty \frac{u^{\theta-1} e^{-u}}{\Gamma(\theta)} \tilde{\mathbf{p}} \left(u\mathbf{I} - \tilde{\mathbf{Q}} \right)^{-1} \tilde{\mathbf{r}}\, du \right) \quad (62)$$

$$= -\frac{1}{\theta} \ln \left(\int_0^\infty \frac{u^{\theta-1} e^{-u}}{\Gamma(\theta)} \frac{\tilde{p}(u)}{\tilde{q}(u)}\, du \right). \quad (63)$$

Proof: We have $C_{\text{eff}}^{\text{RA}} = -\frac{1}{\theta} \ln \left(\mathbb{E}\left\{ e^{-\theta \ln(1+z)} \right\} \right)$, where the expectation is

$$\mathbb{E}\left\{ e^{-\theta \ln(1+z)} \right\}$$

$$= \int_0^\infty e^{-\theta \ln(1+z)} \tilde{\mathbf{p}} e^{z\tilde{\mathbf{Q}}} \tilde{\mathbf{r}}\, dz$$

$$\overset{(a)}{=} \int_0^\infty (1+z)^{-\theta} \tilde{\mathbf{p}} e^{z\tilde{\mathbf{Q}}} \tilde{\mathbf{r}}\, dz$$

$$= \int_0^\infty \left(\int_0^\infty \frac{u^{\theta-1}}{\Gamma(\theta)} e^{-(1+z)u}\, du \right) \tilde{\mathbf{p}} e^{z\tilde{\mathbf{Q}}} \tilde{\mathbf{r}}\, dz$$

$$\overset{(b)}{=} \int_0^\infty \frac{u^{\theta-1} e^{-u}}{\Gamma(\theta)} \tilde{\mathbf{p}} \left(u\mathbf{I} - \tilde{\mathbf{Q}} \right)^{-1} \tilde{\mathbf{r}}\, du.$$

Corollary 5.1: When $\tilde{\mathbf{Q}}$ is diagonalizable, $\tilde{\mathbf{V}}\tilde{\mathbf{\Lambda}}\tilde{\mathbf{V}}^{-1} = \tilde{\mathbf{Q}}$, $\tilde{\mathbf{V}}$ is a non-singular Vandermonde matrix, and all eigenvalues $\lambda_j,\ j \in \{1, 2, \ldots J\}$, are real negative, then

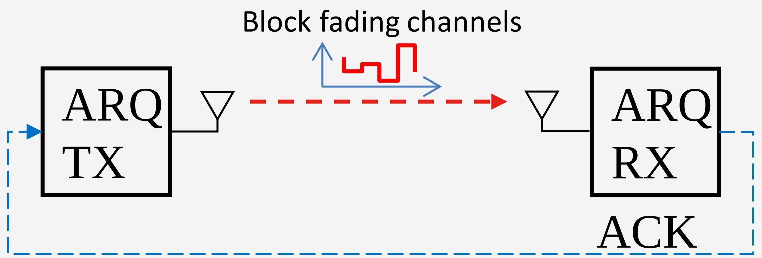
$$\tilde{\mathbf{\Xi}} = \operatorname{diag}\{\xi_1, \xi_2, \ldots \xi_J\}, \quad (67)$$

$$\xi_j = (-\lambda_j)^{\theta-1} e^{-\lambda_j} \Gamma(1-\theta, -\lambda_j). \quad (68)$$

$$C_{\text{eff}}^{\text{RA}} = -\frac{1}{\theta} \ln \left(\tilde{\mathbf{p}} \tilde{\mathbf{T}} \tilde{\mathbf{\Xi}} \tilde{\mathbf{T}}^{-1} \tilde{\mathbf{r}} \right), \quad (64)$$

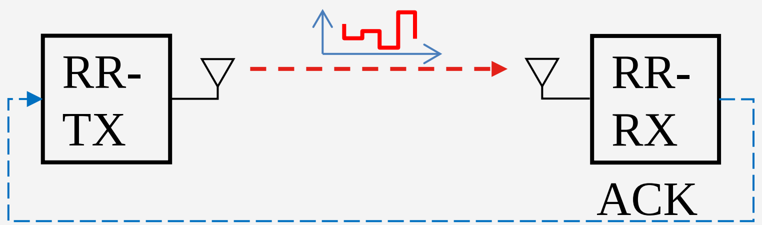
Basic Retransmission-Schemes

- “ARQ”



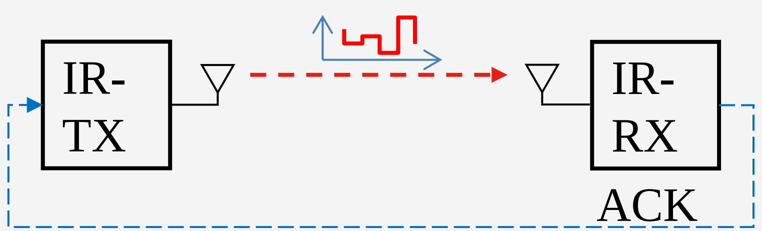
- Retransmit: Repeat packet
- Combine packets: No

- RR-HARQ



- Retransmit: Repeat packet
- Combine packets: Yes

- IR-HARQ



- Retransmit: Incremental blocks
- Combine blocks: Yes

ARQ Throughput

Theorem 5.5: (Outage probability and ARQ throughput for the ME-distributed effective channel) Let the effective channel pdf be $f_Z(z) = \tilde{\mathbf{p}} e^{z\tilde{\mathbf{Q}}} \tilde{\mathbf{r}}$. Then, the throughput of ARQ is

$$T^{\mathrm{ARQ}} = R(1 - \mathbf{E}_{1,d^{\mathrm{I}}}), \tag{78}$$

with $\mathbf{E}_{1,d^{\mathrm{I}}}$ given by Theorem 5.4.

Proof: The throughput is simply $T^{\mathrm{ARQ}} = R(1 - Q_{\mathrm{out}})$, and we then use the outage probability in Theorem 5.4. ■

Truncated-HARQ Throughput Analysis

Theorem 5.6: (Truncated-HARQ throughput for the ME-distributed effective channel) Let the effective channel density be $f_Z(z) = \tilde{\mathbf{p}} e^{z\tilde{\mathbf{Q}}} \tilde{\mathbf{r}}$. Then, the throughput of truncated-HARQ, with a maximum of K transmissions and decoding threshold Θ, is

$$T_K^{\text{HARQ}} = \frac{R(1 - \mathbf{E}_{1,(dK+1)})}{1 + \sum_{k=1}^{K-1} \mathbf{E}_{1,(dk+1)}}, \quad (80)$$

where

$$\mathbf{E} = e^{\Theta \mathbf{Q}^{\text{I}}_{K\circledast}}, \quad (81)$$

and the ME-parameters are

$$\mathbf{p}^{\text{I}}_{K\circledast} = [\tilde{\mathbf{p}}\ \mathbf{0}] \in \mathbb{R}^{1\times d^{\text{I}}}, \quad (82)$$

$$\mathbf{Q}^{\text{I}}_{K\circledast} = \begin{bmatrix} 0 & \mathbf{p}_{K\circledast} \\ \mathbf{0} & \mathbf{Q}_{K\circledast} \end{bmatrix} \in \mathbb{R}^{d^{\text{I}}\times d^{\text{I}}}, \quad (83)$$

$$\mathbf{r}^{\text{I}}_{K\circledast} = \mathbf{e}_{d^{\text{I}}} \in \mathbb{R}^{d^{\text{I}}\times 1}, \quad (84)$$

$$d^{\text{I}} = \tilde{d}K + 1, \quad (85)$$

$$\mathbf{p}_{K\circledast} = [\tilde{\mathbf{p}}\ \mathbf{0}] \in \mathbb{R}^{1\times \tilde{d}K}, \quad (86)$$

$$\mathbf{Q}_{K\circledast} = \begin{bmatrix} \tilde{\mathbf{Q}} & \tilde{\mathbf{P}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{Q}} & \tilde{\mathbf{P}} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{Q}} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \tilde{\mathbf{Q}} & \tilde{\mathbf{P}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \tilde{\mathbf{Q}} \end{bmatrix} \in \mathbb{R}^{\tilde{d}K\times \tilde{d}K}, \quad (87)$$

$$\mathbf{r}_{K\circledast} = \mathbf{e}_{\tilde{d}K} \in \mathbb{R}^{\tilde{d}K\times 1}, \quad (88)$$

$$\tilde{\mathbf{Q}} = \mathbf{S} - \tilde{\mathbf{r}}\tilde{\mathbf{q}} \in \mathbb{R}^{\tilde{d}\times \tilde{d}}, \quad (89)$$

$$\tilde{\mathbf{P}} = \tilde{\mathbf{r}}\tilde{\mathbf{p}} \in \mathbb{R}^{\tilde{d}\times \tilde{d}}. \quad (90)$$

(We know that $T_K^{\text{HARQ}} = R\left(1 - \mathcal{L}_\Theta^{-1}\{s^{-1}F(s)^K\}\right) / \left(1 + \mathcal{L}_\Theta^{-1}\{\sum_{k=1}^{K-1} s^{-1}F(s)^k\}\right)$)

Persistent-HARQ Throughput Analysis

Theorem 5.7: (Persistent-HARQ throughput for the ME-distributed effective channel) Let the effective channel pdf $f_Z(z) = \tilde{\mathbf{p}}e^{z\tilde{\mathbf{Q}}}\tilde{\mathbf{r}}$ have LT $F(s) = \tilde{p}(s)/\tilde{q}(s)$. Then, the throughput, with decoding threshold Θ, is

$$T_{\infty}^{\mathrm{HARQ}} = \frac{R}{1 + \mathbf{E}_{1,d^{\mathrm{I}}}}, \tag{91}$$

where

$$\mathbf{E} = e^{\Theta \mathbf{Q}^{\mathrm{I}}}, \tag{92}$$

$$\mathbf{Q}^{\mathrm{I}} = \begin{bmatrix} 0 & \tilde{\mathbf{p}} \\ \mathbf{0} & \mathbf{S} - \tilde{\mathbf{r}}(\tilde{\mathbf{q}} - \tilde{\mathbf{p}}) \end{bmatrix}. \tag{93}$$

Proof: The mean number of transmissions is

$$\mathfrak{L}_{\Theta}^{-1}\left\{\frac{1}{s}\frac{1}{1-F(s)}\right\} = \mathfrak{L}_{\Theta}^{-1}\left\{\frac{1}{s}\frac{1}{1-\tilde{p}(s)/\tilde{q}(s)}\right\}$$
$$= 1 + \mathfrak{L}_{\Theta}^{-1}\left\{\frac{1}{s}\frac{\tilde{p}(s)}{\tilde{q}(s)-\tilde{p}(s)}\right\}$$
$$= 1 + \mathbf{e}_1 \mathrm{T} e^{\Theta \mathbf{Q}^{\mathrm{I}}} \mathbf{e}_{d^{\mathrm{I}}} = 1 + \mathbf{E}_{1,d^{\mathrm{I}}}, \;\; \mathbf{E} \triangleq e^{\Theta \mathbf{Q}^{\mathrm{I}}}.$$

■

Persistent-HARQ with diversity order 2

Corollary 5.3: (Persistent-HARQ throughput for the ME-distributed wireless channel and N-fold diversity) Let the effective channel pdf $f_Z(z) = \tilde{\mathbf{p}} e^{z\tilde{\mathbf{Q}}}\tilde{\mathbf{r}}$ have LT $F(s) = \tilde{p}(s)/\tilde{q}(s) = (p(s)/q(s))^N, N \in \mathbb{N}^+$. Then, the throughput, with decoding threshold Θ, is

$$T_\infty^{\mathrm{HARQ}} = \frac{R}{1 + \mathbf{E}_{1,d^{\mathrm{I}}}}, \tag{94}$$

where

$$\mathbf{E} = e^{\Theta \mathbf{Q}^{\mathrm{I}}_{N\circledast}}, \tag{95}$$

Example 5.13: (Persistent-HARQ with diversity order 2) Consider Theorem 5.7 with $N = 2$. Then,

$$\mathbf{Q}^{\mathrm{I}}_{2\circledast} = \begin{bmatrix} 0 & \mathbf{p} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} - \mathbf{r}(\mathbf{q} - \mathbf{p}) & \mathbf{rp} \\ \mathbf{0} & \mathbf{0} & \mathbf{S} - \mathbf{r}(\mathbf{q} + \mathbf{p}) \end{bmatrix}, \tag{106}$$

since

$$\begin{aligned} &\mathfrak{L}_\Theta^{-1}\left\{\frac{1}{s}\frac{1}{1 - (p(s)/q(s))^2}\right\} \\ &= 1 + \mathfrak{L}_\Theta^{-1}\left\{\frac{1}{s}\frac{p(s)}{q(s) - p(s)}\frac{p(s)}{q(s) + p(s)}\right\}. \end{aligned}$$

3-phase NC Bidirectional Relaying

Theorem 5.8: Consider the 3-phase NCBR model with nodes ν_1, ν_2, ν_3, in Fig. 3. Let each link between a node pair $\{\nu_i, \nu_j\}$, $\{ij\} = \{13, 32, 23, 31\}$, be characterized by the effective channel SNR r.v. Z_{ij} with pdf $f_Z^{(ij)}(z) = \tilde{\mathbf{p}}_{ij} e^{z\tilde{\mathbf{Q}}_{ij}} \tilde{\mathbf{r}}_{ij}$, $\tilde{\mathbf{p}}_{ij} \in \mathbb{R}^{1\times \tilde{d}_{ij}}$, $\tilde{\mathbf{Q}}_{ij} \in \mathbb{R}^{\tilde{d}_{ij}\times \tilde{d}_{ij}}$, $\tilde{\mathbf{r}}_{ij} = [0\ldots 0\ 1]^{\mathrm{T}} \in \mathbb{R}^{\tilde{d}_{ij}\times 1}$, and the decoding threshold $\Theta_{12} = e^{R_{12}} - 1$, $\Theta_{21} = e^{R_{21}} - 1$. Then, the ETE sum-throughput is

$$T^{\mathrm{NCBR}} = \frac{R_{12}(1-Q_{12}) + R_{21}(1-Q_{21})}{3}, \tag{115}$$

where

$$Q_{ij} = 1 - (1 - \mathbf{E}^{(i3)}_{1,d^{\mathrm{I}}_{i3}})(1 - \mathbf{E}^{(3j)}_{1,d^{\mathrm{I}}_{3j}}), \tag{116}$$

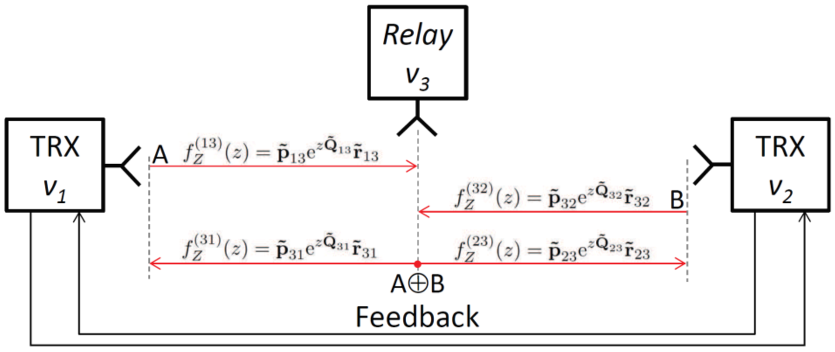
$$\mathbf{E}^{(ij)} \triangleq e^{\Theta_{ij}\mathbf{Q}^{\mathrm{I}}_{ij}}, \tag{117}$$

$$\mathbf{Q}^{\mathrm{I}}_{ij} = \begin{bmatrix} 0 & \tilde{\mathbf{p}}_{ij} \\ \mathbf{0} & \tilde{\mathbf{Q}}_{ij} \end{bmatrix}. \tag{118}$$

Proof: The ETE outage probability is

$$\begin{aligned}
Q_{ij} &= 1 - \mathbb{P}\{\ln(1+\min(Z_{i3}, Z_{3j})) > R_{ij}\} \\
&= 1 - \mathbb{P}\{\min(Z_{i3}, Z_{3j}) > \Theta_{ij}\} \\
&= 1 - \mathbb{P}\{Z_{i3} > \Theta_{ij}\}\mathbb{P}\{Z_{3j} > \Theta_{ij}\} \\
&= 1 - (1 - \mathbb{P}\{Z_{i3} < \Theta_{ij}\})(1 - \mathbb{P}\{Z_{3j} < \Theta_{ij}\}) \\
&= 1 - (1 - \mathbf{E}^{(i3)}_{1,d^{\mathrm{I}}_{i3}})(1 - \mathbf{E}^{(3j)}_{1,d^{\mathrm{I}}_{3j}}),
\end{aligned}$$

with $\mathbf{E}^{(ij)}$ given by (117). ■



System Model: ARQ with Independent Interference

- Signal ME-pdf

$$f_Z(z) = \mathbf{p}\mathrm{e}^{z\mathbf{Q}}\mathbf{r}$$

- Sum-interference

$$Z_\mathrm{I} = \sum_{u=1}^{U} Z_u$$

  - Individual ME-pdfs

$$f_Z^{(u)}(z) \sim \mathbf{p}_u\mathrm{e}^{z\mathbf{Q}_u}\mathbf{r}_u$$

  - Sum-interference pdfs

$$f_{Z_\mathrm{I}}(z_\mathrm{I}) = \mathbf{p}_\mathrm{I}\mathrm{e}^{z\mathbf{Q}_\mathrm{I}}\mathbf{r}_\mathrm{I}$$

- Outage probability

$$P_{\mathrm{Int}}^{\mathrm{ARQ}} = \mathbb{P}\left\{\ln\left(1 + Z/(1 + Z_\mathrm{I})\right) > R\right\} = \mathbb{P}\{Z \leq \Theta(1 + Z_\mathrm{I})\}$$

where $\Theta = \mathrm{e}^R - 1$

ARQ with Independent Interference

Theorem 5.9: (ARQ throughput for iid ME-distributed signal and interferers) Let the signal and sum-interferer be given by the system model. Then, the throughput is

$$T_{\mathrm{Int}}^{\mathrm{ARQ}} = R\left(\mathbf{p}_{\mathrm{I}} \otimes \mathbf{p}\right)\left((\mathbf{Q}_{\mathrm{I}} \oplus \Theta\mathbf{Q})(\mathbf{I} \otimes \mathbf{Q}\mathrm{e}^{-\Theta\mathbf{Q}})\right)^{-1}\left(\mathbf{r}_{\mathrm{I}} \otimes \mathbf{r}\right). \tag{120}$$

ARQ with Independent Interference

Proof: The throughput is $T_{\text{Int}}^{\text{ARQ}} = RP_{\text{Int}}^{\text{ARQ}}$, where $P_{\text{Int}}^{\text{ARQ}} = \mathbb{P}\{Z > \Theta(1+Z_{\text{I}})\}$ is determined via the integral

$$\begin{aligned}
P_{\text{Int}}^{\text{ARQ}} &= \int_0^\infty \int_{\Theta(1+z_{\text{I}})}^\infty \mathbf{p}_{\text{I}} \mathrm{e}^{z_{\text{I}}\mathbf{Q}_{\text{I}}} \mathbf{r}_{\text{I}} \mathbf{p} \mathrm{e}^{z\mathbf{Q}} \mathbf{r} \, \mathrm{d}z_{\text{I}} \, \mathrm{d}z \\
&= -\int_0^\infty \mathbf{p}_{\text{I}} \mathrm{e}^{z_{\text{I}}\mathbf{Q}_{\text{I}}} \mathbf{r}_{\text{I}} \mathbf{p} \mathbf{Q}^{-1} \mathrm{e}^{\Theta\mathbf{Q}} \mathrm{e}^{z_{\text{I}}\Theta\mathbf{Q}} \mathbf{r} \, \mathrm{d}z_{\text{I}} \qquad (121) \\
&\overset{(a)}{=} \left(\mathbf{p}_{\text{I}} \otimes \left(\mathbf{p}\mathbf{Q}^{-1}\mathrm{e}^{\Theta\mathbf{Q}}\right)\right) \left(\mathbf{Q}_{\text{I}} \oplus \Theta\mathbf{Q}\right)^{-1} \left(\mathbf{r}_{\text{I}} \otimes \mathbf{r}\right) \\
&\overset{(b)}{=} \left(\mathbf{p}_{\text{I}} \otimes \mathbf{p}\right) \left(\mathbf{I} \otimes \mathbf{Q}^{-1}\mathrm{e}^{\Theta\mathbf{Q}}\right) \left(\mathbf{Q}_{\text{I}} \oplus \Theta\mathbf{Q}\right)^{-1} \left(\mathbf{r}_{\text{I}} \otimes \mathbf{r}\right) \\
&\overset{(c)}{=} \left(\mathbf{p}_{\text{I}} \otimes \mathbf{p}\right) \left(\mathbf{I} \otimes \mathbf{Q}\mathrm{e}^{-\Theta\mathbf{Q}}\right)^{-1} \left(\mathbf{Q}_{\text{I}} \oplus \Theta\mathbf{Q}\right)^{-1} \left(\mathbf{r}_{\text{I}} \otimes \mathbf{r}\right) \\
&\overset{(d)}{=} \left(\mathbf{p}_{\text{I}} \otimes \mathbf{p}\right) \left((\mathbf{Q}_{\text{I}} \oplus \Theta\mathbf{Q})(\mathbf{I} \otimes \mathbf{Q}\mathrm{e}^{-\Theta\mathbf{Q}})\right)^{-1} \left(\mathbf{r}_{\text{I}} \otimes \mathbf{r}\right),
\end{aligned}$$

where Lemma 5.1 is used in step (a), and the identities $(\mathbf{X}_1 \otimes \mathbf{Y}_1)(\mathbf{X}_2 \otimes \mathbf{Y}_2) = (\mathbf{X}_1\mathbf{X}_2) \otimes (\mathbf{Y}_1\mathbf{Y}_2)$, $(\mathbf{X} \otimes \mathbf{Y})^{-1} = (\mathbf{X}^{-1} \otimes \mathbf{Y}^{-1})$, and $\mathbf{X}^{-1}\mathbf{Y}^{-1} = (\mathbf{Y}\mathbf{X})^{-1}$, are used in step (b)-(d), respectively. ■

Throughput Analysis – Dependent Signal and Interference

Definition 5.1: (Bivariate ME-distribution) We define the joint ME-density of the wireless channel SNR r.v.s (Z_1, Z_2), as $f_{Z_1,Z_2}(z_1, z_2) = \mathbf{p}_1 e^{z_1 \mathbf{Q}_1} \mathbf{P}_{12} e^{z_2 \mathbf{Q}_2} \mathbf{r}_2$, $z_1 \geq 0$, $z_2 \geq 0$, where $\mathbf{p}_1 \in \mathbb{R}^{1\times d_1}$, $\mathbf{Q}_1 \in \mathbb{R}^{d_1 \times d_1}$, $\mathbf{P}_{12} \in \mathbb{R}^{d_1\times d_2}$, $\mathbf{Q}_2 \in \mathbb{R}^{d_2\times d_2}$, $\mathbf{r}_2 \in \mathbb{R}^{d_2 \times 1}$. The parameters defining the joint density are, in a similar manner as for the univariate-ME-distribution, assumed selected to have a corresponding bivariate CDF fulfilling necessary characteristics, e.g. $0 \leq F_{Z_1,Z_2}(z_1, z_2) \leq 1$,

Theorem 5.10: (ARQ throughput solution with Sylvester's equation) Let the signal of interest, and the sum-interference, SNRs have joint density $f_{Z_\mathrm{I},Z}(z_\mathrm{I}, z) = \mathbf{p}_\mathrm{I} e^{z_\mathrm{I} \mathbf{Q}_\mathrm{I}} \mathbf{P}_{12} e^{z\mathbf{Q}} \mathbf{r}$. Then, the throughput is

$$T_{\mathrm{Int}}^{\mathrm{ARQ}} = R\mathbf{p}_\mathrm{I}\mathbf{X}\mathbf{r}, \tag{130}$$

where $\mathbf{X}$ is given by the solution $\mathbf{X}$ to the Sylvester equation

$$\mathbf{Q}_\mathrm{I}\mathbf{X} + \mathbf{X}\Theta\mathbf{Q}_2 = -\check{\mathbf{P}}_{12}, \tag{131}$$

with

$$\check{\mathbf{P}}_{12} \triangleq -\mathbf{P}_{12}\mathbf{Q}^{-1}e^{\Theta\mathbf{Q}}, \tag{132}$$

and $\Theta = e^R - 1$.

Proof: The throughput is $T_{\mathrm{Int}}^{\mathrm{ARQ}} = RP_{\mathrm{Int}}^{\mathrm{ARQ}}$, where, analogously to Theorem 5.9, the decoding probability is

$$\begin{aligned}
P_{\mathrm{Int}}^{\mathrm{ARQ}} &= \int_0^\infty \int_{\Theta(1+z_\mathrm{I})}^\infty \mathbf{p}_\mathrm{I} e^{z_\mathrm{I}\mathbf{Q}_\mathrm{I}} \mathbf{P}_{12} e^{z\mathbf{Q}} \mathbf{r}\, \mathrm{d}z_\mathrm{I}\, \mathrm{d}z \\
&= -\int_0^\infty \mathbf{p}_\mathrm{I} e^{z_\mathrm{I}\mathbf{Q}_\mathrm{I}} \mathbf{P}_{12}\mathbf{Q}^{-1} e^{\Theta(1+z_\mathrm{I})\mathbf{Q}} \mathbf{r}\, \mathrm{d}z_\mathrm{I} \\
&= \int_0^\infty \mathbf{p}_\mathrm{I} e^{z_\mathrm{I}\mathbf{Q}_\mathrm{I}} \check{\mathbf{P}}_{12} e^{z_\mathrm{I}\Theta\mathbf{Q}} \mathbf{r}\, \mathrm{d}z_\mathrm{I},\ \check{\mathbf{P}}_{12} \triangleq -\mathbf{P}_{12}\mathbf{Q}^{-1}e^{\Theta\mathbf{Q}}, \\
&= \mathbf{p}_\mathrm{I}\mathbf{X}\mathbf{r},\ \mathbf{X} \triangleq \int_0^\infty e^{z_\mathrm{I}\mathbf{Q}_\mathrm{I}} \check{\mathbf{P}}_{12} e^{z_\mathrm{I}\Theta\mathbf{Q}}\, \mathrm{d}z_\mathrm{I}.
\end{aligned}$$

Based on Lemma 5.2, we then solve for $\mathbf{X}$ in the Sylvester equation (131). ■

Modulation and Detection

Theorem 5.11: (Differential binary PSK (DBPSK) and FSK BER with non-coherent detection). Let the conditional error probability have the generic form $P(z) = \mathbf{e}^{-az}/2$, where z is the instantaneous SNR, and a is constant for the specific modulation and detection method (DBPSK: $a = 1$, FSK: $a = 1/2$), [3], [4], [24]. Then, the BER can be written as

$$P_b = \int_0^\infty \frac{1}{2}\mathbf{e}^{-az}\tilde{\mathbf{p}}\mathrm{e}^{z\tilde{\mathbf{Q}}}\tilde{\mathbf{r}}\,\mathrm{d}z = \frac{1}{2}\tilde{\mathbf{p}}(a\mathbf{I}-\tilde{\mathbf{Q}})^{-1}\tilde{\mathbf{r}} = \frac{1}{2}\frac{\tilde{p}(a)}{\tilde{q}(a)}. \qquad (153)$$

Theorem 5.12: (Binary PSK (BPSK) and FSK BER with coherent detection). Let the conditional error probability have the generic form $P(z) = Q(\sqrt{2az})$, where z is the instantaneous SNR, and a is constant for the specific modulation and detection method (BPSK: $a = 1$, FSK: $a = 1/2$) [3], [4], [24]. Then, the BER is

$$P_\mathrm{b} = \int_0^\infty Q(\sqrt{2az})\tilde{\mathbf{p}}\mathrm{e}^{z\tilde{\mathbf{Q}}}\tilde{\mathbf{r}}\,\mathrm{d}z = \frac{1}{2}\left(1+\tilde{\mathbf{p}}\tilde{\mathbf{Q}}^{-1}(\mathbf{I}-\tilde{\mathbf{Q}}a^{-1})^{-1/2}\tilde{\mathbf{r}}\right). \qquad (154)$$

Theorem 5.13: (Pairwise error probability). Let the conditional pairwise error probability have the generic form $P(\mathbf{c} \to \mathbf{e}|z_1,\ldots z_N) = Q\left(\sqrt{2\sum_{n=1}^N a_n z_n}\right)$, see e.g. [3, (6.6.9)], [19, (3.84)], or [23, Chap. 13], where $a_n, n \in \{1,2,\ldots N\}$, are constants, and z_n are ME-distributed r.v.s. Then, the average PEP is

$$PEP(\mathbf{c}\to\mathbf{e}) = \frac{1}{\pi}\int_0^{\pi/2}\prod_{n=1}^N \tilde{\mathbf{p}}_n\left(\frac{a_n}{\sin^2(t)}\mathbf{I}-\tilde{\mathbf{Q}}_n\right)^{-1}\tilde{\mathbf{r}}_n\,\mathrm{d}t. \qquad (157)$$

OTHER USES OF THE ME-DISTR.

ME-distributed Time-discrete Signals

- Lloyd-Max Quantization

$$l_q = \frac{1}{2}\left(\hat{u}_{q-1} + \hat{u}_q\right), \; q = \{1, 2, \dots M-1\},$$

$$\hat{u}_q = \frac{\int_{l_q}^{l_{q+1}} t f_T(t)\,\mathrm{d}t}{\int_{l_q}^{l_{q+1}} f_T(t)\,\mathrm{d}t}, \; q = \{0, 1, \dots M-1\},$$

  - with ME-distr.

$$\hat{u}_q = \frac{\int_{l_q}^{l_{q+1}} t\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}\,\mathrm{d}t}{\int_{l_q}^{l_{q+1}} \mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}\,\mathrm{d}t} = \frac{\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\left(t\mathbf{Y}^{-1} - \mathbf{Y}^{-2}\right)\mathbf{z}\big|_{l_q}^{l_{q+1}}}{\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{Y}^{-1}\mathbf{z}\big|_{l_q}^{l_{q+1}}}.$$

- Panter-Dite formula

$$MSE \approx \frac{1}{12M^2}\left(\int_0^\infty \left(\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}\right)^{1/3}\mathrm{d}t\right)^3.$$

$$\begin{aligned}
\mathbf{x}\mathbf{T}^{-1} &= \breve{\mathbf{x}} \otimes \breve{\mathbf{x}} \otimes \breve{\mathbf{x}}, \\
\mathbf{T}\mathbf{Y}\mathbf{T}^{-1} &= \breve{\mathbf{Y}} \oplus \breve{\mathbf{Y}} \oplus \breve{\mathbf{Y}}, \\
\mathbf{T}\mathbf{z} &= \breve{\mathbf{z}} \otimes \breve{\mathbf{z}} \otimes \breve{\mathbf{z}}, \\
&\int_0^\infty \left(\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}\right)^{1/3}\mathrm{d}t \\
&= \int_0^\infty \left(\mathbf{x}\mathbf{T}^{-1}\mathrm{e}^{t\mathbf{T}\mathbf{Y}\mathbf{T}^{-1}}\mathbf{T}\mathbf{z}\right)^{1/3}\mathrm{d}t \\
&= \int_0^\infty \left(\left(\breve{\mathbf{x}} \otimes \breve{\mathbf{x}} \otimes \breve{\mathbf{x}}\right)\mathrm{e}^{t\left(\breve{\mathbf{Y}} \oplus \breve{\mathbf{Y}} \oplus \breve{\mathbf{Y}}\right)}\left(\breve{\mathbf{z}} \otimes \breve{\mathbf{z}} \otimes \breve{\mathbf{z}}\right)\right)^{1/3}\mathrm{d}t \\
&= \int_0^\infty \left(\left(\breve{\mathbf{x}}\mathrm{e}^{t\breve{\mathbf{Y}}}\breve{\mathbf{z}}\right)^3\right)^{1/3}\mathrm{d}t \\
&= -\breve{\mathbf{x}}\breve{\mathbf{Y}}^{-1}\breve{\mathbf{z}}. \qquad (167)
\end{aligned}$$

  - Alternativ Integral repr.

$$\int_0^\infty \left(\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}\right)^{1/3}\mathrm{d}t = \frac{3\sqrt{3}}{2\pi}\int_0^\infty \int_0^\infty \frac{\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}}{u^3 + \mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}}\,\mathrm{d}t\,\mathrm{d}u.$$

ME-distributed Time-discrete Signals

- Entropy

$$h = -\int_0^\infty \mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z} \ln\left(\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}\right) \mathrm{d}t.$$

Not yet solved in closed-form !

$$h = -\int_0^1 \int_0^\infty \frac{\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}\left(\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}-1\right)}{1+u\left(\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}-1\right)} \mathrm{d}t\, \mathrm{d}u, \qquad (159)$$

$$h = \lim_{\theta\to 0} \frac{1}{\theta} \ln\left(\int_0^\infty f_T(t)^{1-\theta}\, \mathrm{d}t\right). \qquad (160)$$

$$\begin{aligned} &\int_0^\infty \left(\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}\right)^{1-\theta} \mathrm{d}t \\ &\overset{(a)}{=} \frac{\sin(\pi(1-\theta))}{\pi(1-\theta)} \int_0^\infty \int_0^\infty \frac{\mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}}{u^{\frac{1}{1-\theta}} + \mathbf{x}\mathrm{e}^{t\mathbf{Y}}\mathbf{z}} \mathrm{d}t\, \mathrm{d}u \\ &\overset{(b)}{=} \frac{\sin(\pi(1-\theta))}{\pi(1-\theta)} \int_0^\infty \int_0^\infty \mathbf{x}\left(u^{\frac{1}{1-\theta}}\mathrm{e}^{-t\mathbf{Y}} + \mathbf{z}\mathbf{x}\right)^{-1} \mathbf{z}\, \mathrm{d}t\, \mathrm{d}u, \end{aligned} \qquad (161)$$

- Mutual Information
  - The signal and noise are assumed ME-distributed

$$y = x + w,\; x \geq 0,\; w \geq 0,$$

$$\begin{aligned} I &= h(y) - h(y|x) = h(y) - h(w) \\ &= \ln\left(S_\mathrm{x}/S_\mathrm{w}\right) - \left(h(y_\mathrm{um}) - h(w_\mathrm{um})\right). \end{aligned}$$

Type-I pdf: Matrix Gaussian-like

Definition 7.1: Let $f_T(t) = c\mathbf{x}\mathrm{e}^{t^2\mathbf{Y}}\mathbf{z}$, $t \in [-\infty, \infty]$, with $c = (\sqrt{\pi}\mathbf{x}\,(-\mathbf{Y})^{-1/2}\,\mathbf{z})^{-1}$, denote the type I pdf.

$$\begin{aligned}
\mathbb{E}\{T^n\} &= c\int_{-\infty}^{\infty} t^n \mathbf{x}\mathrm{e}^{t^2\mathbf{Y}}\mathbf{z}\,\mathrm{d}t,\ n = \{0, 2, 4, \ldots\}, \\
&= 2c\int_0^{\infty} t^n \mathbf{x}\mathrm{e}^{t^2\mathbf{Y}}\mathbf{z}\,\mathrm{d}t \\
&= c\int_0^{\infty} y^{(n-1)/2}\mathbf{x}\mathrm{e}^{y\mathbf{Y}}\mathbf{z}\,\mathrm{d}y \\
&= \frac{2c}{\sqrt{\pi}}\int_0^{\infty}\int_0^{\infty} y^{n/2}\mathrm{e}^{-yx^2}\mathbf{x}\mathrm{e}^{y\mathbf{Y}}\mathbf{z}\,\mathrm{d}y\,\mathrm{d}x \\
&= c\Gamma\left(\frac{n+1}{2}\right)\mathbf{x}\,(-\mathbf{Y})^{-(n+1)/2}\,\mathbf{z}. \qquad (168)
\end{aligned}$$

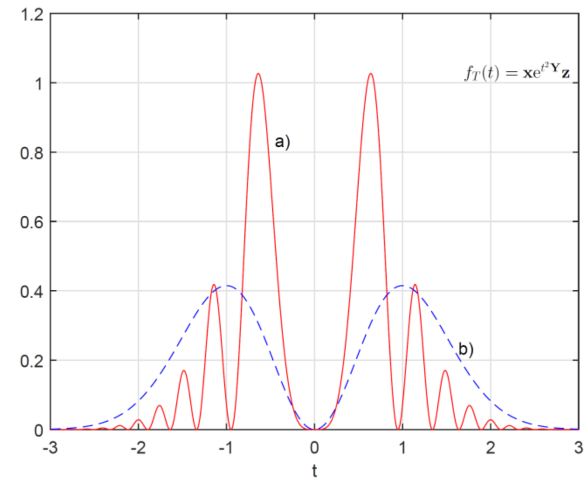


Figure 4. Example of type I distribution for a) $\mathbf{x} = [50\ 0\ 0]$, $\mathbf{y} = [50\ 52\ 3]$, $\mathbf{z} = [0\ 0\ 1]^{\mathrm{T}}$, and b) $\mathbf{x} = [1\ 0]$, $\mathbf{y} = [1\ 2]$, $\mathbf{z} = [0\ 1]^{\mathrm{T}}$, with $\mathbf{Q} = \mathbf{S} - \mathbf{rp}$.

Type-II pdf: Bivariate Matrix Gaussian-like

Definition 7.2: Let $f_{U,V}(u,v) = \frac{1}{\pi}\mathbf{x}\mathrm{e}^{(u^2+v^2)\mathbf{Y}}\mathbf{z}$, $u \in [-\infty,\infty]$, $v \in [-\infty,\infty]$, denote the type II pdf.

$$\begin{aligned}
\mathbb{E}\{U^n V^m\} &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^n v^m \mathbf{x}\mathrm{e}^{(u^2+v^2)\mathbf{Y}}\mathbf{z}\,\mathrm{d}u\,\mathrm{d}v, \\
&\quad n = \{0,2,4,\ldots\},\ m = \{0,2,4,\ldots\}, \\
&= \frac{4}{\pi}\int_{0}^{\infty} a^{(n-1)/2} b^{(m-1)/2}\mathbf{x}\mathrm{e}^{(a+b)\mathbf{Y}}\mathbf{z}\,\mathrm{d}a\,\mathrm{d}b \\
&= \frac{4}{\pi}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{m+1}{2}\right) \\
&\quad \times \mathbf{x}\left(-\mathbf{Y}\right)^{-(n+1)/2}\left(-\mathbf{Y}\right)^{-(m+1)/2}\mathbf{z}. \qquad (169)
\end{aligned}$$

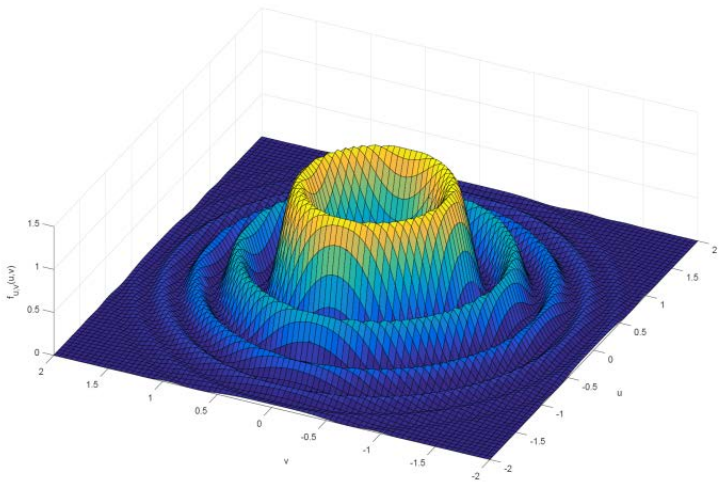


Figure 5. Example of type II distribution for $\mathbf{x} = [50\ 0\ 0]$, $\mathbf{y} = [50\ 52\ 3]$, $\mathbf{z} = [0\ 0\ 1]^{\mathrm{T}}$, and $\mathbf{Y} = \mathbf{S} - \mathbf{z}\mathbf{x}$.

$$\begin{aligned}
f_U(u) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\mathbf{x}\mathrm{e}^{(u^2+v^2)\mathbf{Y}}\mathbf{z}\mathrm{d}v \\
&= \frac{1}{\sqrt{\pi}}\mathbf{x}\mathrm{e}^{u^2\mathbf{Y}}\left(-\mathbf{Y}\right)^{-1/2}\mathbf{z}. \qquad (170)
\end{aligned}$$

Type-III pdf: Matrix Rayleigh-like

Definition 7.3: Let $f_T(t) = 2t\mathbf{x}e^{t^2\mathbf{Y}}\mathbf{z}$, $t \in (0, \infty]$, denote the type III pdf.

$$
\begin{aligned}
\mathbb{E}\{T^n\} &= \int_0^\infty 2t^{n+1}\mathbf{x}e^{t^2\mathbf{Y}}\mathbf{z}\,\mathrm{d}t\, n = \{0, 1, \ldots\} \\
&= \int_0^\infty u^{n/2}\mathbf{x}e^{u\mathbf{Y}}\mathbf{z}\,\mathrm{d}u \\
&= \Gamma\left(\frac{n+2}{2}\right)\mathbf{x}\,(-\mathbf{Y})^{-(n+2)/2}\,\mathbf{z}. \qquad (171)
\end{aligned}
$$

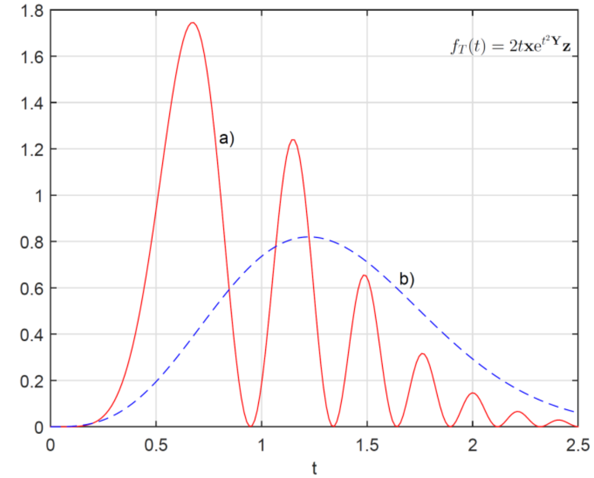


Figure 6. Example of type III distribution for a) $\mathbf{x} = [50\ 0\ 0]$, $\mathbf{y} = [50\ 52\ 3]$, $\mathbf{z} = [0\ 0\ 1]^{\mathrm{T}}$, and b) $\mathbf{x} = [1\ 0]$, $\mathbf{y} = [1\ 2]$, $\mathbf{z} = [0\ 1]^{\mathrm{T}}$, with $\mathbf{Y} = \mathbf{S} - \mathbf{zx}$.

Summary

- structured, refined and extended the ME-distribution approach for perf. analysis of wireless comm. systems with ME-distributed fading SNR.
- New tools derived, new communication cases analyzed, new channel fading models introduced.
- Analyzed:
  - Effective capacity of AMC
  - Throughput of persistent/truncated-(H)ARQ
  - Throughput of 3-phase NCBR
  - SER/BER of Modulation and detection, etc.
  - Throughput of ARQ w (in)dependent Interference
- Widened the use of the ME-distribution to discrete-time r.v. signals: Max-Lloyd Quantization, Panter-Dite formula, Entropy, Mutual information…
- Generalized the ME-distribution to new Matrix distributions.

Conclusion

- The ME-distribution approach can be helpful for wireless system performance analysis in communication theory, information theory, and related areas.

END