Optimal Transmission Rate for MISO Channels with Joint Sum and Per-antenna Power Constraints

Phuong L. Cao *, *Tobias J. Oechtering* *, *Rafael F. Schaefer* †, and *Mikael Skoglund* *

* School of Electrical Engineering and the ACCESS Linnaeus Center
KTH Royal Institute of Technology, Stockholm, Sweden

† Department of Electrical Engineering, Princeton University, New Jersey, USA

***Abstract*—We consider multiple-input single-output (MISO) Gaussian channels with joint sum and per-antenna power constraints. A closed-form solution of the optimal beamforming vector is derived which achieves the maximal transmission rate. The result shows that if the sum power constraint only optimal power allocation violates a per-antenna power constraint then the joint power constraint optimal power allocation is at the intersection of the sum power constraint and the per-antenna power constraints.**

I. Introduction

The optimization problem to find the optimal transmit strategy for a MISO Gaussian channel has been studied subject to sum power constraint or per-antenna power constraints, but never both together. Under the sum power constraint, when the channel state information is known at both transmitter and receiver, the transmission rate is obtained by performing singular value decomposition and applying water-filling on channel eigenvalues [1]-[3]. The per-antenna power constraints problem, which receives more attention recently, results in a different power allocation mechanism since the power can not arbitrarily be allocated among the transmit antennas. Indeed, the capacity of the MISO channel with per-antenna power constraints has been studied for different setups [4]-[7]. In [4], the closed-form solution of the capacity and the optimal signalling scheme has been established for two separate cases: a constant channel and a Rayleigh fading channel. In [5], the problem of transmitter optimization for the multi-antenna downlink is considered. They mainly focus on the minimum-power beam-forming design and the capacity-achieving transmitter design. In [6], an iterative algorithm is proposed for solving the problem of maximizing the weighted sum rate for multiuser system with per-antenna power constraint. The ergodic capacity of the MISO channel with per-antenna power constraint is considered in [4] and [7].

In practice, the individual power constraint reflects constraints on each transmitter chain while the sum power can be a limitation on the allowed radiation from the transmitter or it is used to bound the energy consumption. Although the optimization problem with a separate sum power constraint and/or per-antenna power constraints has been extensively studied, to the best of our knowledge, a combination of both constraints surprisingly has not been considered yet. Since the sum power constraint is not active if the allowed sum power is larger than the sum of the per-antenna power constraints, the problem is only interesting if the sum power constraint is smaller than the sum of the individual power constraints as illustrated in Fig. 1. In this paper, we focus on finding maximal transmission rate and analyzing power allocation behavior for the MISO channel with joint sum and per-antenna power constraints with the assumption of perfect channel knowledge at the transmitter.

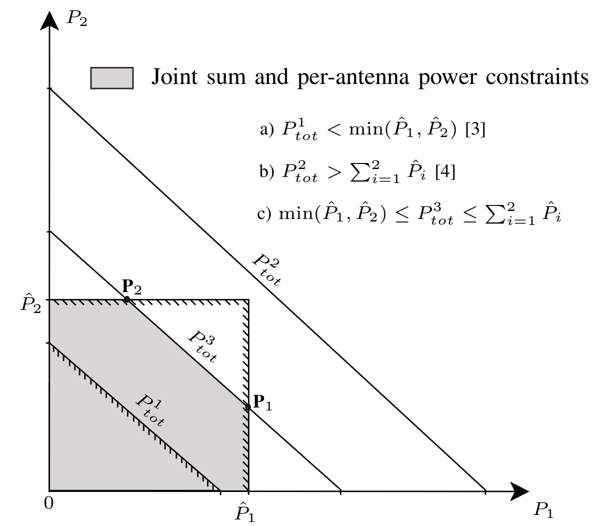

Fig. 1: Feasible power allocation region with joint sum and per-antenna power constraints with a) per-antenna power constraints are inactive, b) sum power constraint is inactive, c) sum and power antenna power constraints are all active.

The paper is organized as follows. In the next section, the system model and power constraints are briefly introduced. Beamforming optimality and optimal transmission rate are considered in the Section III. In Section IV, we analyze the power allocation behavior of MISO channel with joint sum and per-antenna power constraints when the number of transmit antennas equals two with at most one violated per-antenna power constraint. Then, the numerical examples are presented in the next section. Finally, we provide some remarks and conclusions.

Notation: We use bold lower-case letters for vectors, capital letters for matrices. The superscripts $(\cdot)^T$, $(\cdot)^*$ and $(\cdot)^H$ stand for transpose, conjugate, and conjugate transpose; the superscripts $(\cdot)^{\triangle}$, $(\cdot)^{\nabla}$ and $(\cdot)^{\star}$ denote the corresponding optimal values of optimization problems according to the sum power constraint, the per-antenna power constraints, and the joint sum and per-antenna power constraints respectively. We use $\succcurlyeq$ for positive semi-definite relation, $\mathrm{tr}(\cdot)$ for trace, $\mathrm{rank}(\cdot)$ for rank, $\mathrm{diag}\{\cdot\}$ for diagonal matrix. The expectation operator of a random variable is given by $\mathbb{E}[\cdot]$.

II. System Model and Power Constraints

A. System Model

We consider a MISO channel with n transmit antennas and one receive antenna. Further, we assume that channel state information (CSI) is known at both transmitter and receiver. The channel input-output relation of this transmission model can be written as

$$y = \mathbf{x}^T\mathbf{h} + z \tag{1}$$

where $\mathbf{x} = [x_1, ..., x_n]^T \in \mathbb{C}^{n\times 1}$ is a complex transmit signal vector, $\mathbf{h} = [h_1, ..., h_n]^T \in \mathbb{C}^{n\times 1}$ is channel coefficient vector with complex elements and z is zero-mean scalar additive white complex Gaussian noise with power σ^2. Without loss of generality, we assume that $|h_k| > 0, \forall k \in \{1, ..., n\}$, since otherwise we consider a MISO channel with a reduced number of antennas. In the following we focus on achievable rates using Gaussian distribution input. Let $\mathbf{Q} = \mathbb{E}\left[\mathbf{x}\mathbf{x}^H\right]$ be the transmit covariance matrix of the Gaussian input, then the achievable transmission rate is

$$R = f(\mathbf{Q}) = \log\left(1 + \frac{1}{\sigma^2}\mathbf{h}^H\mathbf{Q}\mathbf{h}\right). \tag{2}$$

The question is how to identify the transmit covariance matrix $\mathbf{Q}$ subject to a given power constraint such that the transmission rate in (2) is maximized.

B. Power Constraints

In this part, we formally introduce the sum power, the per-antenna power and the joint sum and per-antenna power constraints problems.

1) Sum Power Constraint: If we consider a sum power constraint, the total transmit power from all antennas is limited by P_{tot}. This power can be allocated arbitrarily among the transmit antennas, and the input covariance matrix has to satisfy the condition $\mathrm{tr}(\mathbf{Q}) \le P_{tot}$. Let $\mathcal{S}_1$ denote the set of all power allocations which satisfy the sum power constraint, then $\mathcal{S}_1$ can be represented as

$$\mathcal{S}_1 := \{\mathbf{Q} \succcurlyeq 0 : \mathrm{tr}(\mathbf{Q}) \le P_{tot}\}.$$

2) Per-antenna Power Constraints: In per-antenna power constraints case [4]-[8], each individual transmit antenna has its own average power limitation $\hat{P}_i$, $\forall i \in \{1, ..., n\}$. In fact, there is no resource allocation flexibility among the transmit antennas. However, the antennas can fully cooperate with each other for the transmission. Thus, for the per-antenna power constraints, the input covariance matrix $\mathbf{Q}$ is formed with diagonal values which have to satisfy $q_{ii} = \mathbf{e}_i^T\mathbf{Q}\mathbf{e}_i \le \hat{P}_i$ with $\mathbf{e}_i = [0, ..., 1, ..., 0]^T$ is the i^{th} Cartesian unit vector. Let $\mathcal{S}_2$ denote the set of all power allocation which satisfy the per-antenna power constraints, then $\mathcal{S}_2$ can be represented as

$$\mathcal{S}_2 := \{\mathbf{Q} \succcurlyeq 0 : \mathbf{e}_i^T\mathbf{Q}\mathbf{e}_i \le \hat{P}_i, i = 1, ..., n\}.$$

3) Joint Sum and Per-antenna Power Constraints: In this case, we combine the sum power and per-antenna power constraints. This means each transmit antenna has the maximum individual transmit power budget of $\hat{P}_i$, $\forall i \in \{1, ..., n\}$ and the sum power condition P_{tot} has to be satisfied as well. Let $\mathcal{S}_3$ denote the set of all power allocations which satisfy the joint sum and per-antenna power constraints, then $\mathcal{S}_3$ can be represented as

$$\begin{aligned}\mathcal{S}_3 &= \mathcal{S}_1 \cap \mathcal{S}_2 \\ &= \{\mathbf{Q} \succcurlyeq 0 : \mathrm{tr}(\mathbf{Q}) \le P_{tot}, \mathbf{e}_i^T\mathbf{Q}\mathbf{e}_i \le \hat{P}_i, i = 1, ..., n\}.\end{aligned}$$

In Fig. 1, the power constraint domains are shown with two given maximum individual powers on each antenna and increasing sum power, i.e., $P_{tot}^1 < \min(\hat{P}_1, \hat{P}_2) \le P_{tot}^3 \le \sum_{i=1}^2 \hat{P}_i < P_{tot}^2$. We have three different cases from Fig. 1 as follows:

- Sum power constraint power domain: this domain exists when $P_{tot} = P_{tot}^1 < \min(\hat{P}_1, \hat{P}_2)$. Then only sum power constraint is activated [3].
- Per-antenna power constraints power domain: this domain exists when $P_{tot} = P_{tot}^2 > \sum_{i=1}^2 \hat{P}_i$. Then only per-antenna power constraints are activated [4].
- Joint sum and per-antenna power constraints power domain: this domain (gray area in Fig. 1) is considered when the power relations satisfy $\min(\hat{P}_1, \hat{P}_2) \le P_{tot} = P_{tot}^3 \le \sum_{i=1}^2 \hat{P}_i$. Both sum power constraint and per-antenna power constraints are activated.

III. Problem Formulations and Solutions

In this section, we derive the new result on the maximum transmission rate of the MISO channel using Gaussian input with joint sum and per-antenna power constraints. First, we review the known results corresponding to optimization problem with sum power constraint and optimization problem with per-antenna power constraints separately. After that, the optimization problem with joint sum and per-antenna power constraints will be studied.

A. Review of Known Results

1) Optimization Problem 1 (OP1) - Sum Power Constraint: This problem aims to find the maximum transmission rate in (2) under the set of power constraint S_1. The optimization problem of transmission rate for our given MISO channel in this case can be written as

$$\begin{aligned} &\text{maximize} && \log\left(1 + \frac{1}{\sigma^2}\mathbf{h}^H\mathbf{Q}\mathbf{h}\right) \\ &\text{subject to} && \mathbf{Q} \in \mathcal{S}_1. \end{aligned} \tag{3}$$

The transmit strategy for the MISO channel is to send the information only in the direction of the channel vector $\mathbf{h}$ [1],[2].

The optimal solution is to perform beamforming using full power P_{tot} in the direction of the channel, i.e., $\mathbf{Q}^{\triangle} = P_{tot}\mathbf{u}_1\mathbf{u}_1^H$ with $\mathbf{u}_1 = \mathbf{h}/\|\mathbf{h}\|$. The MISO channel capacity with a sum power constraint P_{tot} is

$$R^{\triangle} = \log\left(1 + \frac{P_{tot}}{\sigma^2}\sum_{i=1}^{n}|h_i|^2\right) = \log\left(1 + \frac{P_{tot}}{\sigma^2}\|\mathbf{h}\|^2\right). \quad (4)$$

2) Optimization Problem 2 (OP2) - Per-antenna Power Constraints: In [4], Vu established the closed-form expression of the capacity and optimal signaling scheme for the single-user MISO channel with per-antenna power constraints.

The capacity in this situation can be found by solving the optimization problem

$$\begin{aligned} &\text{maximize} && \log\left(1 + \frac{1}{\sigma^2}\mathbf{h}^H\mathbf{Q}\mathbf{h}\right) \\ &\text{subject to} && \mathbf{Q} \in \mathcal{S}_2. \end{aligned} \quad (5)$$

The problem in (5) can be solved by relaxing the semi-definite constraint, reducing the problem to a form solvable in closed-form, and then showing that the optimal solution to the relaxed problem is also the optimal solution to the original problem [4]. In the per-antenna power constraints case, there is no power allocation among the antennas. Therefore, the transmit power from the i^{th} antenna is fixed to be $\hat{P}_i$. The optimal covariance matrix $\mathbf{Q}^{\nabla}$ has rank one with $\mathbf{Q}^{\nabla} = \lambda\mathbf{v}_1\mathbf{v}_1^H$, $\lambda = \sum_i \hat{P}_i$ and the beamforming vector $\mathbf{v}_1$ has the elements given as

$$v_{k1} = \eta_k \frac{h_k^*}{|h_k|} \text{ with } \eta_k = \frac{\sqrt{\hat{P}_k}}{\sqrt{\sum_{i=1}^{n}\hat{P}_i}}, k = 1, ..., n. \quad (6)$$

The capacity with per-antenna power constraints is then given as

$$R^{\nabla} = \log\left(1 + \frac{1}{\sigma^2}\sum_{i=1}^{n}\hat{P}_i|\mathbf{h}^H\mathbf{v}_1|^2\right) \quad (7)$$

$$= \log\left[1 + \frac{1}{\sigma^2}\left(\sum_{i=1}^{n}|h_i|\sqrt{\hat{P}_i}\right)^2\right]. \quad (8)$$

B. Optimization Problem 3 (OP3) - Joint Sum and Per-antenna Power Constraints

In the optimization problem with joint sum and per-antenna power constraints, Gaussian distributed input is optimal, but the proof is not presented here due to space limitation.

1) Problem formulation: The optimization problem to find the capacity is a convex optimization problem given as follows

$$\begin{aligned} &\text{maximize} && \log\left(1 + \frac{1}{\sigma^2}\mathbf{h}^H\mathbf{Q}\mathbf{h}\right) \\ &\text{subject to} && \mathbf{Q} \in \mathcal{S}_3. \end{aligned} \quad (9)$$

The objective function of problem (9) is concave while both constraints $\mathrm{tr}(\mathbf{Q}) \leq P_{tot}$ and $\mathbf{e}_i^T\mathbf{Q}\mathbf{e}_i \leq P_i \; \forall i \in \{1, ..., n\}$ are linear in $\mathbf{Q}$. Furthermore, since $\log\left(1 + \frac{1}{\sigma^2}\mathbf{h}^H\mathbf{Q}\mathbf{h}\right)$ is an increasing function in $\mathbf{h}^H\mathbf{Q}\mathbf{h}$, we can express the optimization problem (9) as

$$\max_{\mathbf{Q}\in\mathcal{S}_3} \log\left(1 + \frac{1}{\sigma^2}\mathbf{h}^H\mathbf{Q}\mathbf{h}\right) = \log\left(1 + \frac{1}{\sigma^2}\max_{\mathbf{Q}\in\mathcal{S}_3}\mathbf{h}^H\mathbf{Q}\mathbf{h}\right). \quad (10)$$

Thus, we can equivalently focus on the following optimization problem

$$\begin{aligned} &\text{maximize} && \mathbf{h}^H\mathbf{Q}\mathbf{h} \\ &\text{subject to} && \mathbf{Q} \in \mathcal{S}_3. \end{aligned} \quad (11)$$

The results in the following propositions will show that the optimal transmit strategy for joint sum and per-antenna power constraint is beamforming; the optimal transmission method is to transmit with full sum power while the per-antenna power constraints have to satisfied. The phase is chosen that match the phase of the channel coefficient. In the following, let $\mathbf{q}$ denote a beamforming vector of a rank one transmit strategy $\mathbf{Q}$, i.e., $\mathbf{Q} = \mathbf{q}\mathbf{q}^H$.

2) Beamforming Optimality:

Proposition 1: For OP3 with $P_{tot} < \sum_{i=1}^{n}\hat{P}_i$ and a given channel $\mathbf{h} \in \mathbb{C}^{n\times 1}$ with $h_i \neq 0, \forall i \in \{1, ..., n\}$, beamforming is the optimal transmit strategy.

Proof:

We denote $\mathbf{P} = \mathrm{diag}\{\hat{P}_i\}$ as diagonal matrix of the per-antenna power constraints, P_{tot} as the total transmit power, $\mathbf{D} = \mathrm{diag}\{\nu_i\}$ as diagonal matrix of Lagrangian multiplier for the per-antenna power constraints, μ as Lagrangian multiplier for the sum power constraint and $\mathbf{K} \succcurlyeq 0$ as Lagrangian multiplier for the positive semi-definite constraint. Then the Lagrangian for problem (11) is given by

$$\mathcal{L} = \mathbf{h}^H\mathbf{Q}\mathbf{h} - \mathrm{tr}[\mathbf{D}(\mathbf{Q}-\mathbf{P})] - \mu(\mathrm{tr}(\mathbf{Q}) - P_{tot}) + \mathrm{tr}(\mathbf{K}\mathbf{Q}). \quad (12)$$

Taking the first derivative and set it equal to zero, we have

$$\frac{\partial\mathcal{L}}{\partial\mathbf{Q}} = \mathbf{h}\mathbf{h}^H - \mathbf{D} - \mu\mathbf{I} + \mathbf{K} \overset{!}{=} 0 \quad (13)$$

$$\mathbf{h}\mathbf{h}^H = \mathbf{W} - \mathbf{K}, \quad (14)$$

where $\mathbf{W} = \mathbf{D} + \mu\mathbf{I}$.

Using the slackness condition $\mathbf{K}\mathbf{Q} = 0$, we obtain

$$\mathbf{h}\mathbf{h}^H\mathbf{Q} = \mathbf{W}\mathbf{Q}. \quad (15)$$

Since $\mathrm{rank}(\mathbf{W}) = \mathrm{rank}(\mathbf{D} + \mu\mathbf{I})$ is full rank, at the optimum, we have

$$\mathrm{rank}(\mathbf{Q}^{\star}) \leq \mathrm{rank}(\mathbf{h}\mathbf{h}^H) = 1. \quad (16)$$

Obviously, since $h_i \neq 0$, $\forall i \in \{1, ..., n\}$, $\mathrm{rank}(\mathbf{Q}^{\star}) = 0$ is not optimal. Therefore, the optimal rank of $\mathbf{Q}^{\star}$ is one, i.e. beamforming is the optimal transmit strategy. ■

Proposition 2: For OP3 with $P_{tot} < \sum_{i=1}^{n}\hat{P}_i$ and a given channel $\mathbf{h} \in \mathbb{C}^{n\times 1}$ with $h_i \neq 0, \forall i \in \{1, ..., n\}$, the maximum transmission rate $R^\star$ is achieved when the optimal transmit strategy $\mathbf{Q}^\star$ uses full power P_{tot}, i.e., $\mathrm{tr}(\mathbf{Q}^\star) = P_{tot}$.

Proof (by Contradiction): Let $f(\mathbf{Q})$ denote the transmission rate expressed as a function of $\mathbf{Q}$

$$f(\mathbf{Q}) = \log(1 + \frac{1}{\sigma^2}\mathbf{h}^H\mathbf{Q}\mathbf{h}). \tag{17}$$

Also, let $P_i^\star$, $i = 1, ..., n$ denote the optimal power allocation of OP3. Suppose there exists an optimal transmit strategy $\mathbf{Q}^\star$ with $\mathrm{tr}(\mathbf{Q}^\star) = \sum_{i=1}^n P_i^\star < P_{tot}$. Then, the maximum transmission rate $R^\star$ can be calculated as

$$\begin{aligned} R^\star = f(\mathbf{Q}^\star) &= \max_{\mathbf{Q}:\{\mathrm{tr}(\mathbf{Q})\leq P_{tot}, q_{ii}\leq \hat{P}_i, \mathbf{Q}\succcurlyeq 0\}} f(\mathbf{Q}) \\ &= \log(1 + \frac{1}{\sigma^2}(\sum_{i=1}^n |h_i|\sqrt{P_i^\star})^2). \end{aligned} \tag{18}$$

Since $\mathrm{tr}(\mathbf{Q}^\star) < P_{tot}$, there exists a k with $P_k^\star < P_k \leq \hat{P}_k$ and $P_{tot} - P_k \geq \sum_{\substack{i=1 \\ i\neq k}}^n P_i^\star$, so that:

$$\begin{aligned} \tilde{R} &= \log(1 + \frac{1}{\sigma^2}(\sum_{\substack{i=1 \\ i\neq k}}^n |h_i|\sqrt{P_i^\star} + |h_k|\sqrt{P_k})^2) \\ &> \log(1 + \frac{1}{\sigma^2}(\sum_{i=1}^n |h_i|\sqrt{P_i^\star})^2) = R^\star. \end{aligned} \tag{19}$$

This contradicts the assumption that $R^\star = f(\mathbf{Q}^\star)$ is the maximum transmission rate. It follows that the optimal transmit strategy $\mathbf{Q}^\star$ allocates full sum power P_{tot}. ■

Next, we focus on characterizing properties of the optimal beamforming vector $\mathbf{q}^\star$.

Lemma 1: Let $\mathbf{q}^\star$ be the optimal beamforming vector corresponding to the optimal covariance matrix $\mathbf{Q}^\star$. Then

$$\mathbf{q}^\star \in \mathbb{Q} := \left\{ \mathbf{q} : \mathbf{q} = \left[\frac{\sqrt{P_1}h_1^*}{|h_1|}, ..., \frac{\sqrt{P_n}h_n^*}{|h_n|}\right]^T, \mathbf{q}\mathbf{q}^H \in \mathcal{S}_3 \right\}.$$

Proof: Consider optimization problem (11) with the optimization domain $\mathcal{S}_3$, we have

$$\begin{aligned} \max_{\mathbf{Q}\in\mathcal{S}_3} \mathbf{h}^H\mathbf{Q}\mathbf{h} &\overset{(1)}{=} \max_{\mathbf{q}:\mathbf{q}\mathbf{q}^H\in\mathcal{S}_3} |\mathbf{h}^H\mathbf{q}|^2 \\ &\overset{(2)}{=} \max_{\mathbf{q}:\mathbf{q}\mathbf{q}^H\in\mathcal{S}_3} |\sum_{i=1}^n |h_i|\sqrt{P_i}e^{j(\varphi_i - \varphi_{h_i})}|^2 \\ &\leq \max_{\mathbf{q}:\mathbf{q}\mathbf{q}^H\in\mathcal{S}_3} (\sum_{i=1}^n |h_i|\sqrt{P_i})^2 \\ &= \max_{\mathbf{q}\in\mathbb{Q}} (\sum_{i=1}^n |h_i|\sqrt{P_i})^2 \\ &= \max_{\mathbf{q}\in\mathbb{Q}} |\sum_{i=1}^n h_i^*\sqrt{P_i}\frac{h_i}{|h_i|}|^2 \\ &= \max_{\mathbf{q}\in\mathbb{Q}} |\mathbf{h}^H\mathbf{q}|^2 \\ &\overset{(3)}{\leq} \max_{\mathbf{q}:\mathbf{q}\mathbf{q}^H\in\mathcal{S}_3} |\mathbf{h}^H\mathbf{q}|^2 \end{aligned} \tag{20}$$

where

(1) follows Proposition 1 and 2,
(2) from the definition $h_i = |h_i|e^{j\varphi_{h_i}}$, $q_i = \sqrt{P_i}e^{j\varphi_i}$ with $\varphi_{h_i}, \varphi_i \in [0, 2\pi]$, and
(3) from the fact that $\mathbb{Q} \subseteq \{\mathbf{q} : \mathbf{q}\mathbf{q}^H \in \mathcal{S}_3\}$.

From (20) it follows that the optimal beamforming vector $\mathbf{q}^\star$ is in $\mathbb{Q}$.

■

The capacity in (2) with joint sum and per-antenna power constraint can be expressed as:

$$R^\star = \log\left(1 + \frac{1}{\sigma^2}\mathbf{h}^H\mathbf{Q}^\star\mathbf{h}\right). \tag{21}$$

3) Optimal Power Allocation for OP3: In joint sum and per-antenna power constraints problem, Proposition 2 states that the capacity achieving strategy always allocates full sum power P_{tot}. The optimal power allocation solution of OP1 sometimes exceeds the maximum allowed per-antenna power of OP3. In the following theorem, we will show how to reallocate the powers for the MISO channel with two antennas to satisfy the constraints. Since we consider $\hat{P}_1 + \hat{P}_2 \geq P_{tot}$, the OP1 solution can violate at most one per-antenna constraint.

Theorem 1: Consider the MISO channel $\mathbf{h} \in \mathbb{C}^{2\times 1}$ and $P_{tot} < \sum_{i=1}^2 \hat{P}_i$. Let $\pi^\triangle$ be an optimal point of the OP1 with the optimal power allocation $[P_1^\triangle, P_2^\triangle]$ under the sum power constraint only. Let $\pi_1 = [\hat{P}_1, P_{tot} - \hat{P}_1]$ and $\pi_2 = [P_{tot} - \hat{P}_2, \hat{P}_2]$ be intersection points of the sum power constraint and per-antenna power constraints. For any optimal point $\pi^\star$ of OP3 with the power allocation pair $[P_1^\star, P_2^\star]$, we have

$$\pi^\star = \begin{cases} \pi_1, & \text{if } P_1^\triangle \geq \hat{P}_1. \\ \pi_2, & \text{if } P_2^\triangle \geq \hat{P}_2. \\ \pi^\triangle, & \text{otherwise.} \end{cases} \tag{22}$$

for $\pi^\triangle, \pi_1, \pi_2, \pi^\star \in \mathbb{R}_+^2$.

Proof: The proof can be found in Appendix. ■

IV. Numerical Example

For numerical example, we first provide a MISO 2×1 system with the complex channel is given as $h = [0.9572 + 0.8003i, 0.4854 + 0.1419i]^T$, $\sigma^2 = 1$. We choose the maximum power on each antenna $\hat{P}_1 = 7$, $\hat{P}_2 = 10$, and the total transmit power $P_{tot} = 13$. The relationship between sum and per antenna power constraints in the joint scenario is shown in Fig. 1 and Fig. 2. Let $\pi_1 = (\hat{P}_1, P_{tot} - \hat{P}_1) = (7, 6)$ and $\pi_2 = (P_{tot} - \hat{P}_2, \hat{P}_2) = (3, 10)$ be the intersection points of sum and per-antenna power constraints when each antenna 1 and 2 transmits full individual power, then the transmission rate at π_1 and π_2 are calculated as $R_{(7,6)}$ and $R_{(3,10)}$. The optimal point $\pi^\triangle$ of OP1 can be found with the optimal power allocation $[P_1^\triangle, P_2^\triangle] = [11, 2]$. However, at the optimum of OP1, the transmit power at antenna 1 violates the per-antenna power constraint of antenna 1. Following Theorem 1, once $P_1^\triangle \geq \hat{P}_1$, π_1 will be the optimal point of OP3 instead of $\pi^\triangle$. Due to the limitation of the transmit power on each antenna

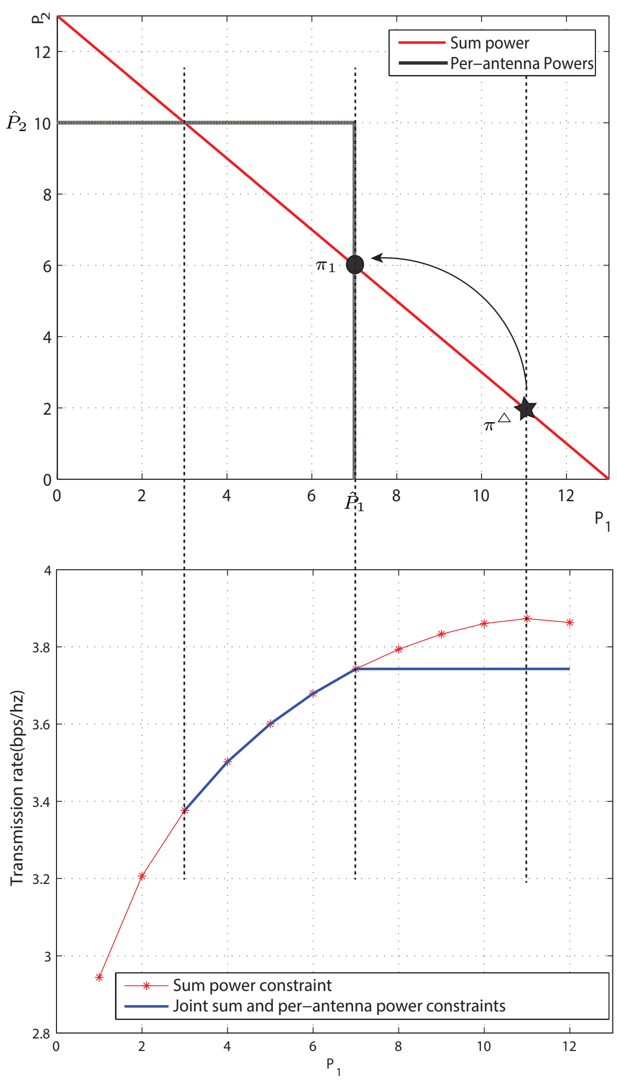

Fig. 2: Power allocation behaviour and transmission rate when $\hat{P}_1 = 7$, $\hat{P}_2 = 10$ and $P_{tot} = 13$.

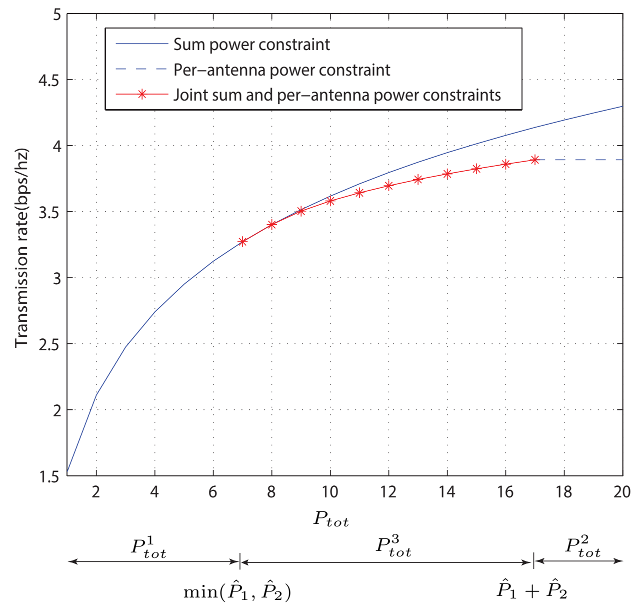

Fig. 3: Transmission rates in different power constraint domains.

and the sum power in the joint sum and per-antenna power constraints problem, the optimal transmission rate of OP3 in this example is achieved at the power allocation $P_1 = \hat{P}_1 = 7$, $P_2 = P_{tot} - \hat{P}_1 = 6$. The plot of this numerical example in Fig. 2 shows the trend of transmission rate and power allocation behaviour. We can observe that the optimal transmission rate of OP3 is the same as optimal transmission rate of OP1 when both sum power constraint and per-antenna power constraints are satisfied. When the optimization solution of OP1 violates the per-antenna power constraints, the transmission rate of OP1 will reduce and set equal as the maximum transmission rate of an intersection point of the sum and per-antenna power constraints.

In Fig. 3, the optimal transmission rates with respect to sum power constraint, per-antenna power constraints and joint sum and per-antenna power constraint are shown with the given transmit power per antenna, i.e., $\hat{P}_1 = 7$, $\hat{P}_2 = 10$ and increasing of total transmit power P_{tot}. From the Fig. 3, we see that with the different range of total transmit power, the optimization problem will be subjected to different power constraints. When the total transmit power $P_{tot} = P^1_{tot} < \min(\hat{P}_1, \hat{P}_2) = 7$, there is only the sum power constraint active. In this case, the optimal transmission rate of the channel is the channel capacity with sum power constraint. When the power relations satisfy $\min(\hat{P}_1, \hat{P}_2) = 7 \leq P_{tot} \leq \sum_{i=1}^{2} \hat{P}_i = 17$, both constraints can be active. In case $P_{tot} = P^3_{tot} > \sum_{i=1}^{2} \hat{P}_i = 17$, only the per-antenna power constraints are active. In this case, whatever the total transmit power increases, the optimal transmission rate will keep the same value $R_{(7,10)}$.

V. Conclusions

In this paper, we derived the solution in finding optimal transmission rate for MISO channel with joint sum and per-antenna power constraints. In our work, we find a procedure how to obtain the optimal solution for MISO channel with joint sum and per-antenna power constraints from two previous different optimization problems with sum power constraint and with per-antenna power constraints. It is shown that beamforming is the optimal transmit strategy and that it is always optimal to use full power.

Furthermore, it has been shown that if the sum power only optimal power allocation violates a per-antenna power constraint, then the optimal power allocation is at the intersection point of the violated per-antenna power allocation and sum power constraint. Thus, the solution of the joint constraint optimal power allocation can be followed from the individual constraint problems.

Appendix

Proof of Theorem 1

For the proof of the theorem we need the following lemma.

Lemma 2: Consider the function

$$f(P_1) = \log\left(1 + \frac{1}{\sigma^2}\mathbf{h}^H \mathbf{Q}^{\triangle}(P_1)\mathbf{h}\right) \tag{23}$$

with $\mathbf{h} = [|h_1|u_1, |h_2|u_2]^T$, $u_i = h_i/|h_i|$, $i = 1, 2$ and

$$\mathbf{Q}^{\triangle}(P_1) = \begin{bmatrix} P_1 & q_{12}^* \\ q_{12} & P_{tot} - P_1 \end{bmatrix} \tag{24}$$

where $q_{12} = \sqrt{P_1(P_{tot} - P_1)}u_1^* u_2$.

For any $P_1 \in [P_1^{\triangle}, P_{tot}]$ with

$$P_1^{\triangle} = \frac{|h_1|^2}{|h_1|^2 + |h_2|^2} P_{tot},$$

$f(P_1)$ is a strictly decreasing function.

Proof: Consider the function $R = f(P_1)$, where $P_1^{\triangle} < P_1 < P_{tot}$.

$$f(P_1) = \log\left(1 + \frac{1}{\sigma^2}\begin{bmatrix}|h_1|u_1\\|h_2|u_2\end{bmatrix}^H \begin{bmatrix} P_1 & q_{12}^H \\ q_{12} & P_{tot} - P_1\end{bmatrix}\begin{bmatrix}|h_1|u_1\\|h_2|u_2\end{bmatrix}\right) \tag{25}$$

where $q_{12} = \sqrt{P_1(P_{tot} - P_1)}u_1^* u_2$.

Taking the derivative of (25), we obtain

$$\begin{aligned}\frac{\partial f(P_1)}{\partial P_1} &= \frac{\frac{1}{\sigma^2}(|h_1|\sqrt{P_1} + |h_2|\sqrt{P_{tot} - P_1})}{1 + \frac{1}{\sigma^2}(|h_1|\sqrt{P_1} + |h_2|\sqrt{P_{tot} - P_1})^2} \\ &\quad \times \left(\frac{|h_1|}{\sqrt{P_1}} - \frac{|h_2|}{\sqrt{P_{tot} - P_1}}\right) \\ &= A\left(\frac{|h_1|}{\sqrt{P_1}} - \frac{|h_2|}{\sqrt{P_{tot} - P_1}}\right),\end{aligned} \tag{26}$$

where

$$A = \frac{\frac{1}{\sigma^2}(|h_1|\sqrt{P_1} + |h_2|\sqrt{P_{tot} - P_1})}{1 + \frac{1}{\sigma^2}(|h_1|\sqrt{P_1} + |h_2|\sqrt{P_{tot} - P_1})^2} > 0. \tag{27}$$

Also, we have

$$\begin{aligned} P_1^{\triangle} &= \frac{|h_1|^2}{|h_1|^2 + |h_2|^2} P_{tot} < P_1 < P_{tot} \\ \Rightarrow &\frac{|h_1|}{\sqrt{P_1}} < \frac{|h_2|}{\sqrt{P_{tot} - P_1}}. \end{aligned} \tag{28}$$

From (28) and (26) we get the inequality $\frac{\partial f(P_1)}{\partial P_1} < 0$. Since $\frac{\partial f(P_1)}{\partial P_1} < 0$, $f(P_1)$ is a strictly decreasing function on $[P_1^{\triangle}, P_{tot}]$. ■

Now, we start to prove the Theorem 1 by applying Lemma 2 directly. Assume that full sum power P_{tot} is used. We can find an optimum point which achieves the capacity, such that $R^{\triangle} = \log\left(1 + \frac{1}{\sigma^2}\mathbf{h}^H \mathbf{Q}^{\triangle}\mathbf{h}\right)$, $\mathrm{tr}(\mathbf{Q}^{\triangle}) = P_1^{\triangle} + P_2^{\triangle} = P_{tot}$.

Let $\pi_2 = [P_{tot} - \hat{P}_2, \hat{P}_2]$ be an intersection point of sum and per-antenna power constraints. Then the corresponding transmit power are represented as

$$\begin{cases} P_1 = & P_{tot} - \hat{P}_2, \\ P_2 = & \hat{P}_2. \end{cases}$$

If $P_1^{\triangle} < P_{tot} - \hat{P}_2$, we examine the function $R = f(P_1)$ with $P_1^{\triangle} < P_1 < P_{tot}$. From Lemma 2, we know that $R = f(P_1)$ is a strictly decreasing function. Therefore, due to power constraints, for any optimal point of OP1 with power allocation pair $[P_1^{\triangle}, P_2^{\triangle}]$, if $P_2^{\triangle} > \hat{P}_2$ then $[P_1, P_2] = [P_1^{\star}, P_2^{\star}] = [P_{tot} - \hat{P}_2, \hat{P}_2]$. Similarly, if $P_1^{\triangle} > \hat{P}_1$ then $[P_1, P_2] = [P_1^{\star}, P_2^{\star}] = [\hat{P}_1, P_{tot} - \hat{P}_1]$. This prove the optimal power allocation (22) of OP3.

Acknowledgement

The work was supported in part by the Strategic Research Agenda Program, Information and Communication Technology The Next Generation (SRA ICT TNG), through the Swedish Government and the Swedish Research Council (VR) through grant C0406401. The work of R. F. Schaefer was supported by the German Research Foundation (DFG) under Grant WY 151/2-1.

References

[1] T. Cover and J. Thomas, *Elements of Information Theory*, 2nd edition. John Wiley and Sons, Inc., 2006.
[2] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, 2005.
[3] I. Telatar, "Capacity of multi-antenna Gaussian channels," *European Trans. Telecommun.*, vol. 10, no. 6, pp. 585-595, Nov. 1999.
[4] Mai Vu, "MISO Capacity with Per-antenna power constraint," *IEEE Trans. on Commun.*, vol. 59, no. 5, May 2011.
[5] W. Yu and T. Lan, "Transmitter optimization for the multi-antenna downlink with per-antenna power constraint", *IEEE Trans. Signal Process.*, vol.55, no. 6, pp. 2646-2660, June 2007.
[6] S. Shi, M. Schubert and H. Boche, "Per-antenna power constrained rate optimization for multiuser MIMO systems", *Proc. International ITG Workshop Smart Antennas.*, 2008.
[7] D. Maamari, N. Devroye, and D. Tuninetti, "The Capacity of the Ergodic MISO channel with Per-antenna Power Constraint and an Application to the Fading Cognitive Interference Channel", *Proc. of International Symposium on Information Theory (ISIT)*, July. 2014.
[8] K. Karakayali, R. Yates, G. Foschini, and R. Valenzuela, "Optimum Zero-forcing Beamforming with Per-antenna Power Constraint", *Proc. of International Symposium on Information Theory (ISIT)*, June. 2007.
[9] J. G. Smith, "The Information Capacity of Amplitude and Variance-Constrained Scalar Gaussian Channel", *Information and Control.*, vol 18, issue 3, pp.203-219, April 1971.
[10] R. Wyrembelski, T. Oechtering, I. Bjelakovic, C. Schnurr and H. Boche, "Capacity of Gaussian MIMO Bidirectional Broadcast channels", *Proc. of International Symposium on Information Theory (ISIT)*, July. 2008.
[11] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed. Cambridge University Press, 2013.
[12] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2009.
[13] Robert B. Ash, *Information Theory*, Dover Publications, Inc., Newyork, 1990.