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# Black's Inverse Investment Problem and Forward Criteria with Consumption*

Sigrid Källblad†

**Abstract.** We study an inverse investment problem proposed by Black and provide necessary and sufficient conditions for a given function to be an admissible indirect utility function in a log-normal market; we also show how to recover the associated utility function. Similar questions are also addressed starting directly from the initial investment choice. In parallel we study so-called forward investment-consumption criteria with the dynamic property that their volatility component is identically zero. We provide a fully forward characterization of such criteria and use it to construct forward preferences. We also provide explicit formulas for the associated optimal strategies and characterize the class of criteria which may be decomposed into a pure forward investment criterion and an infinite horizon Merton problem.

**Key words.** optimal investment and consumption, forward criteria, Black's investment problem, inverse investment problems, dynamic consistency, stochastic utility functions, progressive utilities, infinite horizon Merton criteria

**AMS subject classifications.** 91G10, 91B16, 91G80, 35Q93, 35C15, 49N15, 49N45, 49L20, 60G60

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## 1. Introduction.

The Merton problem is arguably the most classical formulation of the question of how to optimally invest in a financial market. The input to the problem is an a priori fixed time horizon and a utility function specifying the investor's preferences. The investor's aim then is to maximize the expected utility of her terminal wealth while trading in a log-normal market; the output is an optimal investment strategy specifying how to do so. The problem is often addressed via the use of dynamic programming arguments yielding a Hamilton–Jacobi–Bellman (HJB) equation characterising the associated value function from which the optimal investment strategy can be deduced. Over the years there has been an extensive interest in this problem and numerous generalizations of it have been studied; by now there is also a fairly complete theory for very general market models. However, while there is a deep and natural axiomatic foundation for the use of expected utility criteria, there is nonetheless consensus that the utility function representing an investor's preferences is notoriously difficult to specify. It is therefore natural to consider the corresponding inverse questions: Given investors' investment choices, can one determine whether they are optimal with respect to some expected utility criterion? Can one then deduce this utility function

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†KTH Royal Institute of Technology, Department of Mathematics, Lindstedtsvägen 25, SE-100 44 Stockholm, Sweden (sigrid.kallblad@math.kth.se).

and thus determine how to continue to invest in a consistent way? Such problems have attracted recent attention and the aim of this article is to contribute to this literature in two complementary ways.

Our first problem of study is an inverse version of the classical Merton problem. Different variants of such inverse problems have been studied in, e.g., Cox, Hobson, and Obłój [8] and He and Huang [15]; see also Monin [26], Angoshtari, Zariphopoulou, and Zhou [1], and Musiela and Zariphopoulou [29]. We here consider a different input to the problem compared to the existing literature. Specifically, we study the following question: Given an increasing and concave function and a log-normal market, is it possible to deduce whether that function is the value function associated with some utility function and some terminal horizon, and can one then infer the latter from the former? Moreover, is it possible to draw the same conclusions starting directly from a given initial investment profile? We refer to this problem as Black's problem for—to the best of our knowledge—it was first proposed in an early note by Black [5]; see also [4], where the problem was solved, however, only for linear utilities.

Our first main result provides an affirmative answer to this question in that we provide necessary and sufficient conditions for a given function to be the indirect utility associated with some general utility function. In addition, we provide an explicit formula for recovering the utility; the investment strategy for the full investment period may then be derived. We also demonstrate how similar conclusions may be drawn directly from the initial investment profile. The conditions we give naturally involve the characteristics of the given tentative value function or investment profile but also depend on the horizon of the problem as well as the market environment via the Sharpe ratio. We also demonstrate how the question of arbitrary horizons affects the problem and discuss the role of complete monotonicity for investments in log-normal markets.

Our second focus is on a specific class of so-called forward investment-consumption criteria. Forward criteria, also referred to as forward performance processes or progressive utilities, were introduced by Musiela and Zariphopoulou [27]; see also Henderson and Hobson [16] for the closely related notion of horizon-unbiased utilities. We refer to Nadtochiy and Tehranchi [31] for an axiomatic foundation, to El Karoui and Mrad [13, 14] for a study of related SPDEs, to Žitković [38] for the corresponding duality theory, and to [17, 18] for forward criteria under model uncertainty. In a nutshell, forward criteria formalize a notion of consistent preferences. Specifically, within a given market model, a random field $u(x, t)$ taking wealth and time as arguments, is a forward criterion if $u(\cdot, t)$ is increasing and concave a.s. and $u(X_t, t)$, $t \geq 0$, is a supermartingale for any admissible wealth process and a martingale at an optimum. That is, the property of dynamic consistency is taken as the starting point and preferences are denoted feasible if they satisfy this consistency condition. Put differently, since investment choices may be reconsidered at each point in time, the notion may equivalently be understood as a way of successively extending the investment problem while restricting future preferences to those consistent with previous investments. Namely, at any point in time, say $s$, for the upcoming investment period, say $[s, t)$, one searches a criterion such that $\mathrm{E}[u(X_t, t)|\mathcal{F}_s] \leq u(X_s, s)$ for any admissible wealth process with equality for the optimal one. While the investor's terminal horizon is not a priori specified, at any point she chooses to quit the market a posteriori, she has then invested optimally with respect to some time-consistent (stochastic) expected utility criterion. The notion of forward investment-consumption criteria provides the natural extension of forward criteria when also considering utility from consumption; it was introduced by Berrier and Tehranchi [3], where a duality theory and numerous key properties were established. It has recently been further investigated by El Karoui, Hillairet, and Mrad [11], where the authors study related SPDEs and use their results to model long term yield curves; see also El Karoui, Hillairet, and Mrad [12].

Herein, our focus is on a class of forward investment-consumption criteria with the specific property that they are differentiable in time. To understand the role of this specification one should note that a crucial consequence of considering general market models and allowing for stochastic preferences is that there are multiple forward criteria consistent with a given initial condition. Additional requirements need thus be imposed in order to restrict this choice and uniquely specify the preferences. To this end, one may put restrictions on the filtration and consider criteria which are adapted to certain stochastic factors (see Nadtochiy and Tehranchi [31]), or consider criteria of a specific structural form (see Liang and Zariphopoulou [24] and Nadtochiy and Zariphopoulou [32]). Yet an interesting class of preferences results from restricting ourselves to criteria which are differentiable in time; for the pure investment case such criteria were first studied in Berrier, Rogers, and Tehranchi [2] and Musiela and Zariphopoulou [28]. Here, we consider differentiable in time forward investment-consumption criteria; such criteria were first introduced in [3] and similarly to the pure investment case, the differentiability assumption turns out to define an interesting class of preferences. The presence of consumption, however, significantly changes certain key aspects of the theory and introduces new additional features.

We first derive a stochastic partial differential equation (SPDE) which specifies the structure of general forward investment and consumption criteria; in particular, it provides motivation for the specific class of criteria which are differentiable in time. We then complement the findings in [3] in that we provide a new representation of the differentiable criteria. By use of this representation we also provide two different constructions of forward investment-consumption criteria. In addition, we obtain explicit formulas for the optimal investment and consumption strategies. We also study the relation between forward criteria and infinite horizon Merton criteria and characterize the class of forward preferences that can be decomposed into infinite horizon Merton problems and pure forward investment criteria.

We study in this article Black's inverse problem and differentiable in time forward criteria in parallel. Since the problems turn out to be closely related—from a conceptual as well as mathematical perspective—it becomes both illustrative and beneficial to do so. The underlying reason for this connection is that both problems amount to identifying preferences at later times which are consistent with preferences and investment choices specified for previous times; that is, to deduce consistent preferences forwards in time by inverting the operation of maximizing expected utility. The particularly close connection between Black's problem and the specific class of forward criteria we consider relies in addition on the assumption of time differentiability. Indeed, this assumption implies that forward criteria, although set within a general market model, need to feature a specific stochastic structure rendering a behavior resembling that of preferences in log-normal markets. While naturally having distinct features, Black's problem and differentiable in time forward criteria therefore turn out to be connected to similar HJB type equations; in both cases to be solved in their (ill-posed) forward direction. We note that since these equations are closely related to the classical heat equation, our analysis ultimately relies on results by Widder on how to solve the heat equation in its backward (ill-posed) direction or, equivalently, on how to invert the so-called Weierstrass transform.

The remainder of the article is organized as follows: In section 2 we introduce the two investment problems, specify their relation, and formulate our questions of study. In sections 3 and 4 we provide, respectively, the main results for Black's inverse investment problem and the nonvolatile forward investment-consumption criteria. Some auxiliary results and longer derivations are deferred to the appendix.

**2. A tale of two problems: Black's inverse problem and forward criteria.** We here introduce our two main problems of study: the inverse Merton problem proposed by Black, and investment and consumption with respect to forward criteria with zero volatility; we also specify the close relation between the two problems and outline the key questions addressed in the remainder of the article.

**2.1. Black's inverse investment problem.** Our first problem of study is an inverse version of Merton's classical investment problem. To this end, we first recall Merton's problem. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space supporting a Brownian motion $(W_t)_{t\geq 0}$ and suppose that the filtration is the augmented and completed filtration generated by this Brownian motion itself. Let the market consist of a riskless bond with zero interest rate and a risky asset $(S_t)_{t\geq 0}$ with dynamics specified by

$$dS_t = S_t \left(\mu dt + \sigma dW_t\right)$$

for some *constant* coefficients $\mu, \sigma \in \mathbb{R}_+$; for simplicity we restrict ourselves to the one-dimensional case although the results readily generalize to the complete market multiasset case. We denote by $\lambda = \frac{\mu}{\sigma} > 0$ the associated market price of risk. Given a terminal horizon $T > 0$, and initial capital $x > 0$, an investor can then choose to invest in this market by following an investment strategy $(\pi_t)_{0\leq t\leq T}$, which should be a progressively measurable process denoting the amount invested in the risky asset at each time. The associated wealth process in discounted units, $(X_t^\pi)_{0\leq t\leq T}$, is given by

$$X_t^\pi = x + \int_0^t \sigma\pi_s \left(\lambda ds + dW_s\right).$$

We restrict ourselves to strategies such that $X^\pi$ is well defined and remains positive at all times; given capital $x$ at time $t$, we denote the set of such admissible strategies for the remaining time interval by $\mathcal{A}^0_{x,t}$. Meanwhile, the investor's preferences are described by a utility function which is an increasing and strictly concave function $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying the so-called Inada conditions:

$$\lim_{x\to 0} U'(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} U'(x) = 0; \tag{1}$$

we here restrict ourselves to utility functions which are $\mathcal{C}^3$ and use the term *admissible* to refer to a utility function satisfying all these conditions. The classical Merton problem then amounts to maximizing, over admissible strategies, the expected utility of the investor's terminal wealth; the value function $v : \mathbb{R}_+ \times [0, T) \to \mathbb{R}$ associated with this problem is given by

$$v(x, t; T) := \sup_{\pi\in\mathcal{A}^0_{x,t}} \mathrm{E}\left[U(X_T^\pi)\right].$$

It is well known (see, e.g., [22]) that the value function satisfies the equation

$$v_t - \frac{\lambda^2}{2}\frac{v_x^2}{v_{xx}} = 0, \quad (x,t) \in \mathbb{R}_+ \times [0,T), \tag{2}$$

equipped with the terminal condition $v(\cdot,T;T) = U(\cdot)$. Moreover, the optimal strategy for this problem is given in feedback form by $\pi_t^* = \pi(t, X_t^{\pi^*}; T)$ for some function $\pi : \mathbb{R}_+ \times [0,T) \to \mathbb{R}$ which is characterized in terms of $v$.

While there is a solid decision theoretic motivation for the use of expected utility criteria, the utility function is nevertheless an illusive object which investors find notoriously difficult to specify. This motivates the study of inverse investment problems where one take investors' actual investment choices as input to the problem in order to see whether they are consistent with expected utility criteria. Specifically, we are here interested in the following question: Given an (increasing and strictly concave) function, is it possible to determine whether that function is actually the initial value function associated with some Merton problem and can one then infer the associated investment horizon and utility function? Moreover, given an (increasing) function, can one deduce whether that function is actually the optimal initial investment choice for some utility function and can the latter then be deduced from the former? We formalize these questions as follows.

Definition 2.1. *We say that a function $v_0 : \mathbb{R}_+ \to \mathbb{R}$ is a $T$-admissible indirect utility if there exists an admissible utility function such that $v_0(x) = v(x,0;T)$, $x \in \mathbb{R}_+$. Moreover, we say that a function $\pi_0 : \mathbb{R}_+ \to \mathbb{R}$ is a $T$-admissible optimal investment choice if there exists an admissible utility function such that $\pi_0(x) = \pi(x,0;T)$, $x \in \mathbb{R}_+$.*

In section 3, we establish necessary and sufficient conditions for a given function $v_0$ to be $T$-admissible, and for a given function $\pi_0$ to be a $T$-admissible initial investment choice. The conditions notably involve the characteristics of $v_0$ and $\pi_0$, respectively, as well as the horizon $T$ and the market environment via the Sharpe ration $\lambda$. We refer to our problem of study as Black's inverse problem for it was first introduced by Black [4, 5]. The analysis therein is based on the derivation of a PDE for the optimal portfolio function $\pi(x,t)$—it is typically referred to as Black's equation; see also, e.g., [19, 20, 27, 37]. Given an initial investment choice $\pi(\cdot,0)$ (and the full consumption strategy), it is then argued that this equation can be used to characterize and derive the corresponding optimal strategy $\pi(x,t)$ for later times. While the problem in [5] is analytically solved only for the case when $\pi(\cdot,0)$ (and the consumption stream) is a linear function of wealth, we here address the problem for general initial conditions; our approach is notably different.

Various other versions of inverse investment problems have also been previously studied. In [8] and [15] a consistency condition is derived for log-normal markets; specifically, the authors establish Black's equation as a necessary and sufficient condition for optimality. Compared to our problem, their input to the inverse problem is different in that they verify that an investment and consumption strategy, $(\pi(x,t), c(x,t))$ specified for all admissible wealth levels $x$ and time points $t$, is optimal for a Merton problem set with respect to some utility and felicity function, if and only if, it satisfies Black's equation. In [9], the input is rather taken to be the investment strategy along a single trajectory from which the utility function is recovered; see also, e.g., [1] and [26] for related studies.

**2.2. Forward investment-consumption criteria.** Our second problem of study is an investment problem set with respect to a particular class of forward investment-consumption criteria. To this end, we first recall the notion of forward criteria and discuss their general structure and the characteristics of the particular class of criteria with zero volatility. For the forward criteria we consider a more general market model: We let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space supporting a $d$-dimensional Brownian motion $W_t$, $t \geq 0$, and suppose that the filtration is the augmented and completed filtration generated by the Brownian motion itself. The market consists of a riskless bond with zero interest rate and $k$ risky assets $S_t^i$, $t \geq 0$, $i = 1, \dots, k$, with dynamics specified by

$$dS_t^i = S_t^i \left( \mu_t^i dt + \sigma_t^i \cdot dW_t \right),$$

where the coefficients $\mu_t \in \mathbb{R}^k$ and $\sigma_t \in \mathbb{R}^{d\times k}$, $t \geq 0$, are now allowed to be adapted processes; the market is in general incomplete. The vector specifying the market price of risk, $\lambda_t \in \mathbb{R}^d$, is defined by $\lambda_t := (\sigma_t^T)^+ \mu_t$, where $(\sigma_t^T)^+$ denotes the Moore–Penrose pseudoinverse of the matrix $\sigma_t^T$; in particular, it then holds that $\sigma_t^T \lambda_t = \mu_t$.

In this market environment the investor chooses an investment strategy $\pi_t$, $t \geq 0$, and a consumption stream $c_t$, $t \geq 0$. These are progressively measurable processes denoting, respectively, the amount invested and consumed at time $t$. The associated wealth process in discounted units, $X_t^{\pi,c}$, $t \geq 0$, is given by

$$X_t^{\pi,c} = x + \int_0^t \sigma_s \pi_s \cdot (\lambda_s ds + dW_s) - \int_0^t c_s ds, \tag{3}$$

where $x$ denotes the initial wealth. We restrict the choice to investment-consumption pairs $(\pi, c)$ for which the above wealth process is well defined and remains positive at all times; we denote the set of such admissible pairs by $\mathcal{A}$.

Within this market environment, investors may choose to measure their performance using so-called forward criteria. When allowing for consumption, such criteria are characterized by a pair of functions $(u, u^c)$ satisfying certain properties; the following definition was given in [3] and provides the natural extension of classical forward criteria when including consumption.

Definition 2.2. *A pair of adapted functions $(u, u^c)$ defined on $\mathbb{R}_+ \times [0, \infty) \times \Omega$ is a forward investment and consumption criterion if*

(i) *$u(\cdot, t, \omega)$ and $u^c(\cdot, t, \omega)$ are increasing and strictly concave;*

(ii) *for all $0 \leq t \leq T$ and $\mathcal{F}_t$-measurable $\xi$,*

$$u(\xi, t) \geq \mathrm{E}\left[ u\left(X_T^{\pi,c}, T\right) + \int_t^T u^c(c_s, s) ds \middle| \mathcal{F}_t \right], \quad a.s., \tag{4}$$

*for all admissible investment and consumption strategies $\pi$ and $c$ such that $X_t^{\pi,c} = \xi$;*

(iii) *equality holds in (4) for some admissible pair $(\pi^*, c^*)$.*

*We say that a pair $(u, u^c)$ is a local forward criterion*[1] *if it satisfies the above properties with the exception that the supermartingale property in* (ii) *is only required to hold locally.*

[1]Note that our definition of local forward criteria is stronger than, e.g., Definition 2 in [28] (for the pure investment case) in that we still require the true martingale property to hold at an optimum.

For the classical utility maximization problem, the well-known *martingale optimality principle* holds (see [10]); using the notation from section 2.1 (suitably modified to allow for consumption), it states that $v(X_\cdot, \cdot; T) + \int_0^\cdot U^c(c_s, s)ds$ is a supermartingale on $[0, T]$ for every admissible pair $(X, c)$, and a martingale when evaluated at an optimum. The definition of forward criteria relies on an extension of this property to the entire positive timeline. While the martingale optimality principle or, equivalently, dynamic consistency, is a consequence of the problem formulation for the classical problem, in the forward context it is taken as the starting point. In effect, the notion characterizes the set of all time-consistent preferences and describe their evolution over time without prespecifying a future date at which $u(\cdot, t)$ should take a specific (deterministic) form. However, over any finite interval $[0, T)$, the first component of a given criterion $(u, u^c)$ coincides with the value function associated with the classical problem set with respect to the horizon $T$, the (stochastic) terminal utility function $u(\cdot, T, \omega)$, and the felicity function $u^c(\cdot, t, \omega)$. The classical infinite-horizon Merton problem provides a key example of a forward investment-consumption criterion; general forward investment-consumption criteria need, however, not be of this form; see section 4.3 below.

In order to study the structure of forward criteria it is useful to relate them to a certain SPDE. To this end, since our underlying filtration is Brownian, it is natural to suppose that $u(x, t)$ admits the following Itô decomposition:

$$du(x,t) = b(x,t)dt + a(x,t) \cdot dW_t, \quad t \geq 0, \tag{5}$$

for some (wealth-dependent) coefficients $a$ and $b$ which are progressively measurable processes. The key point is that in order for the martingale optimality principle to hold, the processes $a$ and $b$ need to satisfy a certain relation. Indeed, assuming that all involved quantities are regular enough for the Itô–Ventzell formula to be applied, by extending the dynamics of $u(X_\cdot^{\pi,c}, \cdot) + \int_0^\cdot u^c(c_t, t)dt$, optimizing over $(\pi, c)$, and requiring the drift term to vanish at the optimum, a closed form expression for $b(x, t)$ in terms of $a(x, t)$ can be obtained; we perform this derivation in Appendix A. Substituting this expression for $b(x, t)$ in (5), and writing $\hat{u}^c$ for the convex conjugate of $u^c$, one arrives at

$$du(x,t) = \left[ \frac{1}{2} \frac{\left| \lambda_t u_x(x,t) + \sigma_t \sigma_t^+ a_x(x,t) \right|^2}{u_{xx}(x,t)} - \hat{u}^c\big(u_x(x,t), t\big) \right] dt + a(x,t) \cdot dW_t, \quad t \geq 0. \tag{6}$$

If the felicity function $u^c$ is set to zero, this SPDE reduces to the one introduced for the pure investment case in [30] and further studied in [13, 14]; for the case including consumption, see also [11, 12, 17].

We have thus arrived at an SPDE which, under appropriate regularity assumptions, a pair of processes $(u, u^c)$ should satisfy in order to constitute a forward investment-consumption criterion. It is important to note that even for a fixed initial condition $u(x, 0) = u_0(x)$ and given felicity function $u^c(x, t)$, the SPDE (6) need not admit a unique solution. This is due to the flexibility of the volatility structure. Indeed, note that for a Backward SPDE of the form (6) equipped with a terminal condition (which the value function associated with classical expected utility maximization should satisfy; see [25]), a solution is *a pair* of parameter-dependent processes, $(u(x, t), a(x, t))$, which are simultaneously obtained when solving the equation. Crucially, there might, however, exist multiple such pairs yielding the same initial value $u(x,0)$; hence, when solving forwards in time from a given initial condition, since the volatility process $a(x,t)$ is not a priori fixed, the SPDE (6) might admit multiple solutions. Seen from the investment perspective, an investor considering forward criteria is decided to invest for all future times in a manner which is dynamically consistent with her previous actions; the above discussion highlights that there are multiple ways of doing so and that additional assumptions are required in order to pin down a unique criterion. There are alternative ways of imposing such additional assumptions; various examples were given in section 1.

In this article, we restrict our attention to criteria $(u, u^c)$ with the additional characteristics that the volatility component of $u(x,t)$ is identically zero, that is, $a(x,t) \equiv 0$. This nonvolatility assumption crucially implies that the SPDE (6) reduces to the following PDE with random coefficients:

$$u_t - \frac{|\lambda_t|^2}{2}\frac{u_x^2}{u_{xx}} + \hat{u}^c(u_x, t) = 0, \quad a.s., \quad (x,t) \in \mathbb{R}_+ \times [0,\infty). \tag{7}$$

In consequence, a pair of (adapted) random functions $(u, u^c)$ is exepcted to constitute a *nonvolatile forward criterion*[2] if and only if they provide a solution to (7) for almost all $\omega \in \Omega$. This link was formally established under some additional market assumptions in [3], where nonvolatile forward consumption criteria first appeared; see section 4 below. Notably their proof does not go via the SPDE (6) provided herein but uses different arguments. The second aim of this article is to characterize and study nonvolatile forward criteria via a study of the random PDE (7) and thus provide new results and further insights for this class of investment criteria.

**2.3. The relation between the two problems.** We study in this article Black's inverse Merton problem and the notion of nonvolatile forward criteria in parallel for the two problems turn out to be closely related. From a mathematical perspective, this is evident from a comparison of (2) and (7) which shows that nonvolatile forward criteria will almost surely satisfy essentially the same HJB equation as appears for the log-normal problem; both for the inverse problem and in the forward context to be solved in its ill-posed forward direction. This connection relies on the fact that both problems amount to deducing preferences and investment decisions for future times that are consistent with preferences and investment choices specified for earlier times. More pertinently, while there, in general market models, are multiple ways of placing future investments so as to remain dynamically consistent with the past, the assumption of zero volatility implies that the criteria must be of such a form that the evolution of the criterion compensates the changes in the market rendering an optimal behavior resembling that of a log-normal market; this accounts for the close relation between the two problems.

The difference between the two settings is twofold: First, for Black's problem we are looking for conditions in order to solve over finite time intervals, and the results therefore

[2] If $u^c$ is nonnegative, then so is $\hat{u}^c$ and it follows from (7) that the nonvolatility assumption implies $u_t(x,t) < 0$, a.s. This is always the case for nonvolatile pure investment criteria and it is therefore natural to refer to those as *time-monotone*; see [3, 28]. For general investment-consumption criteria, nonvolatility need, however, not imply time monotonicity.

heavily depend on which horizon we consider. For the forward criteria, even if the additional source terms emerging due to consumption notably appear at finite times, we search instead for solutions well defined for all positive times. While this is a rather weak requirement if allowing for general forward criteria, once restricted to nonvolatile ones it is much more restrictive. Second, in Black's context we are searching for deterministic solutions. In the forward context, however, the utility fields are allowed to be random; meanwhile, they should remain adapted which enforces additional restrictions on their structure. While closely related, the two problems thus feature crucially distinct characteristics.

We finally note that already Black [4, 5] put forth the inverse problem not only as a tool for deducing but also for defining suitable preferences and strategies; see also [29]. Specifically, the author wrote, "*instead of stating his preferences among various gambles, the individual can specify ... how much he would invest currently in the market portfolio as a function of his current wealth ... It may well be more convenient ... than to specify his utility function. In addition, it is possible to go directly [from there] to the investment [strategy] without ever deriving the individual's utility function.*" Put differently, once an initial investment profile has been fixed and committed to, given that it is optimal with respect to some utility criterion, the investor should continue to invest in accordance with the associated investment strategy. For some choices, the strategy may even be deduced from Black's equation without ever passing via the utility function. This way, one ends up investing in a dynamically consistent way for *some* expected utility criterion, which, according to the reasoning by Black, is the main priority. Similar ideas appear also in [8] and [15] where the focus is shifted from identifying an investor with a *specific* utility function, to characterizing investment choices consistent with some utility function. In [15], for example, an investment-consumption strategy is denoted *efficient* if and only if it is optimal for some utility and felicity function and the necessary and sufficient conditions for efficiency are placed directly on the pair $(\pi, c)$. Consistency is thus put forth as a desirable property which in its own right motivates the use of a certain strategy regardless of the characteristics of the corresponding utility function. The notion of forward criteria formalizes the same ideas in a general market environment.

**3. Black's inverse investment problem.** In this section we provide our results for Black's inverse investment problem. We first provide some auxiliary definitions and results. Then we present our first main results providing necessary and sufficient conditions for initial preferences and investment choices to be $T$-admissible. Subsequently we address the question of how requiring existence of solutions for arbitrary horizons and the introduction of consumption affects the problem.

**3.1. Auxiliary results on the Weierstrass transform.** Our analysis relies on the fact that for log-normal models, the HJB equation characterizing the value function can be transformed into the heat equation. The question whether a given function is an admissible indirect utility thus turns out to be closely related to the study of heat conduction problems backwards in time. We here provide some definitions and relevant results.

Given $\tau > 0$ and a function $h : \mathbb{R} \to \mathbb{R}_+$ with $k(z,t) = \frac{1}{\sqrt{2\pi t}} \exp\{\frac{-z^2}{2t}\}$, let

$$h(y,t;\tau) := \int_{-\infty}^{\infty} k\big(y-z, \tau - t\big) h(z) dz, \quad t < \tau. \tag{8}$$

It is then well known that $h(y,t;\tau)$ is a solution to the homogeneous heat equation (cf. (15) with terminal condition $h$ at time $t=\tau$); under suitable conditions it is also the unique solution. Consider now the following problem: For a given function $h_0$, does there exist some $h$ and $\tau$ such that $h_0 = h(\cdot,0;\tau)$? Since $h(\cdot,0;\tau)$ is given by a so-called Weierstrass transform (with respect to an absolutely continuous measure with density $h$) this might equivalently be formulated as to whether the Weierstrass transform can be inverted. Under suitable assumptions it is equivalent to solving the heat conduction problem backwards in time. The backward heat equation is, however, well known to be ill-posed in the sense of Hadamard and the problem is therefore highly nontrivial. Widder [34] provides, however, appropriate conditions under which one can indeed invert the Weierstrass transform; his results make use of the following operator.

Definition 3.1. *Given an entire function $f$ with $f(x) \in \mathbb{R}$ for $x \in \mathbb{R}$, let*

$$e^{-t\mathcal{D}^2} f(x) := \int_{-\infty}^{\infty} k(y,t) f(x+iy)dy, \qquad x \in \mathbb{R}, \tag{9}$$

*whenever the integral on the right-hand side converges.*

*Remark* 3.2. Recall that a function is entire if and only if it is analytic in the whole complex plane.[3] Further, that an analytic function on $\mathbb{R}$ can be uniquely extended to an analytic function in the whole complex plane via analytic continuation. Any analytic function on $\mathbb{R}$ can thus be identified with an entire function defined on $\mathbb{C}$ and we use this convention throughout.

We note that whenever well defined, (9) returns a real-valued function which also admits the following alternative representation (see Theorem 2 in [34]):

$$e^{-t\mathcal{D}^2} f(x) = \int_{-\infty}^{\infty} k(y,t) \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!} f^{(2n)}(x)dy, \qquad x \in \mathbb{R}.$$

Widder [34] then proved that the operator $e^{-t\mathcal{D}^2}$ can be used to invert the Weierstrass transform under suitable conditions; we provide in Appendix B a modified version of this result which will be of frequent use in our subsequent analysis.

**3.2. Necessary and sufficient conditions for solutions on finite horizons.** We are now ready to present our first main result providing necessary and sufficient conditions for existence of solutions to the inverse investment problem over finite horizons. To this end, with any given function $v_0 : \mathbb{R}_+ \to \mathbb{R}$, we associate a function $h_0 : \mathbb{R} \to \mathbb{R}_+$ given by

$$h_0(y) := \left(v_0'\right)^{(-1)}\left(e^{-y}\right), \quad y \in \mathbb{R}. \tag{10}$$

For $h_0$ to be well defined, we need naturally to restrict ourselves to functions $v_0$ which are are positive, strictly concave, and $\mathcal{C}^1$; this is however no restriction since any value function

[3] An entire function is a complex-valued function which is holomorphic (complex differentiable in a neighborhood of every point in its domain) over the whole complex plane. This class of functions coincides with the class of functions being analytic on the whole complex plane; that is, functions that are equal to their Taylor series in a neighborhood of each point in the complex plane; see [6].

associated with an admissible utility function will satisfy this. The next result then characterizes the input for which the inverse investment problem admits a solution; the conditions are crucially placed directly on the exogenous input to the inverse problem, that is on $v_0$.

Theorem 3.3. *Let $T > 0$. A given function $v_0 : \mathbb{R}_+ \to \mathbb{R}$ for which $h_0 : \mathbb{R} \to \mathbb{R}_+$ is well defined, is then a $T$-admissible indirect utility if and only if $h_0$ satisfies the following conditions:*[4]

(i) *$h_0(x)$ is analytic;*
(ii) *$h_0(x+iy) = \mathcal{O}(e^{y^2/2\lambda^2 T})$ as $y \to \pm\infty$, uniformly in $-a \le x \le a$, for every $a > 0$;*
(iii) *$e^{-t\mathcal{D}^2} h_0(x) \ge 0$ for $0 < t < \lambda^2 T$, $x \in \mathbb{R}$,*

*and the limit*

$$h(x) := \lim_{t \nearrow \lambda^2 T} e^{-t\mathcal{D}^2} h_0(x), \quad x \in \mathbb{R}, \tag{11}$$

*defines a function in $\mathcal{C}^2$ with*

$$\lim_{x \to -\infty} h(x) = 0 \qquad \text{and} \qquad \lim_{x \to +\infty} h(x) = \infty. \tag{12}$$

*The associated utility function is then unique; up to an additive constant it is specified by $(U')^{-1}(y) \mathrel{\widehat{=}} h(e^{-y})$. Moreover, the associated value function is $\mathcal{C}^{\infty,1}(\mathbb{R}_+ \times [0,T))$.*

*Proof.* Necessary and sufficient conditions for a given function to be represented as a Weierstrass transform were provided in Theorem 3 in [34]. We provide a modified version of this result adjusted for the present purposes in Corollary B.2. Theorem 3.3 then follows by combining Corollary B.2 with the following claim: A function $v_0 : \mathbb{R}_+ \to \mathbb{R}$ is a $T$-admissible indirect utility function if and only if there exists a function $h : \mathbb{R} \to \mathbb{R}_+$ which is $\mathcal{C}^2$, positive, increasing, satisfies (12), and such that

$$h(y, 0; \tau) = h_0(y), \quad y \in \mathbb{R}, \tag{13}$$

where $h(y,t;\tau)$ is defined by (8) with respect to that $h$ and $\tau := \lambda^2 T$; the utility function is then given by $(U')^{(-1)}(y) = h(e^{-y})$, $y \in \mathbb{R}$.

To verify this claim, suppose first that $v_0$ is a $T$-admissible indirect utility function. Then, there exists an admissible utility function $U(x)$, such that the therewith associated value function $v(\cdot,\cdot;T)$ satisfies $v(\cdot,0;T) = v_0(\cdot)$. According to, e.g., [22], since the utility function is admissible, the value function $v : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ also satisfies the following conditions: $v(\cdot,\cdot;T)$ is $\mathcal{C}^{3,1}(\mathbb{R}_+ \times [0,T])$; for $t \in [0,T]$, $v(\cdot,t;T)$ is increasing, strictly concave, and satisfies (1); $v(\cdot,\cdot;T)$ satisfies (2) equipped with the terminal condition $v(\cdot,T;T) = U(\cdot)$. Define now a function $h : \mathbb{R} \times [0,\tau] \to \mathbb{R}_+$ via the transformation

$$v_x\left(h\big(y, \lambda^2 t\big), t; T\right) \mathrel{\widehat{=}} \exp\Big\{ -y + \frac{1}{2}\lambda^2 t \Big\}, \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{14}$$

[4] The first two conditions, namely, that $f(x)$ is analytic and that $f(x+iy) = \mathcal{O}\big(e^{y^2/2\lambda^2 T}\big)$, imply that the integral in (9) converges for any $x \in \mathbb{R}$ and $0 < t < \lambda^2 T$, and thus that $e^{-t\mathcal{D}^2} f(x)$ is well defined for $0 < t < \lambda^2 T$.

It follows that $h$ satisfies the following: It is $\mathcal{C}^{2,1}(\mathbb{R}_+ \times [0,\tau])$; for $t \in [0,\tau]$, $h(\cdot,t)$ is positive, increasing, and satisfies (12); and $h(\cdot,\cdot)$ satisfies the equation

$$h_t + \frac{1}{2}h_{xx} = 0, \quad (x,t) \in \mathbb{R} \times [0,\tau); \tag{15}$$

we also know that $h(y,0) = h_0(y)$. Indeed, the relation between the equations follows by implicit differentiation and the other conditions are immediate. Since (15) equipped with a terminal condition admits a unique nonnegative solution, it follows that $h(y,t) = h(y,t;\tau)$, $(y,t) \in \mathbb{R}_+ \times [0,\tau)$, where $h(y,t;\tau)$ is given by (8) defined with respect to $h(y) \mathrel{\widehat{=}} h(y,\tau)$. In particular, (13) holds; since $h(\cdot,\tau)$ satisfies the required properties we thus obtain the necessity part of the above claim.

Conversely, suppose that for a given $v_0$ there exists a function $h : \mathbb{R} \to \mathbb{R}_+$ satisfying the properties of the claim. Define $h(y,t) := h(y,t;\tau)$, $(y,t) \in \mathbb{R} \times [0,\tau)$, where $h(y,t;\tau)$ is given by (8) with respect to this $h(y)$. Then $h(y,t)$ solves (15) and $h(y,0) = h_0(y)$ by construction. Moreover, $\lim_{t \nearrow \tau} h(y,t) = h(y)$ and, thus, defining $h(x,\tau) := h(y)$, we have that $h \in \mathcal{C}^{2,1}(\mathbb{R}_+ \times [0,\tau])$. In addition, $h(\cdot,t)$ is positive, increasing, and satisfies (12), for $t \in [0,\tau]$. Next, define $v : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ via (14), with respect to this $h(y,t)$, such that $v(x,0) = v_0(x)$; indeed, for a given $h : \mathbb{R} \times [0,\tau] \to \mathbb{R}_+$, the relation (14) determines $v(x,t)$ up to an additive constant. In turn, let $U(x) := v(x,T)$; it is clearly an admissible utility function due to the properties of $v(x,T)$. We may thus apply again the results in [22] to deduce that the associated value function, say $v(\cdot,\cdot;T)$, is $\mathcal{C}^{3,1}(\mathbb{R}_+ \times [0,T])$ and satisfies (2). Now, since (15) admits a unique nonnegative solution, the first part of this proof implies that (2) equipped with a terminal condition in the form of an admissible utility function, admits a unique solution. Hence, $v(\cdot,t) = v(\cdot,t;T)$, $t \le T$. In consequence, $v_0(\cdot) = v(\cdot,0;T)$, and we easily conclude. ■

If a function $v_0$ is $T$-admissible for some $T > 0$, then it is $t$-admissible for any $t \le T$; indeed, the uniqueness of solutions to the inverse problem implies that if a solution exists for the horizon $T$, say $U(\cdot)$, then the solution for any $t \le T$ is given by the corresponding value function $v(\cdot,t;T)$. According to Theorem 3.3, the maximal future horizon with respect to which a given function is admissible, depends both on the shape of the original input $v_0$ and on the market environment via the Sharpe ratio $\lambda$.

Solving the inverse investment problem in a log-normal market effectively amounts to solving the heat equation backwards; a problem well known to be ill-posed in the sense of Hadamard. The restrictive conditions in Theorem 3.3 are thus imposed in order to ensure existence of a solution on the required interval. For, in general, the solution to the backward heat equation might explode even on finite intervals. Specifically, the fact that $e^{-t\mathcal{D}^2}h_0(x) \ge 0$ is the key condition which ensures that we are dealing with a well-defined solution to the backward heat equation; we refer to [34] for further intuition for this condition. Notably, whenever a solution to the backward heat equation exists, it is unique and given in explicit form by (11). However, even though the above conditions are satisfied and both existence and uniqueness hold, the solution need not depend continuously on the input which makes a numerical treatment difficult.

*Remark* 3.4 (the case including consumption). By use of similar arguments, one expects a given function $v_0 : \mathbb{R}_+ \to \mathbb{R}$, to be the initial value function for some investment-consumption

Merton problem with horizon $T > 0$, if there exists a felicity function $U^c(x,t)$, $(x,t) \in \mathbb{R}_+ \times (0,T)$, such that

$$x \mapsto h_0(x) - \int_0^t \int_{\mathbb{R}} k(z-x;s) h^c(z,s) dz ds \tag{16}$$

satisfies conditions (i)–(iii) of Theorem 3.3, for all $t < \lambda^2 T$, where $h_0$ is given by (10) and $h^c$ is associated with $U^c$ as in (24)–(25) below. The utility function would then similarly be obtained from

$$h(x) := \lim_{t \nearrow \lambda^2 T} e^{-t\mathcal{D}^2} \left\{ h_0(x) - \int_0^t \int_{\mathbb{R}} k(z-x;s) h^c(z,s) dz ds \right\}, \quad x \in \mathbb{R}. \tag{17}$$

In particular, whenever $v_0$ is $T$-admissible, it is also $T$-admissible for the problem including consumption; on the contrary, a function which is not $T$-admissible for the pure investment problem, may nevertheless very well be $T$-admissible for the problem including consumption. To formalize these results for the case including consumption one needs however to impose some additional growth conditions to ensure uniqueness of the associated HJB equation.

*Remark* 3.5 (extended investment horizons). Considering the inverse investment problem for the horizon $T$, we note that the utility function $U(x)$ may equivalently be obtained by solving the problem for some horizon $t$, with $t \leq T$, and, in turn, using that solution as input for the horizon $T - t$. Definition 2.1 may therefore equivalently be interpreted as addressing the question as to whether a given investment problem can be *extended* in a consistent way. Indeed, consider an investor with a fixed terminal horizon $T$ and utility function $U$, who starts to invest according to the associated optimal strategy but then decides to extend trading beyond the initial horizon $T$, up to, say, $\bar{T} > T$. Then the question arises: *Can the investment problem, once formulated, be extended until time $\bar{T} > T$ in a way which is consistent with the original problem?* A natural way to formalize this is to say that the investment problem can be extended to $\bar{T}$ if and only if the utility function $U(x)$ is a $(\bar{T} - T)$-admissible indirect utility function in the given market model. Indeed, denoting by $\bar{U}(x)$ the thus uniquely deduced utility function at time $\bar{T}$, we have that $\bar{v}(x,t;\bar{T}) = v(x,t;T)$, $t \leq T$, where $\bar{v}$ and $v$ are the corresponding value functions. In consequence, the investment strategies of the two problems coincide on $t \in [0,T]$ and the investment decisions the investor has made so far are a posteriori optimal also with respect to the extended investment problem. The question of whether an investment problem can be extended is thus implicitly addressed in Theorem 3.3. In particular, whenever possible, there is a *unique* way of extending the problem.

**3.3. Deducing utility functions directly from given investment policies.** The next result establishes how to deduce directly from an initial investment choice $\pi_0(x)$, $x > 0$, whether it is in accordance with some admissible Merton problem, and how to then recover the utility function. The result thus provides the solution to the problem posed in [4] for general initial conditions when excluding consumption. We note that the assumption that $h_0^\pi$ is well defined provides no restriction since this is the case for any $T$-admissible optimal investment choice.

Theorem 3.6. *Let $T > 0$ and consider an initial investment choice $\pi_0(x)$, $x \in \mathbb{R}_+$, for which the function $h_0^\pi : \mathbb{R} \to \mathbb{R}_+$ is well defined as the inverse of the following function:*

$$h_0^{\pi(-1)}(x) := \frac{\lambda}{\sigma} \int_1^x \frac{dz}{\pi_0(z)}, \qquad x > 0. \tag{18}$$

*Then, the initial investment choice $\pi_0$ is a $T$-admissible optimal investment choice if and only if $h_0^\pi$ satisfies the conditions of Theorem 3.3. Moreover, in that case, the associated admissible utility function is unique up to affine transformations and given as in Theorem 3.3.*

*Proof.* We note that the optimal strategy for the Merton problem can be written in the following feedback form:

$$\pi^*(x,t) \;=\; -\frac{\lambda}{\sigma}\frac{v_x(x,t;T)}{v_{xx}(x,t;T)} \;=\; \frac{\lambda}{\sigma} H_y(H^{(-1)}(x,t),t), \quad (x,t) \in \mathbb{R}_+ \times [0,T),$$

where $H^{(-1)}$ denotes the spatial inverse of $H : \mathbb{R} \times [0,T] \to \mathbb{R}_+$, given by $H(y,t) := h(y, \lambda^2 t)$ with $h(y,t)$ defined via (14) with respect to $v(\cdot,\cdot;T)$; see, e.g., [19]. Since $H(y,0) = h(y,0)$, we thus obtain the following relation up to an additive constant:

$$h^{(-1)}(x,0) = \int_{\cdot}^{x} \frac{dz}{h_y(h^{(-1)}(z,0),0)} = \frac{\lambda}{\sigma}\int_{\cdot}^{x} \frac{dz}{\pi^*(z,0)}.$$

Since defining $h(y,t)$ up to an additive constant is equivalent to defining $v(x,t;T)$ up to an affine transformation, we may without loss of generality use the convention $h(0,0) = 1$. The result then follows from Theorem 3.3. ■

The result shows that it is possible to determine whether a given initial investment profile is consistent with an expected utility criterion and, if so, to infer the utility function. Specifically, the result explores the information about risk preferences incorporated in the initial investment choice, and shows that this information is, in fact, sufficient to recover the full investment problem. That is, the knowledge of the investor's optimal investments for all wealth levels at a *single* point in time, is thus enough to reveal her optimal investment strategy at *all* times and wealth levels. The investor's initial investment choice $\pi_0(x)$, $x \geq 0$, is an observable in the sense that one may ask the investor how she would invest if she had access to various amounts of liquidity; via Theorem 3.6 one may then check whether that investment choice is in accordance with some utility function and recover the latter.

*Remark* 3.7. If a function $h_0^\pi$ corresponds to an admissible indirect utility function, it must be analytic in $\mathbb{R}$ according to Theorem 3.3. Since an analytic function is uniquely determined by its values along a curve in its area of analyticity (cf. Remark 3.2) $h_0^\pi$ is therefore uniquely determined by $\pi_0(x)$, $a \leq x \leq b$, for any $0 \leq a < b$. In consequence, taking only $\pi_0(x)$, $a \leq x \leq b$, as input to the inverse investment problem, would with obvious modifications lead to the same solution as provided by Theorem 3.6.

**3.4. Solving for arbitrary horizons.** A key feature of Theorem 3.3 is that it specifies the horizon dependence of the inverse investment problem. We now elaborate further on the particular class of functions for which one may solve for arbitrary horizons. To this end, we define the following set of (nonnegative) Borel measures:

$$\mathcal{B}_0 = \Big\{\mu \text{ is a Borel measure on } \mathbb{R}_+ : \int e^{ry}\mu(dr) < \infty \text{ for all } y \in \mathbb{R}\Big\}. \tag{19}$$

It is then straightforward to see that for $\mu \in \mathcal{B}_0$,

$$h^\mu(y,0) = \int_{-\infty}^{\infty} k(y-z,t)h^\mu(z,t)dz \quad \text{if} \quad h^\mu(y,t) := \int_0^\infty e^{ry-\frac{1}{2}r^2 t}\mu(dr), \quad y \in \mathbb{R},\ t \geq 0. \tag{20}$$

Since the function $h^\mu(\cdot,t)$, $t \geq 0$, is positive, increasing, $\mathcal{C}^\infty$, and satisfies (12), we see from the proof of Theorem 3.3 that $v_0$ defined with respect to $h^\mu(\cdot,0)$ via (10) is $T$-admissible for *any* $T > 0$. The sufficiency of the integral representation (20) for the log-normal problem to be well defined for arbitrary horizons was first demonstrated in [16]. According to a result by Widder ([36, p. 235]; see also [31] for a generalization of this result), this is notably the unique class of functions for which this property holds; hence it is also a necessary condition. The necessity was first established in [2, 28] where it was used to characterize pure investment nonvolatile (i.e., time-monotone) forward criteria. Indeed, for the pure investment case there is no difference between solving Black's inverse investment problem for arbitrary horizons and characterizing time-monotone forward criteria; see section 4.3. The form of the corresponding admissible initial conditions and portfolio choices were further specified in Corollary 3.4 in [2] and Theorem 10 in [29]—the next two results refine those; they are here stated within the framework of Black's inverse problem.

Since the notion of complete monotonicity plays a key role we first recall the following definition.

Definition 3.8 (completely monotonic functions). *A function $f \in \mathcal{C}^\infty$ is absolutely monotonic (AM) in the interval $a < x < b$ if it has nonnegative derivatives of all orders there; that is, if $f^{(n)}(x) \geq 0$, $n = 0,1,2,\ldots, a < x < b$. A function is completely monotonic (CM) in the interval $a < x < b$ if and only if $f(-x)$ is AM in the interval $-b < x < -a$.*

Proposition 3.9. *Given a function $v_0 : \mathbb{R}_+ \to \mathbb{R}$, let $h_0$ be given by (10). Then $v_0$ is a $T$-admissible indirect utility function for any $T > 0$ if and only if $h_0$ is AM in $\mathbb{R}$. Moreover, $I := (U')^{-1}$ is then CM and there exists $\nu \in \mathcal{B}_0$ such that*

$$I(y) \;=\; \int_0^\infty y^{-s}\nu(ds), \quad y \in \mathbb{R}. \tag{21}$$

*Proof.* Recall Widder's theorem (p. 235 in [36]) stating that any positive solution to the backward heat equation being well defined for all times, that is, $h_t + \frac{1}{2}h_{yy} = 0$, $y \in \mathbb{R}, t \geq 0$, must be given by $h(y,t) = \int e^{ry-\frac{1}{2}r^2 t}\mu(dr)$ for some positive Borel measure $\mu$ on $\mathbb{R}$ such that $\int e^{ry}\mu(dr) < \infty$, $y \in \mathbb{R}$. Since $h_y(y,t) = \int e^{ry-\frac{1}{2}r^2 t} r\mu(dr)$ solves the same equation, it follows that any increasing positive solution to this equation must admit this representation for some $\mu$ with positive support, that is, be given by (20) for some $\mu \in \mathcal{B}_0$. By use of the same arguments as used to prove Theorem 3.3, we thus obtain that $v_0$ is a $T$-admissible indirect utility function for any $T \geq 0$ if and only if

$$h_0(y) = \int_0^{\infty} e^{yz} \mu(dz), \qquad y \in \mathbb{R}, \tag{22}$$

for some measure $\mu \in \mathcal{B}_0$; see also [2, 29].

In turn, according to Bernstein's theorem (cf. Corollary C.2 in the appendix), (22) holds if and only if $h_0(y)$ is AM in $\mathbb{R}$. Next, let $I_0(y) := (v_0')^{-1}(y)$; the functions $I_0 : \mathbb{R}_+ \to \mathbb{R}_+$ and $h_0 : \mathbb{R} \to \mathbb{R}_+$ are then related via

$$I_0(y) = h_0\left(-\ln y\right), \quad y > 0.$$

Since $x \mapsto -\ln x$, $x > 0$, is CM, it follows that $I_0(y)$ is the composition of an AM and a CM function and, thus, itself CM according to Theorem IV 2.b in [33]. This completes the proof of the first part. Combined with the uniqueness provided by Theorem 3.3, we see that any specific structural properties satisfied by $I_0$ must also be by $I$; we easily conclude. ■

Using the terminology introduced in Remark 3.5, Proposition 3.9 implies that a given investment problem can be extended to *arbitrary* horizons if and only if the initially chosen utility function defines an AM function via (10); recall from Theorem 3.3 that the way the problem can be extended is unique. The investor using such a utility is therefore not restricted to any prespecified time interval, but might for all times continue to invest in a way which is consistent with her previous actions—a comparison with the finite horizon result in Theorem 3.3 illustrates the strength of the condition that a solution should exist for all horizons. We also note that complete monotonicity, of the form (21), of the inverse marginal utility is a property which is *preserved* by the value function. Indeed, this is an immediate consequence of the uniqueness of the extension for whenever the investment problem set with respect to a utility function $U(\cdot)$ might be extended to arbitrary future horizons, so must the problem using the utility $v(\cdot, t; T)$, $t \leq T$. In consequence, the value function associated with a utility function of this form itself belongs to the same class of utility functions. For more results on what properties of the utility function that are inherited by the value function and how the shape of the utility function affects the investment problem, we refer to [19, 20].

Combining Proposition 3.9 with Theorem 3.6 we also obtain the following result; $\mathcal{L}^{-1}$ here denotes the inverse Laplace operator.

Corollary 3.10. *Given a function $\pi_0 : \mathbb{R}_+ \to \mathbb{R}$, let $h_0^{\pi}$ be defined by (18). Then, there exists for any horizon $T > 0$ an admissible utility function for which $\pi_0$ is the associated optimal investment choice if and only if $h_0^{\pi}$ is AM on $\mathbb{R}$. In particular, $h_0^{\pi}$ then admits the representation $h_0^{\pi}(y) = \int_{\mathbb{R}} e^{yz} d\mu(z)$ for some $\mu \in \mathcal{B}_0$; if, in addition, $\mu$ is of the form $d\mu(s) = \alpha(s)ds$, then*

$$\alpha = \mathcal{L}^{-1}\big\{h_0^{\pi}(-s),\ s > 0\big\}. \tag{23}$$

Formula (23) notably gives the explicit form of $\mu$ in terms of $h_0^{\pi}(y)$, $y < 0$, which is the inverse of $h_0^{\pi(-1)} : (0, 1] \to (-\infty, 0]$, and therefore given by (18) using only the knowledge of $\pi_0(x)$, $0 < x < 1$. This is in accordance with the fact that any two AM functions which coincide on the half-line coincide on the entire real line; see Remarks 3.2 and 3.7.

*Remark* 3.11 (higher order stochastic dominance versus horizon flexibility). According to Proposition 3.9, utility functions for which the investment problem can be extended to arbitrary horizons have the property that the *inverse of the marginal utility* is a CM function. The notion of complete monotonicity has previosuly appeared within the theory of portfolio choice: Brocket and Golden [7] study the class of utility functions for which *the marginal utility itself* is a CM function. The motivation in [7] came from the notion of stochastic dominance; their utilities, namely, have the characteristic property that the associated (expected utility) criterion respects stochastic dominance (of terminal wealth) with respect to any order. Notably, these two classes of utility functions, both closely related to complete monotonicity, are effectively orthogonal. Indeed, functions of the form (21) generally do not have a CM inverse; to obtain a counterexample it suffices to consider $I(y) = y^{-\alpha} + y^{-\beta}$ for a suitable choice of $\alpha, \beta > 0$. Conversely, the inverse of a CM function is in general not CM. That these two classes of utilities have complementary structural features is natural since they are characterized by two entirely different properties: While the former respects stochastic dominance of any order, the latter allows the investment problem to be consistently extended to arbitrary horizons; meanwhile, the latter utilities only respect up to third order stochastic dominance. The classical power utility provides a notable exception belonging to both categories. We refer to [19, 20] for more on higher order risk indices and the role of complete monotonicity.

**4. Nonvolatile forward investment-consumption criteria.** We now turn to the study of the forward investment-consumption criteria with zero volatility. In order to formalize their relation to the characterizing random PDE (7) we first specify the setup and refine the market assumptions. To this end, given the underlying market model of section 2.2, we define the processes $(M_t)$ and $(A_t)$:

$$M_t := \int_0^t \lambda_s \cdot dW_s, \quad t \geq 0, \qquad \text{and} \qquad A_t := \langle M \rangle_t = \int_0^t |\lambda_s|^2 ds, \quad t \geq 0.$$

We will work under the following assumption on the market model.

*Assumption* 4.1. The market price of risk $\lambda_t$, $t \geq 0$, is bounded uniformly in $(t, \omega) \in [0, \infty) \times \Omega$, and the process $A_t$, $t \geq 0$, satisfies the condition $\lim_{t\to\infty} A_t = \infty$, a.s.

As specified in section 2.2, we will consider criteria which are in fact differentiable in time; following [3], we impose the following assumption on the utility random fields.

*Assumption* 4.2. For all $(t, \omega) \in [0, \infty) \times \Omega$, the functions $u(\cdot, t, \omega)$ and $u^c(\cdot, t, \omega)$ are increasing, strictly concave and satisfy the Inada conditions (1). Moreover, $u(\cdot, \cdot, \omega) \in \mathcal{C}^{3,1}$ and $u^c(\cdot, t, \omega) \in \mathcal{C}^2$, and $u^c(\cdot, \cdot, \omega)$ and $u^c_x(\cdot, \cdot, \omega)$ are continuous on $\mathbb{R}_+ \times [0, \infty)$ for all $\omega \in \Omega$.

The differentiability in time imposed by Assumption 4.2 implies that $a(x, t) \equiv 0$ in (6); we refer to [3], see also [2], for further details on this and refer in the following to criteria satisfying Assumption 4.2 as nonvolatile. We note that the nonvolatility requirement is placed on the value function part of the forward investment-consumption criterion as opposed to the felicity function. In [3], the link to the random PDE (7) was formalized under the above assumptions: Theorem 4.3 therein states that under Assumption 4.1, for a pair of (adapted) random functions $(u, u^c)$ satisfying Assumption 4.2 to constitute a forward criterion, it is necessary that they provide a solution to (7) for almost all $\omega \in \Omega$. Under further additional assumptions, they also show sufficiency; since we here provide verification results (see section 4.4 below) under a different set of assumptions we do not recall those.

In order to analyze (7), it is convenient to use the standard trick of passing to the dual domain for the convex conjugate of its solutions satisfy a linear equation. More precisely, we will make use here of a specific dual transformation which yields a direct relation to the (backward) heat equation: For a random field $u(x,t,\omega)$, $(x,t,\omega) \in \mathbb{R}_+ \times [0,\infty) \times \Omega$, such that $u(\cdot,t,\omega)$ is increasing and strictly concave, let the associated random functions $H : \mathbb{R} \times [0,\infty) \times \Omega \to \mathbb{R}_+$ and $H^c : \mathbb{R} \times [0,\infty) \times \Omega \to \mathbb{R}_+$ be given by

$$u_x\left(H(y,t),t\right) \mathrel{\widehat{=}} \exp\Big(-y+\frac{1}{2}A_t\Big) \qquad \text{and} \qquad u_x^c\left(H^c(y,t),t\right) \mathrel{\widehat{=}} \exp\Big(-y+\frac{1}{2}A_t\Big); \tag{24}$$

in turn, let $h : \mathbb{R} \times [0,\infty) \times \Omega \to \mathbb{R}_+$ and $h^c : \mathbb{R} \times [0,\infty) \times \Omega \to \mathbb{R}_+$ be given by

$$H(y,t) \mathrel{\widehat{=}} h(y,A_t) \quad \text{and} \quad H^c(y,t) \mathrel{\widehat{=}} |\lambda_t|^2 h^c(y,A_t). \tag{25}$$

Under Assumptions 4.1 and 4.2, the functions $h$ and $h^c$ are well defined for almost all $\omega \in \Omega$, and they are positive, increasing, and of full range. Conversely, via the above relations, any such pair of functions $(h, h^c)$ uniquely defines a pair of functions $(u, u^c)$ (up to additive constants). We also have the following result; although we here consider random functions we note that it follows by standard arguments and implicit differentiation.

Lemma 4.3. *Let Assumption 4.1 hold and let $u(x,t,\omega)$ and $u^c(x,t,\omega)$ be two random functions satisfying Assumption 4.2. Then the pair $(u,u^c)$ satisfies (7) if and only if the associated pair $(h,h^c)$ satisfies the equation*

$$h_t + \frac{1}{2}h_{yy} + h^c = 0, \quad a.s., \quad (y,t) \in \mathbb{R} \times [0,\infty). \tag{26}$$

By use of the suitably chosen transformation introduced above, the search for solutions to the HJB-type equation (7) is thus reduced to the search for solutions $(h,h^c)$ to (26). The latter is a backward inhomogeneous heat equation; while the inhomogeneity stems from the additional source term introduced by the consumption, it is referred to as backward since it corresponds to solving the heat equation in its (typically ill-posed) backward direction. These features it has in common with Black's inverse investment problem. In contrast, we are now looking for solutions which are well defined on the entire timeline. In addition, we recall that (7) is a PDE with random coefficients and we are therefore searching for pairs of random solutions $(h,h^c)$ to (26). In order to correspond to a forward criterion, the pair $(u,u^c)$ however also needs to be adapted. In consequence, we are looking for *adapted random solutions* $(h,h^c)$ to (26); notably, adaptedness for $(h,h^c)$ refers to the time-changed filtration $(\mathcal{G}_t)_{t\geq 0}$ given by

$$\mathcal{G}_t := \mathcal{F}_{C_t} \quad \text{with} \quad C_s := \inf\{t : A_t > s\}, \tag{27}$$

where we recall that $\lim_{t\to\infty} A_t = \infty$, a.s., under Assumption 4.1. We adopt the convention of implicitly referring to this filtration whenever discussing adaptedness of solutions to (26). Note also that criteria satisfying Assumption 4.2 are continuous in time and adaptivity is therefore equivalent to progressive measurability; without further notice we will use both terms interchangeably. The fact that we are searching for adapted solutions to (26) enforces a crucial additional condition and considerably restricts the set of solutions; in particular, for a general random solution $(h, h^c)$ to the inhomogeneous heat equation, the fact that $h^c$ is adapted is not enough to ensure that $h$ is.

**4.1. The optimal investment strategies and consumption streams.** We first turn to the optimal investment behavior associated with nonvolatile forward investment-consumption criteria. The following result provides explicit formulas for the optimal investment strategies, consumption streams, and wealth processes. While the result is of independent interest, it will also be of frequent use in the upcoming analysis.

Proposition 4.4. *Suppose that Assumption 4.1 holds and let $(u, u^c)$ be a forward criterion which satisfies Assumption 4.2. Let $(h, h^c)$ be the associated pair of dual functions and denote by $x$ the investor's initial wealth. Then, the associated optimal strategy $\pi^*$, consumption stream $c^*$, and wealth process $X^*$, are given by*

$$\left\{\begin{aligned} \pi_t^* &= \sigma_t^+ \lambda_t h_y\left(h^{(-1)}(x,0) + M_t + A_t, A_t\right), \\ c_t^* &= |\lambda_t|^2 h^c\left(h^{(-1)}(x,0) + M_t + A_t, A_t\right), \\ X_t^* &= h\left(h^{(-1)}(x,0) + M_t + A_t, A_t\right). \end{aligned}\right. \tag{28}$$

*Proof.* Since $(u, u^c)$ is a forward criterion, by assumption there exists an optimal strategy $(\pi^*, c^*) \in \mathcal{A}$ for which the associated optimal wealth process is well defined. Suppose first that $X_t^*$, $t > 0$, is given by $X_t^* = h\left(N_t, A_t\right)$ for some appropriately defined process $N_t$, $t > 0$. According to (39) below, it then holds that the optimal strategy $\pi^*$ associated with $(u, u^c)$ is given by

$$\pi_t^* = -\sigma_t^+ \lambda_t \frac{u_x\left(X_t^*, t\right)}{u_{xx}\left(X_t^*, t\right)} = \sigma_t^+ \lambda_t H_y\big(H^{(-1)}\left(X_t^*, t\right), t\big) = \sigma_t^+ \lambda_t h_y\left(N_t, A_t\right), \quad t \geq 0,$$

and the consumption stream $c^*$ by

$$c_t^* = I^c\left(u_x\big(H\left(N_t, t\right), t\big), t\right) = I^c\big(e^{-N_t + \frac{1}{2}A_t}, t\big) = H^c\left(N_t, t\right) = |\lambda_t|^2 h^c\left(N_t, A_t\right), \quad t \geq 0,$$

where we used that $I^c(y,t) = -\hat{u}_y^c(y,t) = H^c(-\ln y + \frac{1}{2}A_t, t)$. In particular, we thus obtain that

$$\begin{aligned} dX_t^{(\pi^*, c^*)} &= \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t) - c_t^* dt \\ &= \lambda_t h_y\left(N_t, A_t\right) \cdot (\lambda_t dt + dW_t) - |\lambda_t|^2 h^c\left(N_t, A_t\right) dt \\ &= \big(|\lambda_t|^2 h_y\left(N_t, A_t\right) - |\lambda_t|^2 h^c\left(N_t, A_t\right)\big) dt + \lambda_t h_y\left(N_t, A_t\right) \cdot dW_t, \quad t \geq 0. \end{aligned} \tag{29}$$

It now suffices to find a process $N_t$, $t \geq 0$, such that $h(N_t, A_t)$ solves the SDE (29). We now argue that this holds true for the process

$$N_t = h^{(-1)}(x,0) + M_t + A_t, \quad t \geq 0.$$

Indeed, for this particular choice of $N_t$, $t \geq 0$, application of Itô's formula yields

$$\begin{aligned} dh\left(N_t, A_t\right) &= h_y\left(N_t, A_t\right) dN_t + \frac{1}{2} h_{yy}\left(N_t, A_t\right) d\langle N\rangle_t + h_t\left(N_t, A_t\right) dA_t \\ &= h_y\left(N_t, A_t\right)\left(dM_t + dA_t\right) + \Big(\frac{1}{2} h_{yy}\left(N_t, A_t\right) + h_t\left(N_t, A_t\right)\Big) dA_t \\ &= h_y\left(N_t, A_t\right)\left(|\lambda_t|^2 dt + \lambda_t \cdot dW_t\right) - h^c\left(N_t, A_t\right)|\lambda_t|^2 dt, \quad t \geq 0, \end{aligned}$$

which coincides with the expression given in (29); we easily conclude. ■

The result shows how the optimal investment and consumption strategies and the resulting wealth process can all be expressed as compilations of the stochastic market input represented by the processes $A_t$ and $M_t$, and the pair of (random) functions $(h, h^c)$ characterizing the investor's preferences. In the particular case when $h^c$ and $\mu$ are deterministic (see below), the stochasticity of the strategies enters solely via the processes $A_t$ and $M_t$.

**4.2. The structure of nonvolatile forward criteria.** In this section we discuss the structure of nonvolatile forward investment-consumption criteria; we suppose throughout that Assumption 4.1 holds and refer to forward criteria $(u, u^c)$ satisfying Assumption 4.2 as nonvolatile.

By use of the transformations introduced above, a characterization of such criteria is equivalent to a characterization of all adapted solutions to the inhomogeneous (backward) heat equation (26); in order to study its solutions, we first introduce some well-known functions. First, let $(k * h^c) : \mathbb{R} \times [0, \infty) \times \Omega \to \mathbb{R}_+$ be the convolution with respect to both time and space of $h^c(y, t, \omega)$ and the fundamental solution to the classical heat equation, $k(y, t)$; that is,

$$\left(k * h^c\right)(y, t) := \int_t^\infty \int_{-\infty}^\infty k(y - z; s - t) h^c(z, s) dz ds.$$

The function is well known to be a (random) solution to (26). Further, recall that we are working on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $\overline{\mathcal{B}}$ the set of random measures on $\mathcal{B}(\mathbb{R}_+)$; we refer to [21] for further details on random measures. In turn, recalling the definition of $\mathcal{B}_0$ from (19), let

$$\mathcal{B} := \left\{\mu \in \overline{\mathcal{B}} : \ \mu \in \mathcal{B}_0, \ \text{a.s.}\right\};$$

in analogy with the deterministic case, for $\mu \in \mathcal{B}$, we then define the function $h^\mu : \mathbb{R} \times [0, \infty) \times \Omega \to [0, \infty]$ by (20). According to Widder's theorem (p. 235 in [36]) (see also [2, 28] and section 3.4), any (deterministic) positive and increasing solution to the homogeneous backward heat equation defined for all times—(26) with $h^c \equiv 0$—is given by $h^\mu$ for some measure $\mu \in \mathcal{B}_0$. In consequence, any random positive solution to this (homogeneous) equation is given by $h^\mu$ for some $\mu \in \mathcal{B}$.

We are now ready to present two complementary representations—(30) and (31)—of nonvolatile forward investment-consumption criteria. To this end, let $(u, u^c)$ be such a criterion. According to Theorem 4.3 in [3] and Lemma 4.3, the associated pair $(h, h^c)$ is then a random solution to (26); that is, for almost all $\omega \in \Omega$, $(h(\cdot, \cdot, \omega), h^c(\cdot, \cdot, \omega))$ solves this equation. In [3], the crucial observation was then made that such critera admit a certain representation in terms of the functions defined above; specifically, it was argued on [3, p. 11] that there exists a measure $\mu \in \mathcal{B}$ such that the pair $(h, h^c)$ admits the representation

$$h(y,t,\omega) = \big(k * h^c\big)(y,t,\omega) + h^\mu(y,t,\omega), \quad a.s.,\ t \geq 0. \tag{30}$$

In particular, transformed into expressions for the convex conjugate $(\hat{u}, \hat{u}^c)$, (30) reads

$$\hat{u}(y,t) = \int_t^\infty \int_{-\infty}^\infty \hat{u}^c\left(y \cdot e^{\sqrt{A_s - A_t}z - (A_s - A_t)/2}, s\right) k(z)dzds - \int_0^\infty \frac{y^{1-r}}{r-1} e^{\frac{1}{2}(r-r^2)A_t} \mu(dr).$$

Here, the first term is of the same form as that appearing for a classical infinite-horizon Merton problem in a log-normal market (cf. pp. 121 and 131 in [23]) and the second term corresponds to a time-monotone forward investment criterion (see, for example, [28]). Hence, within the dual domain, forward investment-consumption criteria correspond, to linear combinations of (random) forward investment criteria and infinite-horizon Merton criteria. Crucially, it is, however, only the *combination* of the two which is adapted which implies that the problem in general cannot be split into two separate components; in section 4.3 we discuss the class of criteria for which such a splitting is possible.

Next, note that given a pair $(h, h^c)$ which satisfies (30), for each fixed $t \geq 0$, it holds for almost all $\omega \in \Omega$, that

$$h(y,0,\omega) = \int_{\mathbb{R}} k(z-y;t)h(z,t,\omega)dz + \int_0^t \int_{\mathbb{R}} k(z-y;s)h^c(z,s,\omega)dzds.$$

By construction, the first term on the right-hand side here lies in the domain of the operator $e^{-t\mathcal{D}^2}$; see the necessity part of Corollary B.2. By rearranging terms and applying the operator $e^{-t\mathcal{D}^2}$, we thus obtain that the functions $h$ and $h^c$ also admit the following alternative representation[5]

$$h(y,t,\omega) = e^{-t\mathcal{D}^2}\left\{h(y,0,\omega) - \int_0^t \int_{\mathbb{R}} k(z-y;s)h^c(z,s,\omega)dzds\right\}, \quad a.s.,\ t \geq 0. \tag{31}$$

The twofold representation (30)–(31) illustrates the intriguing structure of nonvolatile forward investment-consumption criteria which originates from their characterization in terms of functions which are *both* adapted and constitute solutions to an inhomogeneous heat equation. While it is clear that $h(y,t)$ given by (30) provides a random solution to the heat equation, the fact that it is adapted imposes a lot of additional requirements on the pair $(h^\mu, h^c)$. On the other hand, (31) provides a truly forward description of the criterion; if $h(\cdot, 0)$ is deterministic, it is therefore immediate that $h(\cdot, t)$ given by (31) is adapted whenever $h^c(\cdot, t)$ is. A novelty of this forward description is thus that it reduces the question of adaptedness of the pair $(u, u^c)$

[5]When $h(\cdot,0)$ and $h^c(\cdot,s)$ correspond, themselves, to utility functions which are $t$-admissible for any $t \geq 0$, the operator $e^{-t\mathcal{D}^2}$ and the time integral can be interchanged in (31) (cf. formula (37) below). In general, the operator $e^{-t\mathcal{D}^2}$ may, however, *not* be applied to the individual terms on the right-hand side of (31) for those terms may not even lie within the admissible domain. For example, it might be that neither $u(\cdot,0)$ nor $u^c(\cdot,s)$, $s \in I$, are $t$-admissible for all $t \geq 0$, where $I \subset [0,\infty)$, but that $u^c(\cdot,s)$, $s \in [0,\infty) \setminus I$, compensates this so as to ensure that (31) is well defined for all $t \geq 0$.

to the question of adaptedness of $u^c$ and the initial preference function $u(\cdot,0)$. Meanwhile, to ensure that the expression on the right-hand side of (31) lies in the domain of the inversion operator—and thus that $h(y,t,\omega)$ is well-defined—imposes highly nontrivial requirements on the structure of the family $(h^c(\cdot,t))$. In section 4.4 we will use these two complementary representations to obtain two different constructions of nonvolatile forward criteria.

*Remark* 4.5 (admissible initial conditions). The representation (30) implies that $u_0$ corresponds to a nonvolatile forward criterion only if the associated function $h(\cdot,0)$ is given by

$$h(y,0) = \int_0^\infty \int_{-\infty}^\infty k(z,1) h^c\left(y+\sqrt{s}z, s\right) dz ds + \int_0^\infty e^{ry}\mu(dr), \quad y \in \mathbb{R},$$

for some admissible function $h^c$ and measure $\mu \in \mathcal{B}$; since the initial condition is deterministic, this representation must in particular hold for some deterministic $h^c$ and $\mu$. Since the function $h^c$ is not a priori specified, the set of admissible initial conditions is thus considerably larger than for the pure investment case. This is due to the fact that the source terms in (26) (corresponding to utility from consumption) appear at finite times and therefore the requirement that the solution should be well defined for all times does not impose as strong restrictions as in the pure investment case. Moreover, while it holds for the pure investment case that the initial condition $h(\cdot,0)$ uniquely determines $h(\cdot,t)$, $t\geq 0$, this is not true for the investment-consumption case, for different functions $h^c$ might give rise to the same initial function $(k*h^c)(\cdot,0)$. According to the representation (31), the combined knowledge of $h(\cdot,0)$ and $h^c(\cdot,t)$, $t\geq 0$, uniquely determines, however, $h(\cdot,t)$, $t\geq 0$.

**4.3. The class of criteria featuring deterministic $\mu$: A decomposition result.** For the case of forward pure investment criteria, it is well known that a random field $u(x,t)$ corresponds to a time-monotone forward criterion if and only if the associated function $h$ is given by $h=h^\mu$ for some *deterministic* $\mu\in\mathcal{B}$; see [2]. In contrast, when allowing for consumption, we saw in the previous section that the restriction to nonvolatile criteria does not necessarily imply that the corresponding measure $\mu$ is deterministic. In this section we consider the specific class of criteria for which it is; we will see that such nonvolatile criteria admit a certain characterization and decomposition. We also discuss the relation between forward criteria and the infinite horizon Merton problem.

To provide some motivation, consider first an investor maximizing, in the classical sense, expected utility from intermediate consumption and terminal wealth at a fixed terminal horizon. It is then well known (see, e.g., Theorem 3.7.10 in [23]) that there exists a specific decomposition of the initial wealth such that if the corresponding parts, separately, are invested in order to optimize utility from consumption and terminal wealth, then the resulting total investments coincide with the optimal investments for the original problem. The following result shows that forward criteria for which $\mu$ is deterministic similarly can be decomposed into a specific infinite-infinite horizon Merton problem and a pure forward investment criterion.

Theorem 4.6. *Suppose that Assumption 4.1 holds and let $(u,u^c)$ be a forward criterion which satisfies Assumption 4.2. Let $(h,h^c)$ be the associated pair of dual functions which admit the representation (30) for some measure $\mu\in\mathcal{B}$. Let the initial capital be given by $x>0$ and denote by $(\pi^*_t, c^*_t)$, $t\geq 0$, the associated optimal strategies. Then the following hold:*

(i) *The measure $\mu$ is in $\mathcal{B}_0$ if and only if the function $(k * h^c)(y,t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.*

(ii) *Suppose that $\mu \in \mathcal{B}_0$ and define $u_1(x,t)$ with respect to $(k * h^c)(y,t)$ and $u_2(x,t)$ with respect to $h^\mu(y,t)$ via (24)–(25); in turn, let $(\pi^{1,*}, c^{1,*})$ denote the optimal strategy for the investment-consumption criterion $(u_1, u^c)$ given initial wealth $x_1$, and let $\pi^{2,*}$ denote the optimal strategy for the investment criterion $u_2$ given initial wealth $x_2$, where*

$$x_1 := (k * h^c)\left(h^{(-1)}(x,0),0\right) \qquad \text{and} \qquad x_2 := x - x_1. \tag{32}$$

*Then,*

$$\pi^*_t = \pi^{1,*}_t + \pi^{2,*}_t \qquad \text{and} \qquad c^*_t = c^{1,*}_t, \quad \text{a.s.}, \ \ t \geq 0.$$

*Proof.* (i) It follows directly from (30) that if both $h(\cdot,t)$ and $k * h^c(\cdot,t)$ are adapted, then so is $h^\mu(\cdot,t)$. In particular, it then holds that (cf. (20))

$$h^\mu(y,0,\omega) = \int_0^\infty e^{ry}\mu(dr,\omega), \quad y \in \mathbb{R},$$

is deterministic and it follows that $\mu \in \mathcal{B}_0$. Conversely, $\mu \in \mathcal{B}_0$ combined with the fact that $h(\cdot,t)$ is adapted immediately implies that also $k * h^c(\cdot,t)$ is adapted.

(ii) Relying again on the relation (30) between $h^\mu(x,t)$ and $h^c(x,t)$, $(x,t) \in \mathbb{R} \times [0,\infty)$, we first obtain

$$\begin{aligned} h^{(-1)}(x,t) &= (h^\mu)^{(-1)}\big(h^\mu\big(h^{(-1)}(x,t),t\big),t\big) \\ &= (h^\mu)^{(-1)}\big(h\big(h^{(-1)}(x,t),t\big) - (k * h^c)\left(h^{(-1)}(x,t),t\right),t\big) \\ &= (h^\mu)^{(-1)}\big(x - (k * h^c)\left(h^{(-1)}(x,t),t\right),t\big), \qquad t \geq 0. \end{aligned}$$

In consequence, by use of the definition of $x_1$ and $x_2$ given in (32), in turn we obtain

$$(k * h^c)^{(-1)}(x_1,0) \ = \ h^{(-1)}(x,0) \ = \ (h^\mu)^{(-1)}(x_2,0). \tag{33}$$

Next, note that since $(u, u^c)$ is a forward criterion, the existence of the admissible optimal strategy $(\pi^*_t, c^*_t)$, $t \geq 0$, is a consequence of Definition 2.2 and Proposition 4.4. It follows that the pairs $(u_1, u^c)$ and $(u_2, u \equiv 0)$ also satisfy Assumption 4.2 and define admissible forward criteria in the sense of Definition 2.2. Moreover, according to (i), $(k*h^c)(x,0)$ is $\mathcal{F}_0$-measurable and hence so are $x_1$ and $x_2$ given in (32). The optimal strategies associated with initial wealth $x_1$ and $x_2$, respectively, are therefore admissible and given by Proposition 4.4; we denote the latter by $(\pi^{1,*}, c^{1,*})$ and $(\pi^{2,*}, c^{2,*} \equiv 0)$.

By use of the particular form of $(\pi^{1,*}, c^{1,*})$ and $(\pi^{2,*}, c^{2,*} \equiv 0)$ (cf. (28)) combined with (33), we then obtain

$$\begin{aligned} \pi^{1,*}_t + \pi^{2,*}_t &= \sigma^+_t \lambda_t \big[ (k * h^c)_y \left( (k * h^c)^{(-1)}(x_1,0) + M_t + A_t, A_t \right) \\ &\qquad\qquad + h^\mu_y\big((h^\mu)^{(-1)}(x - x_1,0) + M_t + A_t, A_t\big)\big] \\ &= \sigma^+_t \lambda_t \big[ (k * h^c)_y \left(h^{(-1)}(x,0) + M_t + A_t, A_t\right) + h^\mu_y\big(h^{(-1)}(x,0) + M_t + A_t, A_t\big)\big] \\ &= \sigma^+_t \lambda_t h_y\big(h^{(-1)}(x,0) + M_t + A_t, A_t\big), \qquad t \geq 0, \end{aligned}$$

and, furthermore,

$$\begin{aligned} c_t^{1,*}+c_t^{2,*} &= |\lambda_t|^2 h^c\big(\left(k*h^c\right)^{(-1)}(x_1,0)+M_t+A_t,A_t\big) \\ &= |\lambda_t|^2 h^c\big(h^{(-1)}(x,0)+M_t+A_t,A_t\big), \qquad t\geq 0. \end{aligned}$$

Comparing the above expressions with the expression for $(\pi^*,c^*)$ associated with $(u,u^c)$ provided by (28), we easily conclude. ■

The assumptions of Theorem 4.6 are trivially satisfied when $h^\mu\equiv 0$ or $h^c\equiv 0$; that is, for (nonvolatile) infinite-horizon Merton criteria where utility is obtained solely from consumption, and for (time-monotone) forward investment criteria of the type studied in [3, 27, 28]. More generally, the result shows that any forward criterion with $\mu\in\mathcal{B}_0$ may be viewed as a combination of the two, namely, that to invest and consume with respect to such a criterion is equivalent to split the initial wealth into two parts which are then, separately, invested according to an infinite-horizon *pure consumption* criterion (specified by $(u_1,u^c)$) and a *pure investment* forward criterion (specified by $u_2$). Put differently, the notion of forward investment-consumption criteria allows the investor to evaluate utility from both investment and consumption without prespecifying a finite horizon; Theorem 4.6 shows that when $\mu$ is in $\mathcal{B}_0$, this compilation reduces to a deterministic combination of an investment and consumption criterion. Returning to the discussion in section 4.2, we recall from (31) that in the dual domain, nonvolatile forward investment-consumption criteria might always be viewed as (random) linear combinations of infinite-horizon Merton criteria and forward investment criteria. However, unless $\mu$ is deterministic, the corresponding initial splitting of wealth is nondeterministic and therefore the associated strategies nonadmissible; in consequence, the investor's behavior cannot be described as a deterministic combination of the two.

When splitting nonvolatile forward criteria into pure investment and consumption criteria, the emerging pair $(u_1,u^c)$ given by Theorem 4.6 crucially define a *nonvolatile* (stochastic) infinite-horizon Merton problem. To put this into context, recall that in the classical infinite-horizon Merton problem (see, e.g., section 3.9 in [23]), the investor specifies a (random) felicity function $u^c(c,t)$, $t\geq 0$, and initial capital $x$, and then aims at attaining the maximal utility $u(x,0)$ with

$$u(x,t):=\sup_{(\pi,c)}\mathrm{E}\left[\int_t^\infty u^c(c_t,t)ds\big|\mathcal{F}_t\right],\quad x>0,\ t\geq 0,$$

where the supremum is taken over the set of admissible strategies with $X_t^{\pi,c}=x$. The pair $(u,u^c)$ thus defined constitutes a forward investment-consumption criterion in the sense of Definition 2.2. Notably, general forward investment-consumption criteria need, however, not be of this form; indeed, the aim of the notion is to describe preferences which allow the investor to receive utility from the investments themselves as well as from consumption, also when considering arbitrary or infinite horizons. Moreover, we note that infinite-horizon Merton criteria in general do not define nonvolatile forward criteria. Indeed, this is a highly specific property for the pure consumption case too; in particular, it typically does not hold true if the consumption function $u^c$ is deterministic. Put differently, when constructing solutions of the form $(k*h^c,h^c)$ to the random PDE (26), unless $h^c$ is deterministic, the adaptedness requirement is still highly nontrivial.

**4.4. Construction of nonvolatile investment-consumption criteria.** In this section we provide converse results to the representation results presented in section 4.2 in that we construct nonvolatile forward criteria. Recall that we are thus searching for pairs of random functions $(u, u^c)$ which solve (7), which are adapted, and for which the (local) supermartingale property holds for any strategy and the martingale property at an optimum. As we have seen, all three properties are most easily verified for the dual pair $(h, h^c)$ and we will therefore construct forward criteria via the construction of such admissible dual pairs.

Our first construction is based on the representation formula (30).

Theorem 4.7. *Let Assumption 4.1 hold and let $\mu \in \mathcal{B}$ and $h^c : \mathbb{R} \times [0, \infty) \times \Omega \to \mathbb{R}_+$ be such that $h^c(\cdot, t)$ is positive, increasing, continuous, integrable, and of range $(0, \infty)$, a.s. Suppose that $h^c(\cdot, t)$ is adapted. Then, let*

$$h(y, t, \omega) := (k * h^c)(y, t, \omega) + h^\mu(y, t). \tag{34}$$

*Suppose that $h(\cdot, 0)$ thus defined is deterministic and that the following functions are uniformly bounded on finite intervals:*

$$t \mapsto \mathrm{E}\left[\int y e^{yx+\frac{1}{2}y^2 t}\mu(dy)\right] \quad \textit{and} \quad t \mapsto \mathrm{E}\left[(k * h_y^c)^4(x + W_t, t)\right], \quad x \in \mathbb{R}. \tag{35}$$

*Then, the pair $(u, u^c)$ associated with $(h, h^c)$ constitutes a nonvolatile local forward criterion.*

*Proof.* The conditions on $\mu \in \mathcal{B}$ and $h^c$ imply that $h^\mu(y, t)$ and $\big(k * h^c\big)(y, t)$ are well defined on $\mathbb{R} \times [0, \infty)$, a.s. Moreover, note that $h$ is positive and belongs to $\mathcal{C}^{\infty,1}$, and that $h(\cdot, t)$, $t \geq 0$, is increasing and of range $(0, \infty)$, a.s. The associated random functions $u(\cdot, t)$ and $u^c(\cdot, t)$ (cf. (24) and (25)) thus both belong to $\mathcal{C}^2$ and are increasing, strictly concave, and satisfy (1), a.s.; in addition, $u \in \mathcal{C}^{\infty,1}$, a.s. Next, since the pair $(h, h^c)$ satisfies (26), we have that $(u, u^c)$ satisfies (7) (cf. Lemma 4.3). Moreover, by use of the same arguments as in section 4.2, we obtain that the pair $(h, h^c)$ also admit the representation (31). Recall that $h^c(\cdot, t)$ is adapted with respect to the time-changed filtration $(\mathcal{G}_t)_{t\geq 0}$ (cf. (27)); the fact that $h(y, 0)$ is nonrandom combined with the representation (31) thus implies that also $h(\cdot, t)$ is adapted to this filtration. Hence, the pair $(u, u^c)$ is adapted to the original filtration $(\mathcal{F}_t)_{t\geq 0}$.

Next, by use of Itô's formula and the fact that $(u, u^c)$ solves (7), we then obtain (cf. (38) below and (6)), that for any $(\pi, c) \in \mathcal{A}$,

$$du(X_t^{\pi,c}, t) + u^c(c_t, t)dt = k(X_t^{\pi,c}, \pi_t, c_t)dt + u_x(X_t^{\pi,c}, t)\sigma_t \pi_t dW_t$$

for some negative drift term $k(x, \pi, c)$. This gives the local supermartingale property of $u(X_t^{\pi,c}, t) + \int_0^t u^c(c_s, s)ds$, for $t \geq 0$. It remains to argue that the pair $(\pi^*, c^*)$ as provided by Proposition 4.4 is admissible, and that the true martingale property holds for this choice. To this end, suppose first that the pair $(\pi^*, c^*)$ is indeed admissible; then we have the following dynamics (cf. (40) below)

$$du(X_t^*, t) + u^c(c_t^*, t)dt = u_x(X_t^*, t)\sigma_t \pi_t^* dW_t.$$

Note that according to Assumption 4.1, the process $A_\cdot$ has a pathwise well-defined inverse $A_\cdot^{-1}$ for almost all $\omega$. Moreover, $\lambda_\cdot$ is uniformly bounded in $(t,\omega)$, say by $\bar{\lambda}$; we write $\bar{A}_t = \bar{\lambda}^2 t$. By use of Proposition 4.4 and application of Tonelli's theorem and Hölder's inequality, we thus obtain

$$
\begin{aligned}
&\mathrm{E}\int_0^T |u_x(X_t^*,t)\sigma_t\pi_t^*|^2 dt \\
&\qquad = \mathrm{E}\int_0^T e^{-2(h^{-1}(x,0)+M_t+A_t)+A_t} h_y(h^{-1}(x,0)+M_t+A_t,A_t)^2 dA_t \\
&\qquad = \mathrm{E}\int_0^{\bar{A}_T} e^{-2(h^{-1}(x,0)+\beta_t+\frac{t}{2})} h_y(h^{-1}(x,0)+\beta_t+t,t)^2 dt \\
&\qquad \le \int_0^{\bar{A}_T} e^{-2h^{-1}(x,0)-t}\mathrm{E}\big[e^{-2\beta_t}\big((k*h_y^c)^2+2(k*h_y^c)h_y^\mu+(h_y^\mu)^2\big)(R_t,t)\big]dt \\
&\qquad \le \int_0^{\bar{A}_T} e^{-2h^{-1}(x,0)-t}\Big(e^{16t}\mathrm{E}\left[(k*h_y^c)^4(R_t,t)\right]+\mathrm{E}\left[e^{-2\beta_t}(h_y^\mu)^2(R_t,t))\right] \\
&\qquad\qquad + 2\mathrm{E}\left[(k*h_y^c)^2(R_t,t)\right]\mathrm{E}\left[e^{-4\beta_t}(h_y^\mu)^2(R_t,t)\right]\Big)dt,
\end{aligned}
$$

where we used the notation $\beta_t = M_{A_t^{-1}}$ and $R_t = h^{-1}(x,0)+\beta_t+t$, for $t\ge 0$. In order to further estimate the right-hand side, we apply again Tonelli's theorem to obtain

$$
\begin{aligned}
\mathrm{E}\left[e^{-4\beta_t}(h_y^\mu)^2(R_t,t)\right] &= \mathrm{E}\iint y_1y_2 e^{-4\beta_t+(y_1+y_2)(h^{-1}(x,0)+\beta_t+t)-\frac{1}{2}(y_1^2+y_2^2)t}\mu(dy_1)\mu(dy_2) \\
&= e^{8t}\,\mathrm{E}\iint y_1y_2 e^{(y_1+y_2)(h^{-1}(x,0)-3t)+y_1y_2t}\mu(dy_1)\mu(dy_2) \\
&\le e^{8t}\,\mathrm{E}\left(\int y e^{y(h^{-1}(x,0)-3t)+\frac{1}{2}y^2t}\mu(dy)\right)^2.
\end{aligned}
$$

By use of condition (35), the fact that $\beta_t \sim \mathcal{N}(0,t)$, and the above estimates, we then obtain that $\mathrm{E}\int_0^T |u_x(X_t^*,t)\sigma_t\pi_t^*|^2dt<\infty$, for all $T>0$; hence $u(X_t^*,t)$ is a true martingale. Finally, note that according to (28),

$$
\int_0^T |\sigma_t\pi_t^*|^2 dt = \int_0^T h_y(h^{-1}(x,0)+M_t+A_t,A_t)^2 dA_t,\ a.s.
$$

Hence, by straightforward alternations of the above calculations, we obtain $\mathrm{E}\int_0^T|\sigma_t\pi_t^*|^2dt<\infty$, $T>0$, which ensures that $(\pi^*,c^*)$ is indeed an admissible strategy. ■

*Remark* 4.8. Condition (35) is imposed in order to ensure admissibility and the martingale property for the optimal strategy; we note that it ensures the true martingale property only for the optimum (this is also the case when restricting to pure forward investment criteria with $\mu\in\mathcal{B}_0$ and $h^c\equiv 0$; cf. Theorem 4 in [28]). Alternative conditions can be imposed so as to ensure the "true supermartingale property" for any admissible pair $(\pi,c)\in\mathcal{A}$; we refer to Assumption 4.2 in [3] for such a condition but note that in contrast to the conditions in

Theorem 4.7, their assumptions are placed on the field $u(x,t)$ itself and not directly on the exogenous input.

Next we provide an alternative construction which is based on the representation formula (31). While the difficulty in the previous construction lay in ensuring the adaptedness—$h(\cdot,0)$ being deterministic is the key assumption—the difficulty in this case lay in ensuring that the consumption terms are of such a form that the right-hand side of (31) is well defined; cf. section 4.2. To ensure such well posedness, we here restrict ourselves to a class of forward criteria for which *all* utility terms are $t$-admissible for *any* $t \geq 0$; we have the following result.

Theorem 4.9. *Let Assumption 4.1 hold, let $\mu_0 \in \mathcal{B}_0$, and let $h^c : \mathbb{R} \times [0,\infty) \times \Omega \to \mathbb{R}_+$ be given such that $h^c(y,t)$ is adapted and $h^c(\cdot,t) = h^{\mu_t}(\cdot,0)$ for some family $(\mu_t)_{t\geq 0}$ in $\mathcal{B}$. Define*

$$h(y,t,\omega) = h^{\mu_0}(y,t) - \int_0^t h^{\mu_s(\omega)}(y,t-s)ds, \quad a.s., \ t \geq 0. \tag{36}$$

*Given that the associated functions in (35) are uniformly bounded on finite intervals, the pair $(u,u^c)$ associated with $(h,h^c)$ then constitutes a nonvolatile local forward criterion.*

*Proof.* Define first $h(y,t,\omega)$ via (31) with $h(\cdot,0) = h^{\mu_0}(\cdot)$. From the assumptions on $h^c$ it follows that the associated $u^c$ are $t$-admissible utilities for any $t \geq 0$ and therefore the individual terms within the brackets on the right-hand side of (31) are in the domain of $e^{-t\mathcal{D}^2}$, for any $t \geq 0$. Since the operator $e^{-t\mathcal{D}^2}$ is linear on its domain, it follows that

$$h(\cdot,t,\omega) = e^{-t\mathcal{D}^2}h(\cdot,0) - \int_0^t e^{-(t-s)\mathcal{D}^2}h^c(\cdot,s,\omega)ds, \tag{37}$$

which further simplifies to (36). We conclude by applying Theorem 4.7. ■

*Remark* 4.10. Note that the $\mu_0$ appearing in (36) is different from the $\mu$ given in (30); specifically, they admit the following relation:

$$h^{\mu_0}(y,0,\omega) = h^{\mu}(y,0,\omega) + \int_0^\infty \int_{\mathbb{R}} k(z-y;s)h^c(z,s,\omega)dzds, \quad a.s.$$

**Appendix A. The SPDE characterizing forward criteria.** In this section we derive the SPDE provided in section 2.2 which forward investment-consumption criteria are expected to satisfy. Since the filtration is Brownian, given a candidate pair of random functions $(u,u^c)$, it is natural to start with the assumption that $u(x,t)$ admits the Itô decomposition (5) for some (wealth-dependent) coefficients $a$ and $b$ which are progressively measurable processes. We now demonstrate that in order for the martingale optimality principle to hold, that is for $u(X_\cdot,\cdot)+\int^\cdot u^c(c_s,s)ds$ to be a supermartingale for any strategy and a martingale at optimum, the processes $a(x,t)$ and $b(x,t)$ need to satisfy a certain relation.

To this end, assuming that one can apply the Itô–Ventzell formula for all involved quantities (note that our framework is non-Markovian), we have

$$\begin{aligned} du(X_t,t) + u^c(c_t,t)dt = u_x(X_t,t)\left[\sigma_t\pi_t \cdot \left(\lambda_t dt + dW_t\right) - c_t dt\right] + \frac{1}{2}u_{xx}(X_t,t)|\sigma_t\pi_t|^2 dt \\ + b(X_t,t)dt + a(X_t,t)\cdot dW_t + \sigma_t\pi_t \cdot a_x(X_t,t)dt + u^c(c_t,t)dt. \end{aligned}$$

Rearranging yields

(38)
$$du(X_t,t)+u^c(c_t,t)dt=\left[\frac{1}{2}u_{xx}(X_t,t)\left(|\sigma_t\pi_t|^2+2\sigma_t\pi_t\cdot\frac{\lambda_t u_x(X_t,t)+\sigma_t\sigma_t^+a_x(X_t,t)}{u_{xx}(X_t,t)}\right)\right.$$
$$\left.+b(X_t,t)+u^c(c_t,t)-u_x(X_t,t)c_t\right]dt+\Big[u_x(X_t,t)\sigma_t\pi_t+a(X_t,t)\Big]\cdot dW_t,$$

where we recall that $\sigma^+$ denotes the Moore–Penrose pseudoinverse and where the quantity $\sigma\sigma^+$ was introduced as it might be that[6] $a_x\notin Lin(\sigma)$. It follows readily that the drift term attains its maximum for $\pi^*$ and $c^*$ given in feedback form by

$$\pi^*(x,t)=-\sigma_t^+\frac{\lambda_t u_x(x,t)+a_x(x,t)}{u_{xx}(x,t)}\qquad\text{and}\qquad c^*(x,t)=I^c(u_x(x,t),t), \tag{39}$$

where $I^c:\mathbb{R}\times[0,\infty)\times\Omega\to\mathbb{R}_+$ is defined via $u_x^c\left(I^c(y,t),t\right)=y$; we refer to the first and second term of the optimal strategy $\pi^*(x,t)$ as its myopic and nonmyopic parts, respectively. Substitution for these expressions into (38) yields

$$du(X_t,t)+u^c(c_t,t)dt=\left[-\frac{1}{2}\frac{|\lambda_t u_x+\sigma_t\sigma_t^+a_x|^2}{u_{xx}}+b+\hat{u}^c\left(u_x,t\right)\right]dt+\left[u_x\sigma_t\pi_t+a\right]\cdot dW_t, \tag{40}$$

where $\hat{u}^c:\mathbb{R}\times[0,\infty)\times\Omega\to\mathbb{R}_+$ is given by $\hat{u}^c(y,t)=\sup_{x>0}\{u^c(x,t)-xy\}$, and where the argument $(X_t,t)$ is suppressed on the right-hand side. For the required martingale properties to hold, the drift term in (40) needs to vanish. This condition gives an explicit expression of the process $b(x,t)$ in terms of the volatility $a(x,t)$, which when substituted back into (5) leaves us with the (forward) investment-consumption SPDE (6).

## Appendix B. Inversion of the Weierstrass transform.

In the remainder of the paper we provide some well-known results on analytic functions and establish various modified versions thereof which are required for our proofs.

First we recall a result from [34] on the inversion of the so-called Weierstrass transform (see (41) and (42) below). In line with [34], we here use the convention $k(z,t)=(4\pi t)^{-\frac{1}{2}}e^{-\frac{1}{4t}z^2}$ for the operator $e^{-t\mathcal{D}^2}$ introduced in Definition 3.1.

**Theorem B.1** (Widder: Theorem 3 in [34]). *For a function $f(\cdot)$ to be represented in the form*

$$f(x)=c\int_{-\infty}^{\infty}e^{-(x-y)^2/4\tau}d\alpha(y),\qquad c=(4\pi\tau)^{-1/2},\ x\in\mathbb{R}, \tag{41}$$

*where $\alpha(y)$ is a Borel measure with $\alpha(B)\geq 0$, $B\in\mathcal{B}(\mathbb{R})$, it is necessary and sufficient that*

[6]Indeed, it holds that $\sigma\pi\cdot\sigma\sigma^+a_x=\pi^T\sigma^T\sigma\sigma^+a_x=\pi^T\sigma^Ta_x=\sigma\pi\cdot a_x$ and, trivially, $\sigma\sigma^+a_x\in Lin(\sigma)$. Alternatively, the result might be obtained by rewriting the first drift term of (38) as follows: $\ldots=\frac{1}{2}u_{xx}(x,t)(|\sigma_t\pi_t+\frac{\lambda_t u_x(x,t)+a_x(x,t)}{u_{xx}(x,t)}|^2-|\frac{\lambda_t u_x(x,t)+a_x(x,t)}{u_{xx}(x,t)}|^2)|_{x=X_t}$; for simplicity, only the investment part is included. It then follows that the maximum is obtained for the $\pi$ which is the least-squares solution to the linear system $\sigma_t\pi_t=-\frac{\lambda_t u_x(x,t)+a_x(x,t)}{u_{xx}(x,t)}$. That is to say, $\pi^*$ is given by (39).

(i) *$f(x)$ is entire;*
(ii) *$f(x+iy) = \mathcal{O}(e^{y^2/4\tau})$ as $y \to \pm\infty$, uniformly in $-a \le x \le a$ for every $a > 0$;*
(iii) *$e^{-t\mathcal{D}^2} f(x) \ge 0$, for $0 < t < \tau$, $x \in \mathbb{R}$.*

While the result was given for $\tau = 1$ in [34], a study of their proof verifies that the result holds for any $\tau > 0$. In particular, the result directly translates to the time-changed setup (by a factor of two compared to the classical Weierstrass transform) that we consider in the rest of the article.

It is well known that the operator $e^{-t\mathcal{D}^2}$ may be used to invert the Weierstrass transform. Next, we provide a result which combines this property with the above result; see also [34].

Corollary B.2. *Let the operator $e^{-t\mathcal{D}^2}$, $0 < t < \tau$, be as in Definition 3.1 and let $\phi : \mathbb{R} \to \mathbb{R}_+$ be a continuous function. Consider a function $f : \mathbb{R} \to \mathbb{R}_+$. Then,*

$$f(x) = W_\tau[\phi(x)] := c\int_{-\infty}^{\infty} e^{-(x-y)^2/4\tau}\phi(y)dy, \qquad c = (4\pi\tau)^{-1/2},\ x \in \mathbb{R}, \tag{42}$$

*if and only if $f(x)$ satisfies conditions* (i)–(iii) *of Theorem B.1 and, furthermore,*

$$\lim_{t \nearrow \tau} e^{-t\mathcal{D}^2} f(x) = \phi(x). \tag{43}$$

*Proof.* We first show the necessity. To this end, let $f$ be given by (42). In particular, it is then of the form (41) and it follows from Theorem B.1 that conditions (i) to (iii) hold. Hence, for $x \in \mathbb{R}$ and $0 < t < \tau$, the integral in (9) converges and $e^{-t\mathcal{D}^2} f(x)$ is well defined. Specifically, as argued on p. 435 in [34], for $f(x)$ given in this form we have that

$$e^{-t\mathcal{D}^2} f(x) = \frac{c}{(\tau - t)^{1/2}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4(\tau-t)}\phi(y)dy, \qquad 0 < t < \tau.$$

It only remains to argue that the right-hand side converges to $\phi$ as $t \nearrow \tau$. According to Theorem 5 in [35, p. 67], for this it is sufficient that $\phi$ is Lebesgue integrable on every finite interval and that

$$\int_a^x |\phi(y) - \phi(a)|dy = o(|x-a|), \text{ as } x \to a. \tag{44}$$

Since $\phi$ is continuous, both these conditions are satisfied. Indeed, a continuous function is trivially Lebesgue integrable on any finite interval, and for any $\varepsilon > 0$, there exists $\delta$ such that for all $x$ with $|x - a| < \delta$, $|\phi(x) - \phi(a)| < \varepsilon$. Hence, $\int_a^x |\phi(y) - \phi(a)|dy < \varepsilon|x - a|$, which implies (44).

To show the sufficiency, let $f$ satisfy conditions (i)–(iii) of Theorem B.1 along with condition (43). According to Theorem B.1, without loss of generality, we may suppose that $f$ is given by (41) for some Borel measure $\alpha$ such that $\alpha(B) \ge 0$, $B \in \mathcal{B}(\mathbb{R})$. Further, $e^{-t\mathcal{D}^2} f(x)$ is well defined for $x \in \mathbb{R}$ and $0 < t < \tau$; specifically, using again the argument given on p. 435 in [34], we have that

$$e^{-t\mathcal{D}^2} f(x) = \frac{c}{(\tau - t)^{1/2}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4(\tau-t)}\, d\alpha(y), \qquad 0 < t < \tau.$$

Now, by assumption we know that $\lim_{t\nearrow\tau} e^{-t\mathcal{D}^2} f(x)$ exists and is absolutely integrable. Hence we may proceed as follows:

$$
\begin{aligned}
W_\tau\left[\lim_{t\nearrow\tau} e^{-t\mathcal{D}^2} f(x)\right] &= c\int_{-\infty}^{\infty} e^{-(x-y)^2/4\tau} \lim_{t\nearrow\tau} \frac{c}{(\tau-t)^{1/2}} \int_{-\infty}^{\infty} e^{-(y-s)^2/4(\tau-t)} d\alpha(s)\, dy \\
&= c\int_{-\infty}^{\infty} \lim_{t\nearrow\tau} \frac{c}{(\tau-t)^{1/2}} \int_{-\infty}^{\infty} e^{-(y-s)^2/4(\tau-t)} e^{-(x-y)^2/4\tau} dy\, d\alpha(s),
\end{aligned}
\tag{45}
$$

where the order of integration is interchanged by use of Tonelli's theorem. Next, since the function $e^{-(x-s)^2/4\tau}$ is bounded and continuous, it follows from Lebesgue's limit theorem that the inner integral in (45) converges to $e^{-(x-s)^2/4\tau}$ as $t \nearrow \tau$; see [36, p. 61]. Consequently, it follows from (45) that

$$
W_\tau\left[\lim_{t\nearrow\tau} e^{-t\mathcal{D}^2} f(x)\right] = c\int_{-\infty}^{\infty} e^{-(x-s)^2/4\tau} d\alpha(s) = f(x),
$$

and we easily conclude. ■

**Appendix C. On Bernstein's theorem.** Recall the definition of AM and CM functions from Definition 3.8. Next, we recall Bernstein's theorem on the representation of CM functions:

Theorem C.1 (Bernstein: Theorem IV 12 in [33]). *A function $f$ is CM in $(0,\infty)$ if and only if*

$$
f(x) = \int_0^\infty e^{-sx} d\alpha(s), \qquad 0 < x < \infty,
$$

*for some positive measure $\alpha$ such that the integral converges for $0 < x < \infty$.*

We then also have the following result.

Corollary C.2.

(i) *A function $f$ is CM in $(-\infty,\infty)$ if and only if $f(x) = \int_0^\infty e^{-sx} d\alpha(s)$ for $-\infty < x < \infty$ for some positive measure $\alpha$ such that the integral converges for $-\infty < x < \infty$.*

(ii) *A function $f$ is AM in $(-\infty,\infty)$ if and only if $f(x) = \int_0^\infty e^{sx} d\alpha(s)$ for $-\infty < x < \infty$ for some positive measure $\alpha$ such that the integral converges for $-\infty < x < \infty$.*

*Proof.* First, if $f(x)$ is AM for $-\infty < x < \infty$, then $x \mapsto f(-x)$ is CM for $-\infty < x < \infty$. Hence, part (ii) follows immediately from part (i). To argue part (i), let $f(x)$ be CM on $-\infty < x < \infty$. In particular, it is then CM for $x > 0$. Hence, we might apply Theorem C.1 to obtain

$$
f(x) = \int_0^\infty e^{-sx} d\alpha(s), \qquad x > 0, \tag{46}
$$

for some measure $\alpha$, where the integral converges for $x > 0$. Next, let $\eta > 0$ and define the function $g(x)$ via $g(x) := f(x-\eta)$, $x > 0$. Then $g(x)$ is clearly CM for $x > 0$. Thus, according to Theorem C.1, there exists a measure $\mu^\eta$ such that $g(x) = \int_0^\infty e^{-sx} d\mu^\eta(s)$, $x > 0$, where the integral converges for $x > 0$. Since $f(x) = g(x+\eta)$, for $x > -\eta$, this, in turn, implies that

$$
f(x) = \int_0^\infty e^{-s(x+\eta)} d\mu^\eta(s) = \int_0^\infty e^{-sx} d\alpha^\eta(s), \qquad x > -\eta, \tag{47}
$$

where the integral converges for $x > -\eta$ and where $\alpha^\eta(t) := \int_0^t e^{-\eta s} d\mu^\eta(s)$. However, the Laplace integral representation of a function defined on $x > 0$ is unique (Theorem 6.3 in [33, p. 63]). Hence, comparison of formulas (46) and (47) (where both integrals converge for $x > 0$) yields that $\alpha^\eta(s) = \alpha(s)$, $s > 0$. Hence, it follows from (47) that

$$f(x) = \int_0^\infty e^{-sx} d\alpha(s), \qquad x > -\eta,$$

where the integral converges for $x > -\eta$. Since $\eta$ was arbitrarily chosen, this completes the proof. ∎

We also have the following result.

Lemma C.3. *Let $f^1(x)$ and $f^2(x)$ be CM in $-\infty < x < \infty$. Then, it holds that if $f^1(x) = f^2(x)$ for $x \in [a, b]$, $a, b \in \mathbb{R}$, then $f^1(x) = f^2(x)$ for $-\infty < x < \infty$.*

*Proof.* A function which is CM on $\mathbb{R}$ is analytic on $\mathbb{R}$ and, therefore, it can be extended to an analytic function in the entire plane (analytic continuation). Since an analytic function is uniquely determined by its values along a curve in its area of analyticity, the result follows. ∎

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