Analytical Mechanics

Michael A Raadu

Space and Plasma Physics, KTH

Brief Overview

- Generalised coordinates
- Newton’s second law
- Lagrange equations
- Generalised momenta; Legendre transformation
- Hamilton formulation
- Particle motion including E & B forces

Generalised coordinates

Introduce a set of general (space) coordinates (for one particle):

$$\{q_1, q_2, q_3\}$$

Write particle position vector in terms of these and time, *t*:

$$\mathbf{r} = \mathbf{r}(q_1, q_2, q_3, t)$$

The particle velocity is then given by:

$$\mathbf{v} = \dot{\mathbf{r}} = \sum_j \frac{\partial \mathbf{r}}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}}{\partial t}$$

NB treat time derivative of the q's as independent variables e.g.:

$$\mathbf{v} = \mathbf{v}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t)$$

From the equation for the particle velocity note cancelation of dots!

$$\frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}}{\partial q_j}$$

Generalised Force

Generalised force components Q are defined by considering the (virtual) work dW done by the force $\mathbf{F}$ for a spatial displacement:

$$dW = \mathbf{F}.\delta\mathbf{r} = \sum_j \mathbf{F}.\frac{\partial \mathbf{r}}{\partial \mathbf{q}_j}\delta \mathbf{q}_j \equiv \sum_j Q_j \delta \mathbf{q}_j$$

So that.

$$Q_j \equiv \mathbf{F}.\frac{\partial \mathbf{r}}{\partial \mathbf{q}_j}$$

NB If the force is given by a potential V it then follows that:

$$Q_j \equiv -\nabla V.\frac{\partial \mathbf{r}}{\partial \mathbf{q}_j} = -\sum_\alpha \frac{\partial V}{\partial r_\alpha}\frac{\partial r_\alpha}{\partial \mathbf{q}_j} = -\frac{\partial V}{\partial \mathbf{q}_j}$$

Newton's Second Law

Write Newton's second law:

$$m\frac{d^2\mathbf{r}}{dt^2} \equiv m\ddot{\mathbf{r}} = \mathbf{F}$$

in terms of general forces :

$$m\ddot{\mathbf{r}}.\frac{\partial \mathbf{r}}{\partial q_j} = \mathbf{F}.\frac{\partial \mathbf{r}}{\partial q_j} = Q_j$$

Rewrite the left hand as follows (using 'un-cancellation' of dots):

$$m\ddot{\mathbf{r}}.\frac{\partial \mathbf{r}}{\partial q_j} = \frac{d}{dt}\left\{m\dot{\mathbf{r}}.\frac{\partial \mathbf{r}}{\partial q_j}\right\} - m\dot{\mathbf{r}}.\frac{\partial \dot{\mathbf{r}}}{\partial q_j} = \frac{d}{dt}\left\{m\dot{\mathbf{r}}.\frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}_j}\right\} - m\dot{\mathbf{r}}.\frac{\partial \dot{\mathbf{r}}}{\partial q_j}$$

$$= \frac{d}{dt}\left\{\frac{1}{2}m\frac{\partial \left|\dot{\mathbf{r}}\right|^2}{\partial \dot{q}_j}\right\} - \frac{1}{2}m\frac{\partial \left|\dot{\mathbf{r}}\right|^2}{\partial q_j}$$

Lagrange's Equation

Using T the particle kinetic energy

$$T \equiv \frac{1}{2} m \left| \mathbf{v} \right|^2 = \frac{1}{2} m \left| \dot{\mathbf{r}} \right|^2$$

Newton's second law becomes.

$$m\ddot{\mathbf{r}}.\frac{\partial \mathbf{r}}{\partial q_j} = \frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_j}\right\} - \frac{\partial T}{\partial q_j} = Q_j \left\{ = -\frac{\partial V}{\partial q_j} \right\}$$

For forces given by a potential V Lagrange's equation follows:

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_j}\right\} - \frac{\partial L}{\partial q_j} = 0$$ with Lagrangian L : $$L = T - V$$

Lagrangian

The Lagrangian for a particle mass m in a potential V is given by

$$L = T - V = \frac{1}{2} m \left| \mathbf{v} \right|^2 - V\left(\mathbf{r}\right)$$

Writing $\mathbf{r}$ and $\mathbf{v}$ in terms of q's and q-dot's gives the general Lagrangian:

$$L = L\left(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t\right)$$

The equations of motion are given (for j=1,2,3) by Lagrange's equations

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_j}\right\} - \frac{\partial L}{\partial q_j} = 0$$

Generalised Momenta

In the Lagrange equation generalised momenta *p* can be defined by:

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$$; giving for Cartesian coordinates: $$\mathbf{p} = m\mathbf{v}$$

Then from the Lagrange equation:

$$\dot{p}_j \equiv \frac{dp_j}{dt} = \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_j}\right\} = \frac{\partial L}{\partial q_j}$$

Legendre Transformation

To transform the Lagrangian look at its differential:

$$dL = \sum_j \left[\frac{\partial L}{\partial q_j} dq_j + \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right] + \frac{\partial L}{\partial t} dt$$

Using the definition of the generalised momenta p this becomes:

$$dL = \sum_j \left[\frac{\partial L}{\partial q_j} dq_j + p_j d\dot{q}_j \right] + \frac{\partial L}{\partial t} dt$$

The Legendre transformation uses the *p*'s and *q*'s as independent variables with a Hamiltonian function derived from the Lagrangian.

$$H \equiv \sum_j p_j \dot{q}_j - L$$

where $$H = H(p_1, p_2, p_3, q_1, q_2, q_3, t)$$

Hamiltonian

The Hamiltonian function is:

$$H \equiv \sum_j p_j \dot{q}_j - L \qquad = \frac{1}{2} m \left| \mathbf{v} \right|^2 + V\left(\mathbf{r}\right) \quad \text{for Cartesians}$$

The differential of the Hamiltonian *H* is:

$$dH = \sum_j \left[\dot{q}_j dp_j + p_j d\dot{q}_j \right] - dL$$

$$= \sum_j \left[\dot{q}_j dp_j - \frac{\partial L}{\partial q_j} dq_j \right] - \frac{\partial L}{\partial t} dt$$

Using Lagrange's equation with generalised momenta:

$$dH = \sum_j \left[\dot{q}_j dp_j - \dot{p}_j dq_j \right] - \frac{\partial L}{\partial t} dt$$

Hamilton's Equations

From the differential of the Hamiltonian,

$$dH = \sum_j \left[\dot{q}_j dp_j - \dot{p}_j dq_j \right] - \frac{\partial L}{\partial t} dt$$

It follows that the Hamiltonian is a function of the *p*'s,*q*'s and time.

Comparing with the equation (from the chain rule):

$$dH = \sum_j \left[\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right] + \frac{\partial H}{\partial t} dt$$

We find Hamilton's equations:

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

and note that: $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

Electro Magnetic Forces

Now look for a Lagrangian including electrical and magnetic forces.

Use vector potential $\mathbf{A}$, electrostatic potential ϕ and Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial \nabla \times \mathbf{A}}{\partial t} \Rightarrow \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

Write magnetic force on a particle with charge q as:

$$\mathrm{q}\mathbf{v} \times \mathbf{B} = \mathrm{q}\mathbf{v} \times \left(\nabla \times \mathbf{A}\right) = \mathrm{q}\left\{\nabla\left(\mathbf{v}.\mathbf{A}\right) - \mathbf{v}.\nabla \mathbf{A}\right\}$$

Equation of motion for a particle with Cartesian coordinates is then:

$$m\dot{\mathbf{v}} = \mathrm{q}\left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right] = -\mathrm{q}\nabla\left\{\phi \text{-} \mathbf{v}.\mathbf{A}\right\} - q\left[\frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}.\nabla \mathbf{A}\right]$$

The last two terms give the total time derivative following the particle

$$\left[\frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}.\nabla \mathbf{A}\right] = \frac{d\mathbf{A}}{dt} \equiv \dot{\mathbf{A}}$$

Lagrangian with Electromagnetic terms

The equation of motion is now in the form

$$m\dot{\mathbf{v}} + \mathrm{q}\dot{\mathbf{A}} = -\mathrm{q}\nabla\left\{\phi\text{-}\mathbf{v}.\mathbf{A}\right\}$$

The corresponding Lagrange equations (in Cartesian form) are:

$$L \equiv \frac{1}{2}m\left|\mathbf{v}\right|^2 + \mathrm{q}\mathbf{v}.\mathbf{A} - \mathrm{q}\phi$$

The equations of motion are then given by Lagrange's equations

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \mathbf{v}}\right\} - \frac{\partial L}{\partial \mathbf{x}} = 0$$

The generalised momentum is now:

$$\mathbf{p} \equiv \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + \mathrm{q}\mathbf{A}$$

The Hamiltonian

The Hamiltonian including electromagnetic forces follows from L:

$$H = \mathbf{p}.\mathbf{v} - L = \left(m\mathbf{v} + \mathrm{q}\mathbf{A}\right).\mathbf{v} - \left\{\frac{1}{2}m\left|\mathbf{v}\right|^2 + \mathrm{q}\mathbf{v}.\mathbf{A} - \mathrm{q}\phi\right\}$$
$$= \frac{1}{2}m\left|\mathbf{v}\right|^2 + \mathrm{q}\phi$$

The Hamiltonian is given by the sum of kinetic and potential energy.

The Hamilton function in terms of momentum and position is

$$H = \frac{\left|\mathbf{p} - \mathrm{q}\mathbf{A}\right|^2}{2m} + \mathrm{q}\phi$$

Properties of the Hamiltonian

Consider the total time derivative of a Hamiltonian

$$\frac{dH}{dt} = \sum_j \left[\frac{\partial H}{\partial p_j} \frac{dp_j}{dt} + \frac{\partial H}{\partial q_j} \frac{dq_j}{dt} \right] + \frac{\partial H}{\partial t}$$

From Hamilton's equations

$$= \sum_j \left[\frac{\partial H}{\partial p_j} \left\{ -\frac{\partial H}{\partial q_j} \right\} + \frac{\partial H}{\partial q_j} \left\{ \frac{\partial H}{\partial p_j} \right\} \right] + \frac{\partial H}{\partial t}$$

so that:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

The Hamiltonian is then a constant following particle motions if the electric and magnetic fields are constant

Properties of the Lagrangian

Consider the action integral of the Lagrangian over a time interval:

$$S \equiv \int_{t_0}^{t_1} L\left(\mathbf{r}, \dot{\mathbf{r}}, t\right) dt$$

Variation of the path followed by the position $\mathbf{r}$ gives

$$\delta S = \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial \dot{\mathbf{r}}} . \delta \dot{\mathbf{r}} + \frac{\partial L}{\partial \mathbf{r}} . \delta \mathbf{r} \right\} dt$$

Introducing the generalised momentum $\boldsymbol{p}$ and integrating by parts,

$$\delta S = \left[\mathbf{p} . \delta \mathbf{r} \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left\{ -\frac{d\mathbf{p}}{dt} + \frac{\partial L}{\partial \mathbf{r}} \right\} . \delta \mathbf{r} dt$$

For fixed end points S has an extreme value (δS=0) if Lagrange's equations are satisfied:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mathbf{r}}} \right] = \frac{d\mathbf{p}}{dt} = \frac{\partial L}{\partial \mathbf{r}}$$