

OPTIMAL TRANSMIT STRATEGY FOR MIMO CHANNELS WITH JOINT SUM AND PER-ANTENNA POWER CONSTRAINTS

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ABSTRACT

This paper studies optimal transmit strategies for multiple-input multiple-output (MIMO) Gaussian channels with joint sum and per-antenna power constraints. It is shown that if an unconstrained optimal allocation for an antenna exceeds a per-antenna power constraint, then the maximal power for this antenna is used in the constraint optimal transmit strategy. This observation is then used in an iterative algorithm to compute the optimal transmit strategy in closed-form. Finally, a numerical example is provided to illustrate the theoretical results.

Index Terms— MIMO, sum power constraint, per-antenna power constraints, transmit strategy

1. INTRODUCTION

The optimization problem in maximizing the transmission rate for MIMO Gaussian channel has been extensively studied in the last two decades. Under the sum power constraint, the transmission rate is obtained by performing singular value decomposition and applying water-filling on channel eigenvalues [1, 2]. In contrast, the problem in finding maximal transmission rate with per-antenna power constraints results in a different mechanism since the power can not be arbitrarily allocated among the transmit antennas. This problem has been studied in [3–8]. In these works, the authors derived necessary and sufficient conditions for the optimal MIMO transmission schemes and developed an iterative algorithm that converges to the optimal solution. Besides iterative algorithms, a closed-form expression for the capacity of the static Gaussian MIMO channel under per-antenna power constraints is provided in [9].

In a practical system, when joint sum and per-antenna power constraints are considered, the transmitted energy should be limited in total and for each RF chain. The joint sum and per-antenna power constraints setting can be applied either to systems with multiple antennas or to distributed systems with separated energy sources. The optimal transmit strategy problem with the joint sum and per-antenna power constraints has been studied for MISO channels in [10, 11]. Since the sum power constraint is not active if the allowed sum power is larger than the sum of the per-antenna power

constraints, the problem is only interesting if the sum power constraint is smaller than the sum of the individual power constraints. In [12], the optimization problem with the assumption that several antenna subsets are constrained by a sum power constraint while the other antennas are subject to a per-antenna power constraint is studied and a closed-form solution is provided. Unfortunately, the results in that paper can not be directly applied to the case where the transmit powers are jointly constrained by both sum and per-antenna power constraints. To make [12] applicable for the optimization problem with joint sum and per-antenna power constraints, we need to identify for each antenna which constraint is active, which is the key step in this paper.

The contributions of the paper can be summarized as follows: An optimal transmit strategy for MIMO channels with joint sum and per-antenna power constraints with the assumption of perfect channel knowledge at the transmitter is characterized. If an optimal power allocation of an antenna of the unconstrained problem exceeds a per-antenna power constraint, then it is optimal to allocate the maximal power in the constraint optimal transmit strategy. An iterative algorithm, which always converges to the optimum, to compute the optimal solution in closed-form using [12] is proposed.

2. PROBLEM FORMULATION

We consider a MIMO channel with n transmit antennas and m receive antennas. We assume that channel state information (CSI) is known at both transmitter and receiver. The channel input-output relation of this transmission model can be written as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z} \quad (1)$$

where $\mathbf{x} \in \mathbb{C}^{n \times 1}$ is the complex transmit signal vector, $\mathbf{y} \in \mathbb{C}^{m \times 1}$ is the complex received vector. $\mathbf{H} \in \mathbb{C}^{m \times n}$ is the channel coefficient matrix with complex elements. Finally, $\mathbf{z} \in \mathbb{C}^{m \times 1}$ is zero-mean scalar additive white complex Gaussian noise (AWGN), i.e., $\mathbf{z} \sim \mathcal{CN}(0, \sigma^2 \mathbf{I})$. In this paper, for simplicity, we assume that $\sigma^2 = 1$. Let $\mathbf{Q} = \mathbb{E}[\mathbf{x}\mathbf{x}^H]$ be the transmit covariance matrix of the Gaussian input, then the achievable transmission rate is

$$R = f(\mathbf{Q}) = \log \det (\mathbf{I}_m + \mathbf{H}\mathbf{Q}\mathbf{H}^H). \quad (2)$$

The question is how to identify the transmit covariance matrix \mathbf{Q} subject to a given power constraint such that the transmission rate in (2) is maximized.

Let $\mathcal{A} \subseteq \{1, \dots, n\}$ and $\mathcal{S}(\mathcal{A}) := \{\mathbf{Q} \succcurlyeq 0 : \text{tr}(\mathbf{Q}) \leq P_{tot}, P_i = \mathbf{e}_i^T \mathbf{Q} \mathbf{e}_i \leq \hat{P}_i, \forall i \in \mathcal{A}\}$ denote an index set and the set of transmit strategies with total transmit power P_{tot} and per-antenna power constraints $\hat{P}_i, \forall i \in \mathcal{A}$, where $\mathbf{e}_i = [0, \dots, 1, \dots, 0]^T$ is the i -th Cartesian unit vector. Let OP- \mathcal{A} be the optimization problem with optimization domain $\mathcal{S}(\mathcal{A})$ restricted by the per-antenna power constraints with index $i \in \mathcal{A}$. The optimization problem of transmission rate for MIMO channels then will be given as

$$\max_{\mathbf{Q}} f(\mathbf{Q}) \text{ s.t. } \mathbf{Q} \in \mathcal{S}(\mathcal{A}). \quad (3)$$

If $\mathcal{A} = \emptyset$, then (3) corresponds to the sum power constraint only problem which has been well studied in [2]. Therefore in the following we focus on the optimization problem when both sum and per-antenna power constraints are active, i.e., $\mathcal{A} \neq \emptyset$. In the following sections, we first investigate the properties of the power allocation of MIMO channels with joint sum and per-antenna power constraints. After that, an iterative algorithm to find the optimal transmit strategy is proposed.

3. OPTIMAL TRANSMIT STRATEGIES

Depending on the per-antenna power constraints $\hat{P}_i, \forall i = 1, \dots, n$, and the sum power constraint P_{tot} , we can identify three different cases as follows: The first case is when the per-antenna power constraints are never active, i.e., $P_{tot} < \min_i(\hat{P}_i)$. The second case is when the sum power constraint is never active, i.e., $P_{tot} > \sum_{i=1}^n \hat{P}_i$. The most interesting case is when both sum and per-antenna power constraints are active, i.e., $\min_i(\hat{P}_i) \leq P_{tot} \leq \sum_{i=1}^n \hat{P}_i$ [10].

Proposition 1 below shows that the capacity can be achieved with a transmit strategy which allocates full sum power if $\min_i(\hat{P}_i) \leq P_{tot} \leq \sum_{i=1}^n \hat{P}_i$.

Proposition 1. For OP- \mathcal{A} with a given channel $\mathbf{H} \in \mathbb{C}^{m \times n}$ the maximum transmission rate R^* can be achieved when the optimal transmit strategy \mathbf{Q}^* uses full power P_{tot} , i.e., $\text{tr}(\mathbf{Q}^*) = P_{tot}$.

Proof (by Contradiction). Suppose that it is not possible to achieve the maximum transmission rate using full power P_{tot} . This implies that for $\text{tr}(\mathbf{Q}^*) < P_{tot}$, the maximum transmission rate is $R^* = \log \det(\mathbf{I}_m + \mathbf{H} \mathbf{Q}^* \mathbf{H}^H) = \log \prod_{i=1}^{\min(n,m)} (1 + \lambda_i^2 \phi_i)$, where λ_i and ϕ_i are the eigenvalues of \mathbf{H} and \mathbf{Q}^* .

Since $\text{tr}(\mathbf{Q}^*) < P_{tot}$, there exists a positive semi-definite Hermitian matrix $\mathbf{A} \succcurlyeq 0$ with $\mathbf{A} = \mathbf{A}^H$, such that $\mathbf{Q}^* + \mathbf{A} = \mathbf{Q}$ and $\text{tr}(\mathbf{Q}) = P_{tot}$. Following Weyl's theorem [13], we have $\phi_i \leq \psi_i$ for each $i = 1, \dots, \min(n, m)$ where ψ_i are the

eigenvalues of \mathbf{Q} . So that:

$$\begin{aligned} R &= \log \det(\mathbf{I}_m + \mathbf{H} \mathbf{Q} \mathbf{H}^H) = \log \prod_{i=1}^{\min(n,m)} (1 + \lambda_i^2 \psi_i) \\ &\geq \log \prod_{i=1}^{\min(n,m)} (1 + \lambda_i^2 \phi_i) = R^*. \end{aligned} \quad (4)$$

This contradicts the assumption that when the transmit strategy uses full transmit power, the maximal transmission rate is not achievable. It follows that there always exists an optimal transmit strategy using full power. \square

Accordingly, it is sufficient for the optimization to consider only transmit strategy which allocate full power P_{tot} , i.e., the sum power constraint is always active. However, it may happen that the optimal transmit powers of the per-antenna unconstrained optimal solution using full transmit power may exceed the maximum allowed per-antenna powers. Therefore, we can distinguish between the two following cases: (i) All per-antenna power constraints are satisfied; (ii) There exists at least one power exceeds the maximum allowed per-antenna power. In the next lemma we study the properties of those two cases in more detail.

Lemma 1. Let $\mathcal{A}' \subseteq \mathcal{A} := \{1, \dots, n\}$, $\mathcal{S}(\mathcal{A}') := \{\mathbf{Q} \succcurlyeq 0 : \text{tr}(\mathbf{Q}) \leq P_{tot}, P_j^{S(\mathcal{A}')} = \mathbf{e}_j^T \mathbf{Q} \mathbf{e}_j \leq \hat{P}_j, j \in \mathcal{A}'\}$, and $\mathcal{P} := \{i \in \mathcal{A}^c : P_i^{S(\mathcal{A}')} > \hat{P}_i\}$ with $\mathcal{A}^c = \mathcal{A} \setminus \mathcal{A}'$. Then, for OP- \mathcal{A} , the optimal power can be allocated as

$$\begin{cases} P_i^* = P_i^{S(\mathcal{A}')} & \forall i \in \mathcal{A}^c \quad \text{if } \mathcal{P} = \emptyset, \\ P_i^* = \hat{P}_i & \forall i \in \mathcal{P} \quad \text{otherwise,} \end{cases} \quad (5)$$

with $P_i^* = \mathbf{e}_i^T \mathbf{Q}^* \mathbf{e}_i$.

Proof. The proof of this lemma can be divided into two parts.

First, we show that if $\mathcal{P} = \emptyset$ the $P_i^* = P_i^{S(\mathcal{A}')} & \forall i \in \mathcal{A}^c$. Since $\mathcal{S}(\mathcal{A}) \subseteq \mathcal{S}(\mathcal{A}')$, $\max_{\mathbf{Q} \in \mathcal{S}(\mathcal{A})} f(\mathbf{Q}) \leq \max_{\mathbf{Q} \in \mathcal{S}(\mathcal{A}')} f(\mathbf{Q})$. If $\mathbf{Q}^{S(\mathcal{A}')} \in \mathcal{S}(\mathcal{A})$ then $\mathbf{Q}^{S(\mathcal{A}')}$ is also the optimal transmit strategy for OP- \mathcal{A} , i.e. $\mathbf{Q}^* = \mathbf{Q}^{S(\mathcal{A}')}$ or $P_i^* = P_i^{S(\mathcal{A}')} & \forall i \in \mathcal{A}^c$.

Next, we prove that if $\mathcal{P} \neq \emptyset$ then $P_i^* = \hat{P}_i, \forall i \in \mathcal{P}$. This part can be proved by applying [10, Lemma 2] directly to an optimization function $f(\mathbf{Q})$ given in (2). For a given $\mathcal{B} = \mathcal{A}' \cup \mathcal{P}$ with $\mathcal{A}' \subseteq \mathcal{A}$, if $\mathcal{P} \neq \emptyset$ then $P_i^{S(\mathcal{B})} = \hat{P}_i, \forall i \in \mathcal{P}$. Thus, by simply setting $\mathcal{A}' = \mathcal{A} \setminus \mathcal{P}$ we have $P_i^* = \hat{P}_i, \forall i \in \mathcal{P}$. \square

Note that, with Lemma 1, if $\mathcal{A}' = \emptyset$, then for $\mathcal{P} = \emptyset$, P_i^* is also the optimal power allocation of the optimization problem with sum power constraint only.

Lemma 1 leads to an iterative algorithm to compute the optimal transmit strategy \mathbf{Q}^* in closed-form. To see this we

consider the following sequence of optimization problems:

$$\begin{aligned}
\max_{\mathbf{Q} \in \mathcal{S}(\emptyset)} f(\mathbf{Q}) &= \max_{\mathbf{Q} \in \mathcal{S}(\emptyset) \cap \{[\mathbf{Q}]_{ii} \leq \hat{P}_i : \forall i \in \mathcal{P}(1)\}} f(\mathbf{Q}) \\
&\geq \max_{\mathbf{Q} \in \mathcal{S}(\emptyset) \cap \{[\mathbf{Q}]_{ii} \leq \hat{P}_i : \forall i \in \mathcal{P}(2)\}} f(\mathbf{Q}) \\
&\dots \\
&\geq \max_{\mathbf{Q} \in \mathcal{S}(\emptyset) \cap \{[\mathbf{Q}]_{ii} \leq \hat{P}_i : \forall i \in \mathcal{P}(K-1)\}} f(\mathbf{Q}) \\
&\geq \max_{\mathbf{Q} \in \mathcal{S}(\emptyset) \cap \{[\mathbf{Q}]_{ii} \leq \hat{P}_i : \forall i \in \mathcal{P}(K)\}} f(\mathbf{Q}) \\
&= \max_{\mathbf{Q} \in \mathcal{S}(\mathcal{A})} f(\mathbf{Q}), \tag{6}
\end{aligned}$$

where $\mathcal{P}(k)$ is the set of indices of powers which violate the per-antenna power constraints in the k -th iteration with initialization $\mathcal{P}(1) = \emptyset$. The update of $\mathcal{P}(k)$ in each iteration is done using Lemma 1 and can be found in Section 4. The optimization problem in each iteration can be solved using the closed-form solution in [12] because every iteration in (6) can be related to an optimization problem with a total power constraint and a limited number of per antenna power constraints.

4. ITERATIVE ALGORITHM

We now show in detail how to implement the iterative algorithm. If we assume that the iterative algorithm to find the optimal solution for OP-A has K iterations in total, then at k -th iteration, $k = 1, \dots, K$, the set of violated power can be calculated as $\mathcal{P}(k+1) = \mathcal{P}(k) \cup \{i \in \mathcal{P}^c(k) : \mathbf{e}_i^T \mathbf{Q}^*(k) \mathbf{e}_i > \hat{P}_i\}$ with $\mathcal{P}^c(k) = \mathcal{A} \setminus \mathcal{P}(k)$. Note that, $\mathcal{P}(1) = \emptyset$ and if we denote $\mathcal{P}^* := \{i \in \{1, \dots, n\} : P_i^* = \mathbf{e}_i^T \mathbf{Q}^* \mathbf{e}_i > \hat{P}_i\}$ as a set of all violated power of OP-A, then $\mathcal{P}^* = \bigcup_{k=1}^K \mathcal{P}(k)$. Following this formulation, it is clear to obtain that if we consider an arbitrary index i , if $i \in \mathcal{P}(k)$ then $i \in \mathcal{P}(k+1)$.

Remark 1. *The maximum number of violated per-antenna power constraints is $n - 1$, which also corresponds to the maximal number of iterations, i.e., $K \leq n - 1$.*

Therefore, we can, without loss of generality, re-assign the antenna coefficient order corresponding to the number of iteration such that the first $|\mathcal{P}(k)|$ coefficient are in $\mathcal{P}(k)$, i.e., $\mathcal{P}(k) = \{1, \dots, |\mathcal{P}(k)|\}$. Therewith, the optimal transmit strategy at the k -iteration, we can be written as

$$\mathbf{Q}^*(k) = \begin{bmatrix} \mathbf{Q}_P(k) & \mathbf{q}^H(k) \\ \mathbf{q}(k) & \mathbf{Q}_S(k) \end{bmatrix}, \tag{7}$$

with $\mathbf{Q}_P(k)$ is a $|\mathcal{P}(k)| \times |\mathcal{P}(k)|$ matrix which contains $P_i^* = \hat{P}_i, \forall i \in \mathcal{P}(k)$, and $\mathbf{Q}_S(k)$ is a $(n - |\mathcal{P}(k)|) \times (n - |\mathcal{P}(k)|)$ matrix which satisfies the condition that $\text{tr}(\mathbf{Q}_S(k)) = P_{tot}(k)$ where $P_{tot}(k) = P_{tot} - \sum_{i \in \mathcal{P}(k)} \hat{P}_i$. This implies that the diagonal elements of $\mathbf{Q}^*(k)$ can be formed as $P_i^*(k) = \mathbf{e}_i^T \mathbf{Q}_P(k) \mathbf{e}_i = \hat{P}_i, \forall i \in \mathcal{P}(k)$ and $P_j(k) = \mathbf{e}_j^T \mathbf{Q}_S(k) \mathbf{e}_j, \forall j \in \mathcal{P}^c(k)$. To find the remaining optimal power allocation

$P_j(k), \forall j \in \mathcal{P}^c(k)$ in $\mathbf{Q}_S(k)$, we consider following reduced optimization problem

$$\max_{\mathbf{Q}(k)} f(\mathbf{Q}(k)) \text{ s.t. } \mathbf{Q}(k) \in \mathcal{S}(\mathcal{P}(k)) \tag{8}$$

where $f(\mathbf{Q}(k)) = \log \det(\mathbf{I}_m + \mathbf{H}\mathbf{Q}(k)\mathbf{H}^H)$, $\mathcal{S}(\mathcal{P}(k)) = \{\mathbf{Q}(k) \succeq 0 : P_i(k) = \hat{P}_i, \forall i \in \mathcal{P}(k), \sum_{j \in \mathcal{P}^c(k)} P_j(k) \leq P_{tot}(k)\}$ with $P_i(k) = \mathbf{e}_i^T \mathbf{Q}(k) \mathbf{e}_i$ is the transmit power of i -th antenna at k -th iteration.

The Lagrangian of (8) is given as follows

$$\begin{aligned}
\mathcal{L}(\mathbf{Q}(k), \mathbf{D}, \mathbf{M}) &= f(\mathbf{Q}(k)) + \sum_{j \in \mathcal{P}^c(k)} [\mathbf{D}]_{j,j} P_{tot}(k) \\
&\quad - \sum_{i \in \mathcal{A}} \{\text{tr}(\mathbf{D}\mathbf{Q}(k))\} + \text{tr}(\mathbf{M}\mathbf{Q}(k)). \tag{9}
\end{aligned}$$

Following [12], the optimal transmit strategy $\mathbf{Q}(k)$ of the optimization problem (8) can be calculated in closed-form. For simplicity, we here assume that the number of nonzero singular values of \mathbf{H} is $L = \min(n, m)$. The detail about the more general case can be found in [12].

Lemma 2 ([12]). *The optimal solution of the transmit strategy in (8) denoted by $\mathbf{Q}^*(k)$ is given by*

$$\mathbf{Q}^*(k) = (\mathbf{D}^{-\frac{1}{2}} [\mathbf{U}]_{:,1:L} [\mathbf{U}]_{:,1:L}^H \mathbf{D}^{-\frac{1}{2}} - [\mathbf{U}]_{:,1:L} \mathbf{\Lambda}^{-1} [\mathbf{U}]_{:,1:L}^H)^+, \tag{10}$$

where diagonal matrix $\mathbf{\Lambda}$ and the first L columns of a unitary matrix $[\mathbf{U}]_{:,1:L}$ are obtained from eigenvalue decomposition $\mathbf{H}^H \mathbf{H} = \mathbf{U} \begin{bmatrix} \mathbf{\Lambda} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{U}^H$. The diagonal elements of $L \times L$ diagonal matrix $\mathbf{\Lambda}$ are positive real values in decreasing order.

The operation “+” is to guarantee that the solution is positive-semi definite, i.e., the negative eigenvalues of $\mathbf{Q}^*(k)$ are forced to be zero. At high SNR, the elements of the diagonal \mathbf{D} can be computed as

$$[\mathbf{D}]_{i,i} = \frac{[[\mathbf{U}]_{:,1:L} [\mathbf{U}]_{:,1:L}^H]_{i,i}}{\hat{P}_i + [[\mathbf{U}]_{:,1:L} \mathbf{\Lambda}^{-1} [\mathbf{U}]_{:,1:L}^H]_{i,i}} \text{ if } i \in \mathcal{P}(k) \tag{11}$$

and

$$[\mathbf{D}]_{j,j} = \frac{\sum_j [[\mathbf{U}]_{:,1:L} [\mathbf{U}]_{:,1:L}^H]_{j,j}}{P_{tot}(k) + \sum_j [[\mathbf{U}]_{:,1:L} \mathbf{\Lambda}^{-1} [\mathbf{U}]_{:,1:L}^H]_{j,j}}, \tag{12}$$

if $j \in \mathcal{P}^c(k)$.

From the result in Lemma 2, the value of $\mathbf{Q}_S(k)$ and the optimal transmit power on j -th antennas at k -th iteration, $P_j(k)$ can be obtained as $P_j(k) = [\mathbf{Q}^*(k)]_{j,j}$ or $P_j(k) = \mathbf{e}_j^T \mathbf{Q}_S(k) \mathbf{e}_j, \forall j \in \mathcal{P}^c(k)$ respectively. However, it occurs that $P_j(k)$ may violate the per-antenna power constraint $\hat{P}_j(k)$ for some $j \in \mathcal{P}^c(k)$. According to the Lemma 1, that

power has to be set equal to the per-antenna power constraint. Consequently, for the OP- \mathcal{A} solution, a new iteration has to be performed with a new re-assigned covariance matrix which contains a $\mathbf{Q}_P(k+1)$ with diagonal elements contains powers that set equal to per-antenna power constraints and $\mathbf{Q}_S(k+1)$ with trace equal new reduced total power $P_{tot}(k+1)$. The power allocations $\mathbf{P}^*(k) = [P_i^*(k), i \in \mathcal{A}]$, $\mathbf{P}_P(k) = [\hat{P}_i, i \in \mathcal{P}(k)]$, $\mathbf{P}_S(k) = [P_j(k), j \in \mathcal{P}^c(k)]$ which are corresponding to $\mathbf{Q}^*(k)$, $\mathbf{Q}_P(k)$ and $\mathbf{Q}_S(k)$ therefore are updated as follows:

$$\dots \mathcal{P}(k) \xrightarrow{(a)} \mathbf{P}_P(k) \xrightarrow{(b)} \mathbf{P}(k) \rightarrow \mathbf{P}_S(k) \xrightarrow{(c)} \mathcal{P}(k+1) \dots$$

where (a) follows Lemma 1, (b) follows Lemma 2 and (c) follows the limitation of per-antenna power on the antennas. This updated sequence stops when there is no per-antenna power constraint violated, i.e., $\mathcal{P}(k+1) = \mathcal{P}(k)$. The optimal transmit strategy of the optimization problem with joint sum and per-antenna power constraints then can be determined as $\mathbf{Q}^* = \mathbf{Q}^*(K)$. From the discussion above, we can summarize on the iterative algorithm to compute the optimal solution of OP- \mathcal{A} in Algorithm 1.

As a basic result of the algorithm, one can verify that $f(\mathbf{Q}^*(1)) \geq \dots \geq f(\mathbf{Q}^*(k)) \geq \dots \geq f(\mathbf{Q}^*(K))$. Since the reduce optimization problem is convex at every iteration, every limit point of the sequence $\{\mathbf{Q}^*(k)\}$ generated by Algorithm 1 is a KKT point of the optimization problem (3), i.e., the global convergence of the proposed iteration method is guaranteed after at most $n-1$ iterations [14].

Remark 2. (10) is a generalized water-filling solution. In particular, if $\mathcal{P}^* = \emptyset$, \mathbf{D} is the proportional to an identity matrix, i.e., $\mathbf{D} = \mu \mathbf{I}$, and (10) reduces to the standard water-filling solution of the optimization problem with sum power constraint only [2].

5. NUMERICAL EXAMPLE

For numerical example, we provide the transmission rate of the optimization problem OP- \mathcal{A} with a 3×3 complex channel $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]$ with

$$\begin{aligned} \mathbf{h}_1 &= [1.1356e^{0.9653j}, 0.9284e^{0.4658j}, 0.9553e^{-0.4193j}]^T, \\ \mathbf{h}_2 &= [0.9640e^{-0.9996j}, 1.2905e^{-0.9527j}, 1.0384e^{-0.4533j}]^T, \\ \mathbf{h}_3 &= [0.6110e^{-0.9156j}, 1.0559e^{-12103j}, 0.7126e^{-0.3535j}]^T. \end{aligned}$$

Fig. 1 shows the choices of constraints result of the optimization problem OP- \mathcal{A} with $P_{tot} = 25$. We perform the example by adjusting per-antenna power constraint on antenna 1 from 0 to 14 and setting per-antenna power constraint configurations on antennas 2 and 3 as follows: (i) Per-antenna power constraints on antennas 2 and 3 are active; (ii) Only per-antenna power constraint on antenna 3 is active; (iii) Per-antenna power constraints on antennas 2 and 3 are not active. Setting $P_{tot} = \hat{P}_i$ in the numerical experiments illustrated in Fig. 1 denotes the case without a power constraint for the i -th antenna. The plot of the sum power constraint only solution, which corresponds to case when all per-antenna power constraints are never active, is also shown in the figure. We can

Algorithm 1: OA($\mathbf{H}, P_{tot}, \hat{\mathbf{P}}$): Iterative algorithm with $P_{tot} \leq \sum_{i=1}^n \hat{P}_i$, $\mathcal{A} := \{1, \dots, n\}$.

Output: \mathbf{Q}^*

- 1 Initialize $k = 1$, $\mathcal{P}(1) = \emptyset$
 - 2 Compute $\mathbf{Q}^*(k)$ using Lemma 2
 - 3 $\mathcal{P}(k+1) = \mathcal{P}(k) \cup \{i \in \mathcal{P}^c(k) : P_i(k) > \hat{P}_i\}$,
 - 4 **if** $\mathcal{P}(k+1) = \mathcal{P}(k)$ **then**
 - 5 $\mathbf{Q}^* \leftarrow \mathbf{Q}^*(k)$,
 - 6 **Break.**
 - 7 **else**
 - 8 **for** $i \in \mathcal{P}(k+1)$ **do**
 - 9 $k \leftarrow k+1$,
 - 10 **end for**
 - 11 $P_i(k) \leftarrow \hat{P}_i$ according to Lemma 1,
 - 12 $P_{tot}(k) \leftarrow P_{tot} - \sum_{i \in \mathcal{P}(k)} \hat{P}_i$,
 - 13 **end if**
 - 14 **Return to 2.**
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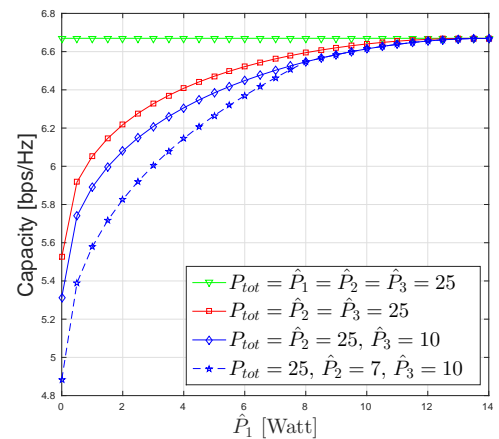


Fig. 1: Capacity under different power constraint settings

see from the figure that the more restricted per-antenna power constraint, the less optimal transmission rate. This happens because of the fact we have less freedom to allocate the optimal transmit power when the optimization domain is limited by adding more per-antenna power constraints.

6. CONCLUSIONS

In this paper, we present an iterative algorithm to find the optimal transmit strategy in closed-form for a MIMO channel with joint sum and per-antenna power constraints using generalized water-filling solution. The algorithm exploits the fact that if an unconstrained optimal power allocation of an antenna exceeds a per-antenna power constraint, then it is optimal to allocate the maximal power in the constraint optimal transmit strategy including the per-antenna power constraints which then enables us to use closed-form solution from [12] in an iterative algorithm to compute the optimal transmit strategy satisfying both sum and per-antenna power constraints.

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