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OPTIMAL SKOROKHOD EMBEDDING GIVEN FULL MARGINALS AND AZÉMA–YOR PEACOCKS[1]

BY SIGRID KÄLLBLAD*,‡, XIAOLU TAN† AND NIZAR TOUZI‡

Vienna University of Technology, University of Paris-Dauphine† and École Polytechnique‡*

We consider the optimal Skorokhod embedding problem (SEP) given full marginals over the time interval [0, 1]. The problem is related to the study of extremal martingales associated with a peacock ("process increasing in convex order," by Hirsch, Profeta, Roynette and Yor [*Peacocks and Associated Martingales, with Explicit Constructions* (2011), Springer, Milan]). A general duality result is obtained by convergence techniques. We then study the case where the reward function depends on the maximum of the embedding process, which is the limit of the martingale transport problem studied in Henry-Labordère, Obłój, Spoida and Touzi [*Ann. Appl. Probab.* **26** (2016) 1–44]. Under technical conditions, we then characterize the optimal value and the solution to the dual problem. In particular, the optimal embedding corresponds to the Madan and Yor [*Bernoulli* **8** (2002) 509–536] peacock under their "increasing mean residual value" condition. We also discuss the associated martingale inequality.

1. Introduction. Given a probability measure μ on $\mathbb{R}$, centered and with finite first moment, the Skorokhod embedding problem (SEP) consists in finding a stopping time T for a Brownian motion W, such that $W_T \sim \mu$ and the stopped process $(W_{T\wedge\cdot})$ is uniformly integrable. We consider here an extended version. Let $(\mu_t)_{t\in[0,1]}$ be a family of probability measures that are all centered, have finite first moments and are nondecreasing in convex order, that is, $t \mapsto \mu_t(\phi) := \int_{\mathbb{R}} \phi(x)\mu_t(dx)$ is nondecreasing for every convex function $\phi : \mathbb{R} \to \mathbb{R}$. The extended Skorokhod embedding problem is then to find a nondecreasing family of stopping times, $(T_t)_{t\in[0,1]}$, for a Brownian motion W, such that $W_{T_t} \sim \mu_t$, for all $t \in [0, 1]$, and the stopped process $(W_{T_1\wedge\cdot})$ is uniformly integrable. Specifically, we study an optimal Skorokhod embedding problem which consists in maximizing a reward value among the class of all such extended embeddings.

It follows from Kellerer's theorem (see, e.g., Kellerer [31] or Hirsch and Roynette [21]) that for a family $\mu = (\mu_t)_{t\in[0,1]}$ satisfying the above conditions,

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there exists at least one (Markov) martingale whose one-dimensional marginal distributions coincide with μ. Assume in addition that $t \mapsto \mu_t$ is right-continuous, then any associated martingale admits a right-continuous modification. Since, according to Monroe [34], any right-continuous martingale can be embedded into a Brownian motion with a nondecreasing family of stopping times, this implies that the collection of solutions to the extended Skorokhod embedding problem is nonempty. Moreover, the full marginal optimal SEP is thus related to the study of extremal martingales associated with peacocks. A peacock (or PCOC "Processus Croissant pour l'Ordre Convexe") is a continuous time stochastic process whose one-dimensional marginal distributions are nondecreasing in convex order according to Hirsch, Profeta, Roynette and Yor [20]. Since Kellerer's theorem ensures the existence of martingales with given one-dimensional marginal distributions (see also [7, 22, 32] for new and insightful proofs of this result), the interesting subject is to construct these associated martingales; we refer to the book [20] and the references therein for various techniques. We mention that when the marginal distributions are those of a Brownian motion, such an associated martingale is also referred to as a fake Brownian motion; see, e.g., [1, 14, 27, 37].

Our problem of study is motivated by financial applications; specifically, by the problem of model-independent pricing of exotic options. Indeed, the knowledge of prices of a continuum of call options with a given maturity, allows one to recover the marginal distribution of any market model consistent with those prices (see, e.g., Breeden and Litzenberger [9]). Financial considerations further imply that any feasible price process should be a martingale. Maximizing the expected value of an exotic (path-dependent) option over the class of martingales fitting the given marginals—therefore gives an upper bound on arbitrage-free prices of the option which are consistent with the market data.

The problem was initially studied under one marginal constraint using the SEP approach; first by Hobson [25] and later by many others. This approach is based on the fact that any continuous martingale can be viewed as a time-changed Brownian motion; we refer to the survey papers of Obłój [35] and Hobson [23]. More recently, it has also been studied using the so-called martingale transport approach introduced in Beiglböck, Henry-Labordère and Penkner [6] and Galichon, Henry-Labordère and Touzi [15]. Since then, there has been an intensive development of the literature on martingale optimal transport and the connection with model-free hedging in finance. In the present context of full marginals constraint, Henry-Labordère, Tan and Touzi [19] considered reward functions satisfying the so-called martingale Spence-Mirrlees condition, and solved the martingale transport problem via a quasi-explicit construction of the corresponding martingale peacock and the optimal semi-static hedging strategy. In Hobson [24], yet a martingale peacock with a certain optimality property has been constructed.

Here, we study the full marginal problem from the SEP perspective. First, taking the limit of a duality result established for a general optimal SEP under finitely many marginal constraints in [16] and [5] (extending a duality result in Beiglböck, Cox and Huesmann [4]), we obtain a general duality result for the optimal SEP under full marginal constraints. Thereafter, we study the case when the reward function depends on the realized maximum of the embedding process. For the case of one marginal constraint, this problem is solved by the Azéma–Yor [2] embedding; see also Hobson [25]. For multiple marginal constraints which are *increasing in mean residual value* [see (3.13) below], Madan and Yor [33] studied an iteration of the Azéma–Yor embedding. They also provided the limiting object in the full-marginal case and characterized the resulting peacock. When the marginals are not increasing in mean residual value, the multi-marginal problem becomes significantly more involved. The two-marginal case was solved in Brown, Hobson and Rogers [10] (see also Hobson [26]). For a finite number of marginal constraints, an iterated Azéma–Yor embedding was proposed in Obłój and Spoida [36] under an additional technical assumption. In the accompanying paper by Henry-Labordère, Obłój, Spoida and Touzi [18], this embedding was proven to be optimal for the realized maximum. They also provided the solution to the associated dual problem and characterized the optimal value of the problem. By applying limiting arguments to these results, under a certain technical assumption we here obtain an explicit characterization of the optimal value and of the primal and dual optimizers for the corresponding full marginal optimal SEP. In particular, this defines a peacock, which we refer to as the Azéma–Yor peacock, and which coincides with the one given in [33] under the additional condition that the marginals are increasing in mean residual value.

We also provide further intuition for our results and relate them to the corresponding martingale optimal transport (i.e., pricing) problems. Specifically, we show that for a class of payoffs which are invariant under time changes, our full marginal SEP provides the limit of the value (i.e., the price) of the multi-marginal problem as the number of marginals tends to infinity. Meanwhile, the peacock corresponding to the optimizer of the full-marginal SEP is in general not a solution to the full marginal martingale optimal transport problem.

The rest of the paper is organized as follows. The main results are presented in Section 2: in Section 2.1, we formulate our optimal SEP given full marginals; in Section 2.2, we provide the general duality result; in Section 2.3, we focus on the class of maximal reward functions for which we specify the value of the problem and give the explicit form of a dual optimizer; and in Section 2.4 we present an associated martingale inequality. In Section 3, we provide further discussion of our results and relate them to the finite-marginal SEP and to martingale optimal transport. The proofs are completed in Section 4.

Notation. (i) Let $\Omega := C(\mathbb{R}_+, \mathbb{R})$ denote the canonical space of all continuous paths ω on $\mathbb{R}_+$ with $\omega_0 = 0$, let B be the canonical process and $\mathbb{P}_0$ the Wiener measure under which B is a standard Brownian motion. Further, let $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t\geq 0}$ denote the canonical filtration generated by B, and $\mathbb{F}^a = (\mathcal{F}_t^a)_{t\geq 0}$ the augmented filtration under $\mathbb{P}_0$.

We equip Ω with the compact convergence topology (see, e.g., Whitt [39] or Stroock and Varadhan [38]):

$$\rho(\omega,\omega') := \sum_{n\geq 1} \frac{1}{2^n} \frac{\sup_{0\leq t\leq n}|\omega_t-\omega'_t|}{1+\sup_{0\leq t\leq n}|\omega_t-\omega'_t|}, \qquad \forall \omega,\omega'\in\Omega. \tag{1.1}$$

Then (Ω,ρ) is a Polish space (separable and complete metric space).

(ii) Let $\mathbb{V}_r^+ = \mathbb{V}_r^+([0,1],\mathbb{R}_+)$ denote the space of all nondecreasing càdlàg functions on $[0,1]$ taking values in $\mathbb{R}_+$. Similarly, let $\mathbb{V}_l^+ = \mathbb{V}_l^+([0,1],\mathbb{R})$ denote the space of all nondecreasing càglàd functions on $[0,1]$ taking values in $\mathbb{R}$.

We equip $\mathbb{V}_r^+$ and $\mathbb{V}_l^+$ with the Lévy metric: for all $\theta,\theta'\in\mathbb{V}_r^+$,

$$d(\theta,\theta') := \inf\{\varepsilon>0 : \theta_{t-\varepsilon}-\varepsilon \leq \theta'_t \leq \theta_{t+\varepsilon}+\varepsilon, \forall t\in[0,1]\}, \tag{1.2}$$

where we extend the definition of θ to $[-\varepsilon, 1+\varepsilon]$ by letting $\theta_s := \theta_0$ for $s\in[-\varepsilon,0]$ and $\theta_s := \theta_1$ for $s\in[1,1+\varepsilon]$. For $m\in\mathbb{R}$, denote by $\mathbb{V}_l^{+,m}\subset\mathbb{V}_l^+$ the subset of functions θ such that $\theta(1)\leq m$. Then both $\mathbb{V}_r^+$ and $\mathbb{V}_l^{+,m}$ are Polish spaces (see Remark A.1).

(iii) We define an enlarged canonical space by $\overline{\Omega} := \Omega\times\mathbb{V}_r^+$, where the canonical process is denoted by $\overline{B}=(B,T)$. We introduce the enlarged canonical filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\geq 0}$, where $\overline{\mathcal{F}}_t$ is generated by $(B_s)_{s\in[0,t]}$ and all the sets $\{T_r\leq s\}$ for $s\in[0,t]$ and $r\in[0,1]$. In particular, all the canonical variables $(T_r)_{r\in[0,1]}$ are $\overline{\mathbb{F}}$-stopping times.[2] We notice that the σ-field $\overline{\mathcal{F}}_\infty := \bigvee_{t\geq 0}\overline{\mathcal{F}}_t$ coincides with the Borel σ-field of the Polish space $\overline{\Omega}$ (see Lemma A.2).

For a set $\mathcal{P}$ of probability measures on $\overline{\Omega}$, we say that a property holds $\mathcal{P}$-quasi-surely (q.s.) if it holds $\mathbb{P}$-a.s. for all $\mathbb{P}\in\mathcal{P}$.

(iv) Let $\mathcal{C}_b$ denote the space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and $\mathcal{C}_1$ the space of all functions $f:\mathbb{R}\to\mathbb{R}$ such that $\frac{f(x)}{1+|x|}\in\mathcal{C}_b$.

2. Main results. Throughout the paper, we are given a family of probability measures on $\mathbb{R}$, $\mu=(\mu_t)_{t\in[0,1]}$, satisfying the following condition:

ASSUMPTION 2.1. *The family of marginal distributions, $\mu=(\mu_t)_{t\in[0,1]}$, satisfies*

$$\int_{\mathbb{R}} |x|\mu_t(dx) < \infty \quad \text{and} \quad \int_{\mathbb{R}} x\mu_t(dx) = 0, \qquad t\in[0,1]. \tag{2.1}$$

Furthermore, $\mu_0=\delta_{\{0\}}$, $t\mapsto\mu_t$ is càdlàg w.r.t. the weak convergence topology, and μ is nondecreasing in convex order, that is, for every convex function $\phi:\mathbb{R}\to\mathbb{R}$,

$$\mu_s(\phi)\leq\mu_t(\phi) := \int_{\mathbb{R}}\phi(x)\mu_t(dx) \qquad \text{whenever } s\leq t. \tag{2.2}$$

[2] This definition of filtration follows the idea introduced in [12, 13] to study a general mixed stochastic control/stopping problem.

2.1. *The optimal SEP given full marginals.* Let $\mathcal{P}(\overline{\Omega})$ denote the collection of all Borel probability measures on the canonical space $\overline{\Omega}$, and define

$$\mathcal{P} := \{\mathbb{P} \in \mathcal{P}(\overline{\Omega}) : B \text{ is a } \overline{\mathbb{F}}\text{-Brownian motion and} \\ B_{\cdot \wedge T_1} \text{ is uniformly integrable under } \mathbb{P}\}. \tag{2.3}$$

For the given marginals $\mu = (\mu_t)_{t\in[0,1]}$, we then define

$$\mathcal{P}(\mu) := \{\mathbb{P} \in \mathcal{P} : B_{T_t} \sim^{\mathbb{P}} \mu_t, \forall t \in [0,1]\}.$$

LEMMA 2.2. *Suppose that Assumption 2.1 holds, then $\mathcal{P}(\mu)$ is nonempty.*

PROOF. Since the marginal distributions $\mu = (\mu_t)_{t\in[0,1]}$ satisfy Assumption 2.1, it follows from Kellerer's theorem (see, e.g., Kellerer [31] or Hirsch and Roynette [21]) that there is a martingale M such that $M_t \sim \mu_t$ for all $t \in [0,1]$. Since $t \mapsto \mu_t$ is right-continuous, the martingale M can be chosen to be right-continuous. It follows from Theorem 11 in Monroe [34], that there is a Brownian motion W and a family of nondecreasing and right-continuous stopping times $(\tau_t)_{t\in[0,1]}$, such that $(W_{\tau_1\wedge\cdot})$ is uniformly integrable and $(W_{\tau_\cdot})$ has the same finite-dimensional distributions as $(M_\cdot)$. In consequence, the probability induced by $(W_\cdot, \tau_\cdot)$ on $\overline{\Omega}$ belongs to $\mathcal{P}(\mu)$. □

The main objective of the paper is to study the following optimal Skorokhod Embedding Problem (SEP) under full marginal constraints:

$$P(\mu) := \sup_{\mathbb{P}\in\mathcal{P}(\mu)} \mathbb{E}^{\mathbb{P}}[\Phi(B_\cdot, T_\cdot)], \tag{2.4}$$

where $\Phi : \overline{\Omega} \to \mathbb{R}$ is the reward function.

The optimal SEP (2.4) under full marginal constraints is given in a weak formulation. Indeed, for given marginals $\mu = (\mu_t)_{t\in[0,1]}$, let a μ-embedding be a term

$$\alpha = (\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{F}^\alpha = (\mathcal{F}_t^\alpha)_{t\geq 0}, \mathbb{P}^\alpha, (W_t^\alpha)_{t\geq 0}, (T_s^\alpha)_{s\in[0,1]}), \tag{2.5}$$

such that in the filtered space $(\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{F}^\alpha, \mathbb{P}^\alpha)$ the following holds: $W_\cdot^\alpha$ is a Brownian motion; $T_\cdot^\alpha$ is a nondecreasing càdlàg family of stopping times; the stopped process $(W^\alpha_{T_1^\alpha\wedge\cdot})$ is uniformly integrable; and $W^\alpha_{T_t^\alpha} \sim^{\mathbb{P}^\alpha} \mu_t$ for every $t \in [0,1]$. Denote by $\mathcal{A}(\mu)$ the collection of all such μ-embeddings α. It is then clear that every μ-embedding $\alpha \in \mathcal{A}(\mu)$ induces a probability measure $\mathbb{P} \in \mathcal{P}(\mu)$. On the other hand, every $\mathbb{P} \in \mathcal{P}(\mu)$ together with the canonical space $\overline{\Omega}$, the canonical process $\overline{B}$ and the canonical filtration, forms an embedding term in $\mathcal{A}(\mu)$. In consequence, the set $\mathcal{P}(\mu)$ is the collection of all probability measures $\mathbb{P}$ on $\overline{\Omega}$ induced by the embeddings $\alpha \in \mathcal{A}(\mu)$. As a direct consequence, the optimal SEP (2.4) admits the following equivalent formulation:

$$P(\mu) = \sup_{\alpha\in\mathcal{A}(\mu)} \mathbb{E}^\alpha[\Phi(W_\cdot^\alpha, T_\cdot^\alpha)]. \tag{2.6}$$

2.2. *Duality for the full marginal optimal SEP.* In order to introduce the dual problem, we let $\mathbb{L}^2_{\text{loc}}$ denote the space of all $\overline{\mathbb{F}}$-progressively measurable processes, say $H = (H_t)_{t\geq 0}$, defined on the enlarged canonical space $\overline{\Omega}$, and such that

$$\int_0^t H_s^2\, ds < \infty, \qquad \mathbb{P}\text{-a.s., for all } t > 0 \text{ and } \mathbb{P} \in \mathcal{P}.$$

Then, for every $H \in \mathbb{L}^2_{\text{loc}}$ and $\mathbb{P} \in \mathcal{P}$, the stochastic integral of H w.r.t. the canonical process B under $\mathbb{P}$, denoted by $(H \cdot B)_\cdot$, is well defined. An adapted process $M = (M_t)_{t\geq 0}$ defined on $\overline{\Omega}$ is called a strong supermartingale under $\mathbb{P}$ if M_τ is integrable for all $\overline{\mathbb{F}}$-stopping times $\tau \geq 0$, and for any two $\overline{\mathbb{F}}$-stopping times $\tau_1 \leq \tau_2$, we have that $\mathbb{E}^{\mathbb{P}}[M_{\tau_2}|\overline{\mathcal{F}}_{\tau_1}] \leq M_{\tau_1}$. We then define $\mathcal{H}$ by

$$\mathcal{H} := \{H \in \mathbb{L}^2_{\text{loc}} : (H \cdot B)_\cdot \text{ is a strong supermartingale under every } \mathbb{P} \in \mathcal{P}\}.$$

We let $M([0, 1])$ denote the space of all finite signed measures on $[0, 1]$ and note that it is a Polish space under the weak convergence topology. Further, we denote by Λ the space of all $\lambda : \mathbb{R} \to M([0, 1])$ admitting the representation

$$\lambda(x, dt) = \lambda_0(x, t)\bar{\lambda}(dt),$$

for some finite positive measure $\bar{\lambda} \in M([0, 1])$ and some locally bounded measurable function $\lambda_0 : \mathbb{R} \times [0, 1] \to \mathbb{R}$. For $\mu = (\mu_t)_{t\in[0,1]}$, we define

$$\Lambda(\mu) := \left\{\lambda \in \Lambda : \mu(|\lambda|) := \int_0^1 \int_{\mathbb{R}} |\lambda_0(x, t)|\mu_t(dx)\bar{\lambda}(dt) < \infty\right\},$$

and

$$\mu(\lambda) := \int_0^1 \int_{\mathbb{R}} \lambda(x, dt)\mu_t(dx) = \int_0^1 \int_{\mathbb{R}} \lambda_0(x, t)\mu_t(dx)\bar{\lambda}(dt), \ \forall \lambda \in \Lambda(\mu). \tag{2.7}$$

With the notation $\lambda(\overline{B}) := \int_0^1 \lambda_0(B_{T_s}, s)\bar{\lambda}(ds)$, we finally set

$$\mathcal{D}(\mu) := \{(\lambda, H) \in \Lambda(\mu) \times \mathcal{H} : \lambda(\overline{B}) + (H \cdot B)_{T_1} \geq \Phi(\overline{B}_\cdot), \mathcal{P}\text{-q.s.}\}. \tag{2.8}$$

The dual problem for the optimal SEP (2.4) under full marginal constraints is then defined as follows:

$$D(\mu) := \inf_{(\lambda,H)\in\mathcal{D}(\mu)} \mu(\lambda). \tag{2.9}$$

Our first main result is the following.

THEOREM 2.3. *Let Assumption 2.1 hold true. Suppose in addition that* $\Phi : \overline{\Omega} \to \mathbb{R}$ *is upper semicontinuous and bounded from above, and satisfies* $\Phi(\omega, \theta) = \Phi(\omega_{\theta_1\wedge\cdot}, \theta)$ *for all* $(\omega, \theta) \in \overline{\Omega}$. *Then there exists a solution* $\widehat{\mathbb{P}} \in \mathcal{P}(\mu)$ *to problem* $P(\mu)$ *in (2.4) and we have the duality*

$$\mathbb{E}^{\widehat{\mathbb{P}}}[\Phi(B_\cdot, T_\cdot)] = P(\mu) = D(\mu).$$

EXAMPLE 2.4. The maximum reward function given in (2.11) below aside, the following are also examples of reward functions which satisfy the conditions of Theorem 2.3 (see also Remark A.1 on the Lévy metric on $\mathbb{V}_r^+$):

- the weighted variance $\Phi(\omega,\theta) = \int_0^1 f(t)\,d\theta_t$, where $f : [0,1] \to \mathbb{R}$ is bounded and continuous;
- the knock-in reward function $\Phi(\omega,\theta) = \omega_{\theta_1} \mathbf{1}_{\{\omega^*_{\theta(1)} \geq \kappa\}}$, for some constant $\kappa \in \mathbb{R}$;
- the weighted average value function $\Phi(\omega,\theta) := \int_0^1 \omega_{\theta_t} f(t)\,dt$ for some nonnegative weight function $f \in \mathbb{L}^1([0,1])$.

We also introduce the weaker version of the dual problem:

$$D_0(\mu) := \inf_{(\lambda,H)\in\mathcal{D}_0(\mu)} \mu(\lambda), \tag{2.10}$$

with $\mathcal{D}_0(\mu)$ given by

$$\mathcal{D}_0(\mu) := \{(\lambda, H) \in \Lambda(\mu) \times \mathcal{H} : \lambda(\overline{B}) + (H \cdot B)_{T_1} \geq \Phi(\overline{B}_\cdot), \mathcal{P}(\mu)\text{-q.s.}\}.$$

As a consequence of Theorem 2.3, we have the following result.

COROLLARY 2.5. *Under the same conditions as in Theorem 2.3, it holds that*

$$P(\mu) = D_0(\mu) = D(\mu).$$

PROOF. Let $(\lambda, H) \in \mathcal{D}_0(\mu)$. For any $\mathbb{P} \in \mathcal{P}(\mu)$, taking expectation over the inequality in the definition of $\mathcal{D}_0(\mu)$, one obtains $\mu(\lambda) \geq \mathbb{E}^{\mathbb{P}}[\Phi(B_\cdot, T_\cdot)]$. Hence, $\mu(\lambda) \geq P(\mu)$, which yields the weak duality $D_0(\mu) \geq P(\mu)$. Since $\mathcal{P}(\mu) \subseteq \mathcal{P}$, it follows that $D_0(\mu) \leq D(\mu)$. In consequence, the result follows from Theorem 2.3. □

2.3. *Maximum maximum given full marginals.* We now restrict to the case where

$$\Phi(\omega,\theta) = \phi(\omega^*_{\theta_1}) \qquad \text{with } \omega^*_t := \max_{0\leq s\leq t} \omega_s,\ t \geq 0, \tag{2.11}$$

for some bounded, nondecreasing and upper semicontinuous (or equivalently càdlàg) function $\phi : \mathbb{R}_+ \to \mathbb{R}$. According to Theorem 2.3 and Lemma 4.3 below, the duality $P(\mu) = D(\mu)$ holds for this reward function. The aim of this section is to compute this optimal value and characterize a solution to the dual problem (2.10). To this end, we introduce some further conditions on the marginals μ; let $c(t,x) := \int_{\mathbb{R}} (y-x)^+ \mu_t(dy)$ for every $(t,x) \in [0,1] \times \mathbb{R}$:

ASSUMPTION 2.6. (i) The function c is differentiable in t and the derivative function $\partial_t c$ is continuous, that is, $\partial_t c(t,x) \in C([0,1] \times \mathbb{R})$.

(ii) There exists a sequence of discrete time grids $(\pi_n)_{n\geq 1}$ with $\pi_n = (0 = t_0^n < t_1^n < \cdots < t_n^n = 1)$, such that $|\pi_n| \to 0$ and, for all $n \geq 1$, the finite family of marginals $(\mu_{t_i^n})_{i=1}^n$ satisfies Assumption ⊛ in [36].

For further discussion of Assumption ⊛, see Section 3.1.3 below.

For every fixed $m \geq 0$, we introduce a minimization problem: with the convention that $\frac{0}{0} = 0$ and $\frac{c}{0} = \infty$ for $c > 0$, let

$$C(m) := \inf_{\zeta \in \mathbb{V}_l^+ : \zeta \leq m} \left\{ \frac{c(0, \zeta_0)}{m - \zeta_0} + \int_0^1 \frac{\partial_t c(s, \zeta_s)}{m - \zeta_s} ds \right\}. \tag{2.12}$$

Our first result is on the value of the optimal SEP (2.4).

THEOREM 2.7. *Let Φ be given by* (2.11) *for some bounded, nondecreasing and càdlàg function ϕ. Suppose that Assumptions* 2.1 *and* 2.6(i) *hold true. Then*

$$P(\mu) = D(\mu) \leq \phi(0) + \int_0^\infty C(m)\, d\phi(m). \tag{2.13}$$

Suppose in addition that Assumption 2.6(ii) *holds, then equality holds in* (2.13).

Our second result is on the existence and characterization of a specific dual optimizer. To this end, we first introduce a specific class of dual objects. For $m > 0$ and $\zeta \in \mathbb{V}_l^+$ such that $\zeta(1) < m$, let the functions $\lambda_c^{\zeta,m}$ and $\lambda_d^{\zeta,m}$ be given by

$$\lambda_c^{\zeta,m}(x,t) := \frac{m - x}{(m - \zeta_t)^2} \mathbf{1}_{\{x \geq \zeta_t\}} \mathbf{1}_{D_m^c}(t)$$

and

$$\lambda_d^{\zeta,m}(x,t) := \frac{1}{\Delta\zeta_t} \left(\frac{(x - \zeta_t)^+}{m - \zeta_t} - \frac{(x - \zeta_{t+})^+}{m - \zeta_{t+}} \right) \mathbf{1}_{D_m}(t) + \frac{(x - \zeta_1)^+}{m - \zeta_1} \mathbf{1}_{\{t=1\}},$$

where $\Delta\zeta_t := \zeta_{t+} - \zeta_t$, $D_m := \{t \in [0,1) : \Delta\zeta_t > 0\}$ and $D_m^c := [0,1) \setminus D_m$. We then define the static term

$$\lambda^{\zeta,m}(x, dt) := \left(\lambda_c^{\zeta,m}(x,t) + \lambda_d^{\zeta,m}(x,t) \right) d\zeta_t. \tag{2.14}$$

It is clear that $\lambda^{\zeta,m} \in \Lambda$. Next, let $\theta^{-1} : \mathbb{R}_+ \to [0,1]$ be the right-continuous inverse of $s \mapsto \theta_s$; that is, $\theta_s^{-1} := \sup\{r \in [0,1] : \theta_r \leq s\}$. We note that $\theta_s^{-1}(\bar{\omega})$ is $\overline{\mathcal{F}}_s$-measurable for fixed s. Hence, it is càdlàg and $\overline{\mathbb{F}}$-adapted and therefore $\overline{\mathbb{F}}$-progressively measurable. With I^- and I^+, both functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, given by $I^-(s) := \theta(\theta^{-1}(s)-)$ and $I^+(s) := \theta(\theta^{-1}(s))$, and with $\tau_m(\bar{\omega}) = \inf\{t \geq 0 : \omega_{\theta(t)} \geq m\}$, we then define the dynamic term:

$$H_s^{\zeta,m}(\omega,\theta) := \frac{\mathbf{1}_{[\tau_m, I^+(\tau_m)]}(s)}{m - \zeta_{\theta^{-1}(\tau_m)}} + \frac{\mathbf{1}_{\{m \leq \omega^*_{I^-(s)}; \zeta_{\theta^{-1}(s)} \leq \omega_{I^-(s)}\}}}{m - \zeta_{\theta^{-1}(s)}}. \tag{2.15}$$

Finally, given $\zeta : [0,1) \times \mathbb{R}_+ \to \mathbb{R}$ with $\zeta_\cdot^m \in \mathbb{V}_l^+$ and $\zeta_1^m \leq m$, for all $m > 0$ where $\zeta_\cdot^m = \zeta(\cdot, m)$, and such that $\int_0^\infty \frac{d\phi(m)}{(m - \zeta_1^m)^2} < \infty$, we define the following dual object:

$$\lambda^\zeta(x, dt) := \int_0^\infty \lambda^{\zeta^m, m}(x, dt)\, d\phi(m) \quad \text{and} \quad H_s^\zeta := \int_0^\infty H_s^{\zeta^m, m}\, d\phi(m). \tag{2.16}$$

The construction of the dual optimizer below is based on evaluating (λ^ζ, H^ζ) at an optimizer of the minimization problem (2.12). Hence, we first establish the existence of such a solution.

LEMMA 2.8. *Let Assumptions 2.1 and 2.6(i) hold true. Then there exists a measurable function $\hat{\zeta} : [0,1) \times \mathbb{R}_+ \to \mathbb{R}$ such that $\hat{\zeta}^m_\cdot \in \mathbb{V}_l^+$ is a solution to (2.12), for all $m > 0$.*

THEOREM 2.9. *Suppose that $\phi : \mathbb{R}_+ \to \mathbb{R}$ is nondecreasing and that Assumptions 2.1 and 2.6 hold true. Let $\hat{\zeta} : [0,1) \times \mathbb{R}_+ \to \mathbb{R}$ be a measurable function such that, for all $m > 0$, $\hat{\zeta}^m_\cdot \in \mathbb{V}_l^+$ is a solution to (2.12) and*

$$\int_0^\infty \frac{d\phi(m)}{(m - \hat{\zeta}_1^m)^2} < \infty. \tag{2.17}$$

Then $(\hat{\lambda}, \widehat{H}) := (\lambda^{\hat{\zeta}}, H^{\hat{\zeta}}) \in \Lambda(\mu) \times \mathcal{H}$. Suppose in addition that $\phi : \mathbb{R}_+ \to \mathbb{R}$ is bounded and continuous and that, for all $t \in [0,1]$,

$$\mu_t \text{ is atomless and } \hat{\zeta}_t^m \text{ and its inverse are both continuous in } m. \tag{2.18}$$

Then $(\hat{\lambda}, \widehat{H})$ is a dual optimizer for the problem $D_0(\mu)$ in (2.10). That is, with Φ given in (2.11), it holds that

$$\mu(\hat{\lambda}) = D_0(\mu) \quad \text{and} \quad \hat{\lambda}(\overline{B}) + (\widehat{H} \cdot B)_{T_1} \geq \Phi(B_\cdot, T_\cdot), \quad \mathcal{P}(\mu)\text{-q.s.} \tag{2.19}$$

REMARK 2.10. The condition (2.18) is needed to argue the convergence to $(\hat{\lambda}, \widehat{H})$, in an appropriate sense, of the corresponding dual optimizers for the finite marginal case (see Lemma 4.4). As seen from the proof, if $\hat{\zeta}^m_\cdot$ can be represented as a countable sum, that is,

$$\hat{\zeta}_s^m = \sum_{k=0}^\infty \zeta_k^m \mathbf{1}_{(t_k, t_{k+1}]}(s), \tag{2.20}$$

for some $(\zeta_k^m)_{k\geq 0}$, then $(\hat{\lambda}, \widehat{H})$ is a dual optimizer regardless of condition (2.18).

2.4. *An associated martingale inequality.* In this section, we establish a closely related martingale inequality. We stress that this result does not require Assumption 2.6.

PROPOSITION 2.11. *Let $(M_t)_{t\in[0,1]}$ be a right continuous martingale, $\phi : \mathbb{R}_+ \to \mathbb{R}$ a nondecreasing and càdlàg function, and take $\zeta : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ such that for each $m > 0$, $\zeta^m_\cdot \in \mathbb{V}_l^+$ and $\zeta_1^m < m$, and the set*

$$\{t \in [0,1) : x \longmapsto \mathbb{P}[M_t \leq x] \text{ is discontinuous at } x = \zeta_t^m\}$$

is a $d\zeta_t^{m,c}$*-null set, where* $\zeta_t^{m,c}$ *is the continuous part of* $t \mapsto \zeta_t^m$. *Then, with* $M_t^* := \max_{0\le s\le t} M_s$, *it holds that*

$$\mathbb{E}[\phi(M_1^*)] \le \phi(0) + \int_0^\infty \mathbb{E}\Big[\int_0^1 \tilde{\lambda}^m(M, dt)\Big] d\phi(m),$$

where [*cf.* (2.14)] $\tilde{\lambda}^m(x, dt) := \lambda^{\zeta^m, m}(x, dt)$ *so that*

$$\begin{aligned}\mathbb{E}\Big[\int_0^1 \tilde{\lambda}^m(M, dt)\Big] = &\int_0^1 \frac{\mathbb{P}[M_t > \zeta_t^m](m - \zeta_t^m) - \mathbb{E}[(M_t - \zeta_t^m)^+]}{(m - \zeta_t^m)^2} d\zeta_t^{m,c} \\ &+ \frac{\mathbb{E}[(M_1 - \zeta_1^m)^+]}{m - \zeta_1^m} \\ &+ \sum_t \Big[\frac{\mathbb{E}[(M_t - \zeta_t^m)^+]}{m - \zeta_t^m} - \frac{\mathbb{E}[(M_t - \zeta_{t+}^m)^+]}{m - \zeta_{t+}^m}\Big].\end{aligned}$$

We conclude this section with a remark on an alternative version of the above martingale inequality.

REMARK 2.12. Suppose that $(M_t)_{t\in[0,1]}$ is a càdlàg martingale such that the function $c_M(t, x) := \mathbb{E}[(M_t - x)^+]$ is $\mathcal{C}^1$ in t. Further, let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be nondecreasing and càdlàg, and take $\zeta : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}$ such that $\zeta_\cdot^m \in \mathbb{V}_l^+$ and $\zeta_1^m < m$, for all $m > 0$. Then

$$\mathbb{E}[\phi(M_1^*)] \le \phi(0) + \int_0^\infty \Big(\frac{\mathbb{E}[(M_0 - \zeta_0^m)^+]}{m - \zeta_0^m} + \int_0^1 \frac{\partial_t c_M(t, \zeta_s^m)}{m - \zeta_s^m} ds\Big) d\phi(m).$$

Indeed, due to Monroe [34], there is $\mathbb{P} \in \mathcal{P}(\mu)$ such that

$$\mathbb{E}[\phi(M_1^*)] = \mathbb{E}^{\mathbb{P}}\Big[\phi\Big(\max_{0\le t\le 1} B_{T_t}\Big)\Big] \le \mathbb{E}^{\mathbb{P}}[\phi(B_{T_1}^*)], \tag{2.21}$$

where the inequality follows as ϕ is nondecreasing and $\max_{0\le t\le 1} B_{T_t} \le B_{T_1}^*$. The above inequality is therefore an immediate consequence of Theorem 2.7.

3. Further discussion. In this section, we discuss the relation between the optimal SEP and the martingale transport problem, and specify how the optimal SEP given full marginals can be considered as the limit of the approximating problem defined by a finite subset of marginals. We also discuss the relation to the full marginal martingale optimal transport problem. Further, we provide a numerical scheme for the problem $C(m)$ introduced in (2.12).

3.1. *The optimal SEP given finitely many marginals.* In this section, we consider the optimal SEP given *finitely* many marginals and recall some results established in previous works.

For $n \geq 1$, let $\pi_n = \{t_0^n, \dots, t_n^n\}$ be a discrete time grid on $[0, 1]$ such that $0 = t_0^n < t_1^n < \cdots < t_n^n = 1$. Recalling the definition of $\mathcal{P}$ from (2.3), let

$$\mathcal{P}_n(\mu) := \{\mathbb{P} \in \mathcal{P} : B_{T_{t_k^n}} \sim^{\mathbb{P}} \mu_{t_k^n}, k = 1, \dots, n\}.$$

That is, the set $\mathcal{P}_n(\mu)$ consists of all Skorokhod embeddings of the n marginals $(\mu_{t_k^n})_{k=1,\dots,n}$. Let $\Phi_n : \Omega \times (\mathbb{R}_+)^n \to \mathbb{R}$ be a reward function. The associated optimal SEP is then given by

$$P_n(\mu) := \sup_{\mathbb{P} \in \mathcal{P}_n(\mu)} \mathbb{E}^{\mathbb{P}}[\Phi_n(B_{\cdot \wedge T_1}, T_{t_1^n}, \dots, T_{t_n^n})]. \tag{3.1}$$

3.1.1. *The duality result.* In Guo, Tan and Touzi [16], a duality result is established for the optimal SEP (3.1). Let us define

$$\begin{aligned} D_n(\mu) := \inf \Bigg\{ & \sum_{k=1}^n \mu_{t_k^n}(\lambda_k) : (\lambda_1, \dots, \lambda_n, H) \in (\mathcal{C}_1)^n \times \mathcal{H} \text{ such that} \\ & \sum_{k=1}^n \lambda_k(B_{T_{t_k^n}}) + (H \cdot B)_{T_1} \geq \Phi_n(B_{\cdot \wedge T_1}, T_{t_1^n}, \dots, T_{t_n^n}), \mathcal{P}\text{-q.s.} \Bigg\}. \end{aligned} \tag{3.2}$$

One of the main results in [16] is the following duality result (see also [5] for a similar result), which is a cornerstone in our proof of Theorem 2.3.

PROPOSITION 3.1. *Suppose that Assumption 2.1 holds true and that Φ_n is upper semicontinuous and bounded from above. Then $P_n(\mu) = D_n(\mu)$ and the supremum in problem $P_n(\mu)$ in (3.1) is attained.*

3.1.2. *Optimal SEP and martingale transport problems.* One of the main motivations for studying the optimal Skorokhod embedding problem is the fact that any continuous local martingale can be seen as a time-changed Brownian motion. It is therefore natural to relate the optimal SEP to the martingale optimal transport (MOT) problem.

Let $\tilde{\Omega} := C([0, 1], \mathbb{R})$ denote the canonical space of all continuous paths on $[0, 1]$, with canonical process X and canonical filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0 \leq t \leq 1}$. Let $\mathcal{M}$ denote the collection of all martingale measures on $\tilde{\Omega}$, that is, the probability measures $\tilde{\mathbb{P}}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}_1)$ under which X is a martingale. We recall that there exists a nondecreasing $\tilde{\mathbb{F}}$-progressively measurable process $\langle X \rangle$ which coincides with the quadratic variation of X under every martingale measure $\tilde{\mathbb{P}} \in \mathcal{M}$ (see, e.g., Karandikar [30]). Let

$$\langle X \rangle_s^{-1} := \inf\{t : \langle X \rangle_t \geq s\}.$$

Then, under every $\tilde{\mathbb{P}} \in \mathcal{M}$, the process $(X_{\langle X \rangle_s^{-1}})_{s \geq 0}$ is a Brownian motion, and for every $t \geq 0$, $\langle X \rangle_t$ is a stopping time w.r.t. the filtration $(\tilde{\mathcal{F}}_{\langle X \rangle_s^{-1}})_{s \geq 0}$. Let $\mu =$

$(\mu_t)_{0\le t\le 1}$ be the given family of marginals satisfying Assumption 2.1. For $n\ge 1$ and a discrete time grid $\pi_n : 0 = t_0^n < t_1^n < \cdots < t_n^n = 1$, we denote

$$\mathcal{M}_n(\mu) := \{\tilde{\mathbb{P}} \in \mathcal{M} : X_{t_k^n} \sim^{\tilde{\mathbb{P}}} \mu_{t_k^n}, k = 1, \dots, n\}.$$

For a reward function $\xi : \tilde{\Omega} \to \mathbb{R}$, we then define the MOT problem

$$\tilde{P}_n(\mu) := \sup_{\tilde{\mathbb{P}}\in\mathcal{M}_n(\mu)} \mathbb{E}^{\tilde{\mathbb{P}}}[\xi(X_\cdot)]. \tag{3.3}$$

The problem has a natural interpretation as a model-independent bound on arbitrage-free prices of the exotic option $\xi(X_\cdot)$. In order to introduce the corresponding dual formulation, let $\widetilde{\mathcal{H}}$ denote the collection of all $\tilde{\mathbb{F}}$-progressively measurable processes $\tilde{H} : [0,1] \times \tilde{\Omega} \to \mathbb{R}$ such that $\int^\cdot \tilde{H}_s\, dX_s$ is a supermartingale under every $\tilde{\mathbb{P}} \in \mathcal{M}$. Then, let

$$\tilde{D}_n(\mu) := \inf\Biggl\{\sum_{k=1}^n \mu_{t_k^n}(\lambda_k) : (\lambda, \widetilde{H}) \in (\mathcal{C}_1)^n \times \widetilde{\mathcal{H}} \text{ such that}$$
$$\sum_{k=1}^n \lambda_k(X_{t_k^n}) + (\widetilde{H}\cdot X)_1 \ge \xi(X_\cdot), \mathcal{M}\text{-q.s.}\Biggr\}.$$

The above dual problem gives the minimal robust super-hedging cost of the exotic option, in the quasi-sure sense, using static strategy λ and dynamic strategy $\widetilde{H}$.

Via the time change argument, the above MOT problem and its dual version are related to the optimal SEP $P_n(\mu)$ and the associated dual formulation $D_n(\mu)$. Specifically, the following result is given in [16].

PROPOSITION 3.2. *Suppose that Assumption 2.1 holds true and that the payoff function $\xi : \tilde{\Omega} \to \mathbb{R}$ is given by*

$$\xi(X_\cdot) = \Phi_n\bigl(X_{\langle X\rangle_\cdot^{-1}\wedge 1}, \langle X\rangle_{t_1^n}, \dots, \langle X\rangle_{t_n^n}\bigr), \tag{3.4}$$

for some Φ_n which is upper semicontinuous and bounded from above. Then,

$$P_n(\mu) = \tilde{P}_n(\mu) = \tilde{D}_n(\mu) = D_n(\mu).$$

3.1.3. *The iterated Azéma–Yor embedding.* We now consider a lookback option, whose payoff function is given by $\xi(X_\cdot) := \phi(X_1^*)$, with $X_1^* := \max_{0\le t\le 1} X_t$ and $\phi : \mathbb{R}_+ \to \mathbb{R}$ a nondecreasing, bounded and càdlàg function. We notice that ξ satisfies the conditions in Proposition 3.2. Indeed, the corresponding Φ is given in (2.11). For the one-marginal case, this problem is solved by the (time-changed) Azéma–Yor embedding. Given multiple marginal constraints with increasing mean residual value [see (3.13) below], this solution can be iterated to solve the problem also for multiple marginals; see [33]. For general marginals, the problem becomes significantly more involved. The two-marginal case is solved in [10] (see also [26]). Under an additional technical assumption, the problem is solved for a finite number of marginals in Henry-Labordère, Obłój, Spoida and Touzi [18] using a certain iterated Azéma–Yor embedding introduced in Obłój and Spoida [36]. We now recall their solution. A first technical step is the following path-wise inequality (see Section 4.1 in [18]):

PROPOSITION 3.3. *Let* $\mathbf{x}$ *be a càdlàg path on* $[0,1]$ *and denote* $\mathbf{x}^*_t := \max_{0\le s\le t}\mathbf{x}_s$. *Then, for every* $m > \mathbf{x}_0$ *and* $\zeta_1 \le \cdots \le \zeta_n < m$:

$$\begin{aligned}\mathbf{1}_{\{\mathbf{x}^*_{t_n}\ge m\}} \le{}& \sum_{i=1}^{n}\left(\frac{(\mathbf{x}_{t_i}-\zeta_i)^+}{m-\zeta_i} + \mathbf{1}_{\{\mathbf{x}^*_{t_{i-1}}<m\le \mathbf{x}^*_{t_i}\}}\frac{m-\mathbf{x}_{t_i}}{m-\zeta_i}\right) \\ &- \sum_{i=1}^{n-1}\left(\frac{(\mathbf{x}_{t_i}-\zeta_{i+1})^+}{m-\zeta_{i+1}} + \mathbf{1}_{\{m\le \mathbf{x}^*_{t_i},\zeta_{i+1}\le \mathbf{x}_{t_i}\}}\frac{\mathbf{x}_{t_{i+1}}-\mathbf{x}_{t_i}}{m-\zeta_{i+1}}\right).\end{aligned} \tag{3.5}$$

As argued in [18], the above inequality implies that also the following inequality holds:

$$\mathbf{1}_{\{\mathbf{x}^*_{t_n}\ge m\}} \le \sum_{i=1}^{n}\left\{\lambda_i^{\zeta,m}(\mathbf{x}_{t_i}) + \int_{t_{i-1}}^{t_i} H_t^{\zeta,m}(\mathbf{x})\,d\mathbf{x}_t\right\}, \tag{3.6}$$

with $T_m(\mathbf{x}) := \inf\{t\ge 0 : \mathbf{x}_t \ge m\}$ and

$$\lambda_i^{\zeta,m}(x) := \frac{(x-\zeta_i)^+}{m-\zeta_i} - \frac{(x-\zeta_{i+1})^+}{m-\zeta_{i+1}}\mathbf{1}_{\{i<n\}}, \qquad x\in\mathbb{R},$$

$$H_t^{\zeta,m}(\mathbf{x}) := -\frac{\mathbf{1}_{(t_{i-1},t]}(T_m(\mathbf{x})) + \mathbf{1}_{[0,t_{i-1}]}(T_m(\mathbf{x}))\mathbf{1}_{\{\mathbf{x}_{t_{i-1}}\ge\zeta_i\}}}{m-\zeta_i}, \qquad t\in[t_{i-1},t_i).$$

Indeed, if $\mathbf{x}$ is continuous at $T_m(\mathbf{x})$, then the inequalities (3.5) and (3.6) coincide. If $\mathbf{x}$ has a jump at $T_m(\mathbf{x})$, then the first component of the dynamic term in (3.6) strictly dominates the corresponding term in (3.5).

Intuitively, the left-hand side of (3.5) can be interpreted as the payoff of a specific exotic option; it serves as the basic ingredient for more general exotic payoffs since any nondecreasing function ϕ admits the representation $\phi(x) = \phi(0) + \int_0^x \mathbf{1}_{\{x\ge m\}}\,d\phi(m)$. The right-hand side of (3.5) can be interpreted as a model-independent super-replicating semi-static strategy, the cost of which can be computed explicitly.

Minimizing the super-hedging cost yields the following optimization problem:

$$C_n(m) := \inf_{\zeta_1\le\cdots\le\zeta_n\le m}\sum_{i=1}^{n}\left(\frac{c_i(\zeta_i)}{m-\zeta_i} - \frac{c_i(\zeta_{i+1})}{m-\zeta_{i+1}}\mathbf{1}_{i<n}\right), \tag{3.7}$$

where $c_k(x) := \int_{\mathbb{R}}(y-x)^+\mu_{t_k}(dy)$. It is argued in [18] that the minimization problem (3.7) admits at least one solution $(\hat\zeta_k(m))_{1\le k\le n}$. An immediate consequence is that

$$D_n(\mu) = \tilde{D}_n(\mu) \le \phi(0) + \int_0^\infty C_n(m)\, d\phi(m). \tag{3.8}$$

Under further conditions (Assumption ⊛ in [36]; see Assumption 3.5 below), Obłój and Spoida [36] provide an iterative way to solve (3.7), and to obtain a family of continuous functions $(\xi_k)_{1\le k\le n}$ such that $\hat{\zeta}_k(m) = \min_{k\le i\le n} \xi_i(m), \forall m \ge 0$. Using the family of functions $(\xi_k)_{1\le k\le n}$, they further define a family of iterated Azéma–Yor embedding stopping times, given by $\tau_0 \equiv 0$ and

$$\tau_k := \inf\{t \ge \tau_{k-1} : B_t \le \xi_k(B_t^*)\}, \qquad k = 1, \dots, n. \tag{3.9}$$

The stopping times $(\tau_k)_{k=1,\dots,n}$ embed the marginals $(\mu_{t_k^n})_{k=1,\dots,n}$. Moreover, it is proven in [18] that the embedding satisfies

$$\mathbb{E}[\phi(W_{\tau_n}^*)] = \phi(0) + \int_0^\infty C_n(m)\, d\phi(m).$$

In consequence, under Assumption ⊛ in [36], it holds that

$$P_n(\mu) = \tilde{P}_n(\mu) = \tilde{D}_n(\mu) = D_n(\mu) = \phi(0) + \int_0^\infty C_n(m)\, d\phi(m). \tag{3.10}$$

We notice that the discrete process $(W_{\tau_k})_{1\le k\le n}$ resulting from this construction is in general not a Markov chain.

REMARK 3.4. In [18], the result (3.10) is formulated for the continuous martingale problem as defined in (3.3). However, it can be easily deduced that the solution is optimal also for the corresponding càdlàg martingale problem. Specifically, let $\tilde{\Omega}^d$ denote the space of all càdlàg functions on [0, 1], X the canonical process, $\tilde{\mathbb{F}}^d$ the canonical filtration and $\mathcal{M}^d$ the space of all martingale measures. Define

$$\tilde{P}_n^d(\mu) := \sup_{\tilde{\mathbb{P}}\in\mathcal{M}_n^d(\mu)} \mathbb{E}^{\tilde{\mathbb{P}}}[\phi(X_1^*)], \quad \text{with } \mathcal{M}_n^d(\mu) := \{\tilde{\mathbb{P}} \in \mathcal{M}^d : X_{t_k} \sim^{\tilde{\mathbb{P}}} \mu_{t_k}, \forall k\}.$$

It is clear that $\tilde{P}_n(\mu) \le \tilde{P}_n^d(\mu)$ since every continuous martingale is a càdlàg martingale. Further, by Monroe's [34] result, every càdlàg martingale can be represented as a time-changed Brownian motion. Since $\max_{0\le t\le 1} \omega_{\theta_t} \le \omega_{\theta_1}^*$ and ϕ is nondecreasing, it follows that $\tilde{P}_n^d(\mu) \le P_n(\mu)$. Therefore, according to (3.10), for the payoff $\xi(X_\cdot) := \phi(X_1^*)$ with $\phi : \mathbb{R} \to \mathbb{R}$ nondecreasing,

$$\tilde{P}_n(\mu) = \tilde{P}_n^d(\mu).$$

We conclude this section with a discussion of Assumption ⊛ in [36]. To this end, we need their explicit definition of the stopping barriers $(\xi_i)_{1\le i\le n}$ [cf. (3.9)].

Recall first the one-marginal Azéma–Yor embedding: let $b(x)$ be the barycenter function of μ given by

$$b(x) = b(x;\mu) := \frac{\int_{[x,\infty)} y\mu(dy)}{\mu([x,\infty))}\mathbf{1}_{\{l_\mu < x < r_\mu\}} + x\mathbf{1}_{\{x \geq r_\mu\}}, \tag{3.11}$$

where l_μ and r_μ is the left and right endpoint of the support of μ: $l_\mu := \sup\{x : \mu([x,\infty)) = 1\}$ and $r_\mu := \inf\{x : \mu((x,\infty)) = 0\}$. Then the Azéma–Yor embedding of μ is given by $\inf\{t \geq 0 : B_t \leq b^{-1}(B_t^*)\}$, where b^{-1} denotes the right-continuous inverse of b. As shown in [10], it holds that

$$b^{-1}(y) = \sup\left\{\operatorname*{argmin}_{\zeta \leq y} \frac{c(\zeta)}{y - \zeta}\right\}.$$

The construction in [36] provides a generalisation of this in that the barriers are defined as follows: with $c_0 = 0$ and $\xi_0 = -\infty$, having previously defined $\xi_1(y), \dots, \xi_{n-1}(y)$, let

$$\zeta_i^k(y) := \min_{i \leq j \leq k} \xi_j(y), \qquad y \geq 0, 1 \leq i \leq k,$$

for $k \leq n-1$, and define the subsequent stopping boundary by $\xi_n(0) = l_{\mu_n}$, and

$$\xi_n(y) := \sup\left\{\operatorname*{argmin}_{\zeta \leq y} \overline{K}_n\big(\zeta_1^{n-1}(y) \wedge \zeta, \dots, \zeta_{n-1}^{n-1}(y) \wedge \zeta, \zeta, y\big)\right\}, \quad y > 0, \tag{3.12}$$

where $\overline{K}_n(\zeta_1, \dots, \zeta_n, y) := \sum_{i=1}^n \frac{c_i(\zeta_i) - c_{i-1}(\zeta_i)}{y - \zeta_i}$, for $\zeta_1, \dots, \zeta_n \in (-\infty, y]$ and $y \geq 0$ [cf. (3.7)].

The relevant quantities for the n-marginal problem are clearly highly nontrivial. They are, however, crucially defined in an inductive manner. It is also clear that for the case of one and two marginals, they reduce to the solutions given in [2] and [10]. The key assumption in [36] now reads as follows:

ASSUMPTION 3.5 (Assumption ⊛ in [36]). The marginals $\mu_1, \dots, \mu_n$ satisfy conditions (2.1) and (2.2) in Assumption 2.1 and $c_{i-1} < c_i$ on (l_{μ_i}, r_{μ_i}), $i = 1, \dots, n$.[3] Further, for all $2 \leq i \leq n$, and all $0 < y < r_{\mu_i}$, the minimization problem in (3.12) admits a unique minimizer ζ^* on (l_{μ_i}, y).

The role of Assumption ⊛ is to rule out a certain interdependence between the marginals. More specifically, it is proven in [36] that under Assumption ⊛, the barriers $y \mapsto \xi(y)$ are continuous and increasing. Notably, for the two-marginal problem where the assumption is not needed, the case when this does not hold true calls for particular attention; see [10]. The assumption is nontrivial to verify and we refer to [36] for examples and further discussion (see Section 4 therein);

[3]Note that under assumption (2.1), (μ_n) is nondecreasing in convex order [i.e., (2.2) holds] if and only if $c_{i-1} \leq c_i$, $i = 1, \dots, n$.

we emphasize that for our results in Section 2.3, we do not need this condition to hold for any partition but it suffices that there is some sequence of partitions along which it holds. We also note that in [3], the existence of barriers $(\xi_i)_{1\le i\le n}$ such that $(\tau_i)_{1\le i\le n}$ given in (3.9) gives an optimal solution to the multi-marginal SEP has been established without Assumption $\circledast$. No explicit solution is however known for the general case.

3.2. *The optimal SEP given full marginals.* We now return to our optimal SEP (2.4) given full marginals. In fact, this problem is obtained as the limit of the problem given finitely many marginals; see the proof of Theorem 2.3. We provide here further discussion of the convergence of various optimal values and the corresponding optimizers.

3.2.1. *The limit of the MOT problem given finitely many marginals.* Our main motivation for studying the optimal SEP is the MOT problem, which has a natural interpretation and application in finance. For the case of finitely many marginal constraints, and for certain payoff functions, the optimal SEP $P_n(\mu)$ in (3.1) is equivalent to the MOT problem $\tilde{P}_n(\mu)$ in (3.3) (see Proposition 3.2).

When the number of marginals turns to infinity, the question arises whether the MOT problem (3.3) converges in some sense. Specifically, we are interested in the convergence of the optimal value and of the optimizer. The following convergence result is an immediate consequence of the proof of Theorem 2.3.

PROPOSITION 3.6. *Suppose that Assumption 2.1 holds true and let $\xi : \tilde{\Omega} \to \mathbb{R}$ be given by*

$$\xi(X_\cdot) = \Phi\big(X_{\langle X\rangle_\cdot^{-1}\wedge 1}, \langle X\rangle_1\big),$$

for some upper semicontinuous and bounded function $\Phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$. Let $\tilde{P}_n(\mu)$, $P_n(\mu)$ and $P(\mu)$ be defined with respect to ξ and Φ, respectively. Then we have the approximation result

$$\lim_{n\to\infty} \tilde{P}_n(\mu) = \lim_{n\to\infty} P_n(\mu) = P(\mu).$$

Further, the optimal transferences converge in the sense of the convergence of Skorokhod embeddings (i.e., the convergence of probability measures on $\overline{\Omega}$).

The above result implies that the full marginal SEP provides the limit of the n-marginal continuous MOT problem (i.e., of the robust prices).

For the lookback option, according to Remark 3.4, $P(\mu)$ is also the limit of the n-marginal càdlàg MOT problem. We note that in [17, 19], it is shown that for a certain class of payoffs, the limit of the n-marginal càdlàg MOT problem equals the value of the full marginal càdlàg MOT problem. Notably, the maximum payoff does not satisfy the assumptions imposed therein; see also Remark 3.7 below.

3.2.2. *The limit of the optimal martingale transference plan*. Our main results characterize, under technical conditions, the optimal value as well as the dual optimizer for the maximum reward function. As for the (primal) optimal Skorokhod embedding under full marginal constraints, we only have the existence result but no explicit construction of the optimal limiting object. However, as we now recall, in the particular setup of Madan and Yor [33] things are more explicit. In particular, one can then characterize the corresponding limiting martingale in an explicit way. To see this, suppose that the family $(\mu_t)_{t\in[0,1]}$ satisfies the so-called property of *increasing mean residual value*:

$$t \mapsto b_t(x) \qquad \text{is nondecreasing for every } x, \tag{3.13}$$

where $b_t(x) = b(x, \mu_t)$ is the barycenter function given in (3.11). For any discrete time grid $\pi_n : 0 = t_0^n < t_1^n < \cdots < t_n^n = 1$, it turns out that the boundary functions $(\xi_k)_{1\le k\le n}$ defined in [36] are then given by $\xi_k = b_{t_k}^{-1}$ (see Example 2.9 in [36]), and that the iterated Azéma–Yor embedding coincides with the Azéma–Yor embedding:

$$\tau_t := \inf\{s \ge 0 : B_s \le b_t^{-1}(B_s^*)\}. \tag{3.14}$$

Recall that the iterated Azéma–Yor embedding induces a continuous martingale, which is the optimal martingale transference for a class of lookback option given finitely many marginal constraints (cf. Proposition 3.2). It follows that under condition (3.13), this optimal martingale transference plan converges to $M = (M_t)_{t\in[0,1]}$, given by

$$M_t := B_{\tau_t}.$$

More precisely, since M may not be right-continuous by its definition, we should say the limiting martingale is the right-continuous modification of M, which does exist since $t \mapsto \mu_t$ is right-continuous. Moreover, it is proven in [33] that M is in fact a Markov process and that one can compute its generator in explicit form.

REMARK 3.7. (i) Similarly to the martingale optimal transport problem $\tilde{P}_n(\mu)$ under finitely many marginal constraints, we can introduce the corresponding full marginal version $\tilde{P}(\mu)$ by considering

$$\mathcal{M}(\mu) := \{\tilde{\mathbb{P}} \in \mathcal{M} : X_t \sim^{\tilde{\mathbb{P}}} \mu_t, t \in [0, 1]\}.$$

However, the convergence of the optimal SEP in Proposition 3.6 does not imply convergence towards this full marginal MOT problem. This since the limiting martingale may be càdlàg, and, crucially, we lose information of the Brownian motion because of the jumps of $t \mapsto B_{\tau_t}$.

(ii) As an example, let us take the maximum reward function $\xi(X) = \phi(X_1^*)$ (with increasing function ϕ). Under the increasing mean residual value condition (3.13), we know from Madan and Yor [33] that the optimal SEP $P(\mu)$

under full marginal constraints is solved by the sequence $(\tau_t)_{0\le t\le 1}$ defined in (3.14). However, it is clear that one has $M_1^* := \sup_{0\le t\le 1} B_{\tau_t} \le B_{\tau_1}^*$ for the resulting martingale $M_t := B_{\tau_t}$ due to the jumps of $t \mapsto \tau_t$. Therefore, in general, $\mathbb{E}[\phi(M_1^*)] \ne \mathbb{E}[\phi(B_{\tau_1}^*)]$ and M need not be an optimal martingale for the MOT problem given full marginals.[4]

3.2.3. *The limit of the pathwise inequality.* The proof of Theorem 2.9 is based on applying limiting arguments to the path-wise inequality (3.5) (see Section 4.3). By use of a similar arguments, we might obtain an almost sure inequality for càdlàg martingales.

PROPOSITION 3.8. *Let $(M_t)_{t\in[0,1]}$ be a right-continuous martingale, take $\zeta : [0,1) \times \mathbb{R}_+ \to \mathbb{R}$ such that $\zeta_\cdot^m \in \mathbb{V}_l^+$ and $\zeta_\cdot^m < m$ and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be bounded, continuous and nondecreasing. Suppose either* (i) *that ζ, ϕ and the marginals of M satisfy the conditions of Theorem 2.9; or* (ii) *that ζ admits the representation* (2.20). *Then, with λ^ζ given in* (2.16), *M satisfies the following inequality*:

$$\phi(M_1^*) \le \int_0^1 \lambda^\zeta(dt, M_t) + \int_0^1 \int_0^\infty \mathbf{1}_{\{m\le M_{t-}^*;\, \zeta_t^m \le M_{t-}\}} \frac{d\phi(m)}{m-\zeta_t^m}\, dM_t, \quad a.s. \tag{3.15}$$

The difference between the right-hand side of (3.15) and (2.19) appear in the dynamic term [cf. (2.15)]. Specifically, for the martingale formulation, the counterpart of the first dynamic term in (2.19) is always negative and thus vanishes from the inequality. This is related to the fact that the limit of the first dynamic component in (3.6) is zero. For continuous martingales, the two inequalities coincide.

3.3. *The resolution of $C(m)$.* Finally, we would like to discuss the resolution of the problem $C(m)$ in (2.12), for the main results in Theorems 2.7 and 2.9 rely on its solution $\hat{\zeta}$.

It is clear that we can decompose the minimization problem $C(m)$ as follows:

$$C(m) = \inf_{x<m} \left\{ \frac{c(0,x)}{m-x} + v(0,x) \right\}; \qquad v(0,x) := \inf_{\zeta\in\tilde{\mathbb{V}}_l^+,\, \zeta_0 = x} \int_0^1 \frac{\partial_t c(s,\zeta_s)}{m-\zeta_s}\, ds.$$

The problem of computing $v(0,x)$ is a standard singular deterministic control problem. When the function $\partial_t c(s,x)$ is continuous, it therefore follows by standard arguments (see, e.g., [11]) that v can be characterized as a viscosity solution

[4]To confirm this point, we did some simulations on M, which is a Markov martingale with generator explicitly given in [33]. Taking the marginals of a Brownian motion on [0, 1], we simulated M on a discrete grid $(t_k)_{0\le k\le n}$ for $t_k := \frac{k}{n}$ and $n = 200$: with $N = 10{,}000$ simulations, the Monte-Carlo estimation of $\mathbb{E}[M_1^*]$ is 0.6199571 with confidence interval [0.6086966, 0.6312176] (and confidence level 95%); for the Brownian motion W itself, however, one has $\mathbb{E}[W_1^*] = \mathbb{E}[|W_1|] \approx 0.7978846$.

to the PDE

$$\max\left\{-\partial_x v(t,x), -\partial_t v(t,x) - \frac{\partial_t c(t,x)}{m-x}\right\} = 0, \tag{3.16}$$

equipped with the terminal condition $v(1,x) = 0$, for all $x < m$.

We now propose a numerical scheme for the problem $C(m)$. To this end, for a given partition $\pi_n = \{t_1^n, \dots, t_n^n\}$, with $0 = t_0^n \le \dots \le t_n^n = 1$, let $\mathbb{V}_l^n$ be the subset of $\mathbb{V}_l^+$ for which ζ is constant on $(t_{i-1}, t_i]$, $i = 1, \dots, n$. Further, let

$$v^n(0,x) := \inf_{\substack{\zeta \in \mathbb{V}_l^n, \\ \zeta_0 = x, \zeta_1 < m}} \int_0^1 \frac{\partial_t c(s, \zeta_s)}{m - \zeta_s}\, ds.$$

For a sequence of partitions such that $|\pi_n| \to 0$, it follows that $v^n(0,x) \to v(0,x)$; cf. the proof of Lemma 4.2 below. On the other hand,

$$v^n(0,x) = \inf_{\substack{\zeta_i, i=1,\dots,n: \\ x \le \zeta_1 \le \dots \le \zeta_n < m}} \sum_{i=1}^n \frac{\Delta c(t_i^n, \zeta_i)}{m - \zeta_i},$$

with $\Delta c(t_i^n, \zeta) := c(t_i^n, \zeta) - c(t_{i-1}^n, \zeta)$. In consequence, $v^n(0,x) = \bar{v}^n(t_0^n, x)$, where $\bar{v}^n(t_k^n, x)$, $k = 0, \dots, n$, is iteratively defined by

$$\begin{cases} \bar{v}^n(t_k^n, x) = \inf_{0 \le y < m - x} \left(\bar{v}^n(t_{k+1}^n, x + y) + \dfrac{\Delta c(t_i^n, x + y)}{m - (x + y)} \right), & k \le n - 1, \\ \bar{v}^n(t_n^n, x) = 0. \end{cases}$$

This yields a scheme for explicit calculation of $v^n(0,x)$ as an approximation of $v(0,x)$.

4. Proofs.

4.1. *Technical lemmas.* In preparation for the first lemma, we define as follows: given $n \ge 1$, a partition $\pi : 0 = t_0^n < t_1^n < \dots < t_n^n = 1$ of $[0,1]$, $s = (s_0, s_1, \dots, s_n) \in (\mathbb{R}_+)^{n+1}$ and $\theta \in \mathbb{V}_r^+$, we define $\hat{\theta}^s = (\hat{\theta}_r^s)_{r \in [0,1]} \in \mathbb{V}_r^+$ by $\hat{\theta}_1^s := \max_{1 \le j \le n} s_j$ and

$$\hat{\theta}_r^s := \max_{0 \le j \le i} s_j \qquad \text{for } r \in [t_i^n, t_{i+1}^n), i = 0, 1, \dots, n-1;$$

further, we let $\hat{\theta}^\theta := \hat{\theta}^{\hat{s}}$, with $\hat{s} = (\hat{s}_i)_{0 \le i \le n}$ given by $\hat{s}_n := \theta_1$ and

$$\hat{s}_i := \frac{1}{t_{i+1}^n - t_i^n} \int_{t_i^n}^{t_{i+1}^n} \theta_s\, ds, \qquad i = 0, \dots, n-1. \tag{4.1}$$

LEMMA 4.1. *Let $\Phi : \overline{\Omega} \to \mathbb{R}$ be upper semicontinuous, bounded and such that $\Phi(\omega, \theta) = \Phi(\omega_{\cdot \wedge \theta_1}, \theta)$, for all $(\omega, \theta) \in \overline{\Omega}$. Further, let $(\pi_m)_{m \ge 1}$ be an increasing* *sequence of partitions of* $[0,1]$ (*i.e.*, $\pi_{m+1} \supset \pi_m$) *such that* $|\pi_m| \to 0$. *Then there exists a subsequence* $(m_n)_{n\geq 1}$ *and a sequence* $(\Phi_n)_{n\geq 1}$ *of bounded continuous functions* $\Phi_n : \Omega \times (\mathbb{R}_+)^{m_n} \to \mathbb{R}$ *such that*

$$\tilde{\Phi}_n(\overline{\omega}) = \Phi_n(\omega_{\cdot\wedge\theta_1}, \hat{s}_1, \dots, \hat{s}_{m_n}) \searrow \Phi(\overline{\omega}) \qquad \textit{as } n \to \infty, \forall \overline{\omega} \in \overline{\Omega}, \tag{4.2}$$

where $\hat{s}_i$ *is defined by* (4.1) *with partition* π_{m_n}. *In particular*, $(\tilde{\Phi}_n)_{n\geq 1}$ *is a nonincreasing sequence of bounded continuous functions defined on* $\overline{\Omega}$.

PROOF. Define $\Phi^k : \Omega \times \mathbb{V}_r^+ \to \mathbb{R}$, $k \in \mathbb{N}$, by $\Phi^k(\omega, \theta) := \widetilde{\Phi}^k(\omega_{\cdot\wedge\theta_1}, \theta)$, with $\widetilde{\Phi}^k(\omega, \theta) := (-k) \vee \sup\{\Phi(\omega', \theta') - kd((\omega, \theta), (\omega', \theta'))\}$, so that Φ^k is bounded and Lipschitz and $\Phi^k \searrow \Phi$ as $k \to \infty$. Next, let $\Phi^{k,m} : \overline{\Omega} \to \mathbb{R}$, $m \in \mathbb{N}$, be given by

$$\Phi^{k,m}(\omega, \theta) := \Phi^k(\omega, \hat{\theta}^\theta),$$

where $\hat{\theta}^\theta$ is defined above (4.1) with partition $\pi_m = \{t_k^m : 0 \leq k \leq m\}$ of $[0,1]$. In particular, $\Phi^{k,m}$ is a bounded continuous function defined on $\overline{\Omega}$ since $\theta \mapsto \theta_1$ and $\theta \mapsto \int_{t_i^m}^{t_{i+1}^m} \theta_s\, ds$ are all continuous. Note further that since $d(\theta, \hat{\theta}^\theta) \leq |\pi_m|$ [cf. (1.2)], we then have

$$\left|\Phi^{k,m}(\omega, \theta) - \Phi^k(\omega, \theta)\right| = \left|\Phi^k(\omega, \hat{\theta}^\theta) - \Phi^k(\omega, \theta)\right| \leq L_k |\pi_m|,$$

with L_k the Lipschitz constant associated with Φ^k. In consequence,

$$\hat{\Phi}^{k,m}(\omega, \theta) := \Phi^{k,m}(\omega, \theta) + L_k|\pi_m| \longrightarrow \Phi^k(\omega, \theta) \qquad \text{as } m \to \infty.$$

Hence, we may choose an increasing sequence $(m_k)_{k\geq 1}$ such that $\hat{\Phi}^{k,m_k}(\omega, \theta) \geq \Phi(\omega, \theta)$ and $\hat{\Phi}^{k,m_k}(\omega, \theta) \longrightarrow \Phi(\omega, \theta)$, as $k \to \infty$. In consequence, defining

$$\tilde{\Phi}_n(\overline{\omega}) := \min_{1\leq k\leq n} \hat{\Phi}^{k,m_k}(\overline{\omega}) \quad \text{and} \quad \Phi_n(\omega, s_1, \dots, s_{m_n}) := \tilde{\Phi}_n(\omega, \hat{\theta}^s),$$

we have that (4.2) holds true. Moreover, since $\Phi^{k,m} : \overline{\Omega} \to \mathbb{R}$ is bounded and continuous, both $\tilde{\Phi}_n$ and Φ_n are also bounded and continuous, which completes the proof. □

PROOF OF LEMMA 2.8. We follow the argument at the beginning of Section 3 in [18]. Let

$$\Psi_m(\zeta) := \frac{c(0, \zeta_0)}{m - \zeta_0} + \int_0^1 \frac{\partial_t c(s, \zeta_s)}{m - \zeta_s}\, ds. \tag{4.3}$$

We first consider a constant function $\hat{\zeta}_\cdot^z \equiv z$ for some constant $z < m$. By direct computation, it is easy to see that

$$\Psi_m(\hat{\zeta}^z) = \frac{c(1, z)}{m - z} \geq C(m).$$

Note that since $z \mapsto c(1,z)$ is convex and $\frac{c(1,z)}{m-z}$ is the slope of the tangent to $z \mapsto c(1,z)$ intersecting the x-axis in m, it follows that $C(m) < 1$.

On the other hand, since $\partial_t c(s,z) \geq 0$, we have

$$\Psi_m(\zeta) \geq \frac{c(0,\zeta_0)}{m-\zeta_0} \to 1 \qquad \text{as } \zeta_0 \to -\infty.$$

For the minimization problem $C(m)$ in (2.12), it is therefore enough to consider the space $\mathbb{V}_l^+([0,1),[K,m))$ for some constant $K \in (-\infty, m)$, that is,

$$C(m) = \inf_{\zeta \in \mathbb{V}_l^+([0,1),[K,m))} \Psi_m(\zeta).$$

Since, for elements in $\mathbb{V}_l^+$, convergence in the Lévy metric implies pointwise convergence at $t=0$, $\zeta \mapsto \Psi_m(\zeta)$ is continuous. Further, $\mathbb{V}_l^+([0,1),[K,m))$ is compact under the Lévy metric. In consequence, for every $m>0$, there exists at least one solution in $\mathbb{V}_l^+$ to (2.12). To conclude, it is enough to use a measurable selection argument to choose a measurable function $\hat{\zeta}$. $\square$

LEMMA 4.2. *Suppose that the function c is differentiable in t and that the derivative function $\partial_t c$ is continuous. Then, for every $m>0$, we have*

$$\lim_{n\to\infty} C_n(m) = C(m).$$

PROOF. Let $\mathbb{V}_l^n$ be the subset of $\mathbb{V}_l^+$ for which ζ is constant on $(t_{i-1}, t_i]$, $i = 1, \dots, n-1$, and on (t_{n-1}, t_n). For n fixed and $\zeta \in \mathbb{V}_l^n$, let $\zeta(t_n) := \zeta(t_n^-)$. Recall the definition of Ψ_m from (4.3) and notice that for every $\zeta \in \mathbb{V}_l^n$,

$$\begin{aligned}
\Psi_m(\zeta) &= \frac{c(0,\zeta_{t_0})}{m-\zeta_{t_0}} + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\partial_t c(s,\zeta_{t_i})}{m-\zeta_{t_i}}\,ds \\
&= \frac{c(0,\zeta_{t_0})}{m-\zeta_{t_0}} + \sum_{i=1}^n \left(\frac{c(t_i,\zeta_{t_i})}{m-\zeta_{t_i}} - \frac{c(t_{i-1},\zeta_{t_i})}{m-\zeta_{t_i}}\right) \\
&= \sum_{i=0}^n \left(\frac{c(t_i,\zeta_{t_i})}{m-\zeta_{t_i}} - \frac{c(t_i,\zeta_{t_{i+1}})}{m-\zeta_{t_{i+1}}}\mathbf{1}_{i<n}\right).
\end{aligned}$$

Since $x_0 \leq m$, it holds that $\frac{c(t_0,\zeta_{t_0})}{m-\zeta_{t_0}} - \frac{c(t_0,\zeta_{t_1})}{m-\zeta_{t_1}} \geq 0$. In consequence, recalling the definition of $C_n(m)$ from (3.7), we obtain

$$\inf_{\zeta \in \mathbb{V}_l^n, \zeta_1 \leq m} \Psi_m(\zeta) = \inf_{\zeta \in \mathbb{V}_l^n, \zeta_1 \leq m} \sum_{i=1}^n \left(\frac{c(t_i,\zeta_{t_i})}{m-\zeta_{t_i}} - \frac{c(t_i,\zeta_{t_{i+1}})}{m-\zeta_{t_{i+1}}}\mathbf{1}_{i<n}\right) = C_n(m).$$

Hence, the $C_n(m)$ are nonincreasing in n and

$$C_n(m) = \inf_{\zeta \in \mathbb{V}_l^n, \zeta_1 \leq m} \Psi_m(\zeta) \geq \inf_{\zeta \in \mathbb{V}_l^+, \zeta_1 \leq m} \Psi_m(\zeta) = C(m).$$

Next, for any $\zeta \in \mathbb{V}_l^+$, by direct truncation we can easily obtain a sequence ζ^n such that $\zeta^n \in \mathbb{V}_l^n$ and $\zeta^n \to \zeta$ under the Lévy metric. Since $\zeta \to \Psi_m(\zeta)$ is continuous, it follows that $C_n(m) \to C(m)$ as $n \to \infty$. □

LEMMA 4.3. *The mapping from* $\overline{\Omega}$ *to* $\mathbb{R}$ *given by*

$$(\omega, \theta) \longmapsto \omega^*\bigl(\theta(1)\bigr) := \sup_{0 \le s \le \theta(1)} \omega(s), \tag{4.4}$$

is continuous with respect to the product topology on $\overline{\Omega}$.

PROOF. Let $(\omega^n(\cdot), \theta^n(\cdot)) \in \overline{\Omega}$, $n \in \mathbb{N}$, be a sequence converging in the product topology to $(\tilde{\omega}(\cdot), \tilde{\theta}(\cdot)) \in \overline{\Omega}$. Recall that $C(\mathbb{R}_+, \mathbb{R})$ is equipped with the metric ρ defined in (1.1), which induces the topology of uniform convergence on compact subsets. Hence,

$$\lim_{n \to \infty} \sup_{0 \le s \le m} \bigl|\omega^n(s) - \tilde{\omega}(s)\bigr| = 0 \qquad \text{for all } m \ge 0. \tag{4.5}$$

Note that

$$\begin{aligned}\Bigl|\sup_{0 \le s \le \theta^n(1)} \omega^n(s) - \sup_{0 \le s \le \tilde{\theta}(1)} \tilde{\omega}(s)\Bigr| &= \Bigl|\sup_{0 \le s \le \theta^n(1)} \omega^n(s) - \sup_{0 \le s \le \theta^n(1)} \tilde{\omega}(s)\Bigr| \\ &\quad + \Bigl|\sup_{0 \le s \le \theta^n(1)} \tilde{\omega}(s) - \sup_{0 \le s \le \tilde{\theta}(1)} \tilde{\omega}(s)\Bigr|.\end{aligned}$$

The first term is dominated by $\sup_{0 \le s \le \theta^n(1)} |\omega^n(s) - \tilde{\omega}(s)|$ which tends to zero as n tends to infinity due to (4.5). Recall that for elements in $\mathbb{V}_r^+$, convergence in the Lévy metric implies pointwise convergence at $t = 1$ (see Remark A.1). Since $\tilde{\omega}(\cdot)$ is a continuous path, this implies that the second term tends to zero. Hence, the mapping in (4.4) is continuous and we conclude. □

LEMMA 4.4. *Let* $\tilde{\Omega}^d := D([0,1], \mathbb{R})$ *be the space of all càdlàg paths on* $[0,1]$ *with canonical process* X, *and* $\mathcal{M}^d$ *the space of all martingale measures on* $\tilde{\Omega}^d$. *We define*

$$\mathcal{M}^d(\mu) := \bigl\{\tilde{\mathbb{P}} \in \mathcal{M}^d : X_t \sim^{\tilde{\mathbb{P}}} \mu_t, \forall t \in [0,1]\bigr\}.$$

Further, let $\zeta : [0,1) \to (-\infty, m)$ *be a nondecreasing càglàd path on* $[0,1)$ *and* $\pi_n : 0 = t_0^n < \cdots < t_n^n = 1$ *a sequence of discrete time grids such that* $|\pi_n| \to 0$ *as* $n \to \infty$. *Let* ζ^c *be the continuous part of* ζ, *and let* $\lambda^{\zeta,m}$ *be given by* (2.14). *Then*:

(i) *If* μ_t *is atomless for all* $t \in [0,1]$, *it holds* $\mathcal{M}^d(\mu)$-*q.s. that as* $n \to \infty$,

$$\sum_{k=1}^n \biggl(\frac{(X_{t_k^n} - \zeta_{t_k^n})^+}{m - \zeta_{t_k^n}} - \frac{(X_{t_k^n} - \zeta_{t_{k+1}^n})^+}{m - \zeta_{t_{k+1}^n}} \mathbf{1}_{\{k<n\}}\biggr) \longrightarrow \int_0^1 \lambda^{\zeta,m}(X_t, dt).$$

(ii) *if* $\zeta_s = \sum_{k=0}^{\infty} \zeta_k \mathbf{1}_{(t_k, t_{k+1}]}(s)$, *then the convergence in* (i) *holds path-wise for all* $\mathbf{x} \in \tilde{\Omega}^d$. *The integral with respect to* $d\zeta^c$ *is then identically zero.*

PROOF. It follows from the definition of $\lambda^{\zeta,m}$, that in order to prove (i), it is sufficient to show that, $\mathcal{M}^d(\mu)$-q.s.,

$$\begin{aligned}&\sum_{k=1}^{n-1}\left(\frac{(X_{t_k^n}-\zeta_{t_{k+1}^n})^+}{m-\zeta_{t_{k+1}^n}}-\frac{(X_{t_k^n}-\zeta_{t_k^n})^+}{m-\zeta_{t_k^n}}\right)\\ &\quad\longrightarrow \int_0^1 \frac{X_t-m}{(m-\zeta_t)^2}\mathbf{1}_{\{X_t\geq\zeta_t\}}\,d\zeta_t^c+\sum_t\left(\frac{(X_t-\zeta_{t+})^+}{m-\zeta_{t+}}-\frac{(X_t-\zeta_t)^+}{m-\zeta_t}\right).\end{aligned} \tag{4.6}$$

Observe that for each path $\mathbf{x} \in D([0,1], \mathbb{R})$, the discrete sum in (4.6) might be written as $\int_0^1 f_n(t;\zeta)\,d\zeta_t$, where

$$f_n(t;\zeta)=\sum_{k=1}^{n-1}\frac{\mathbf{1}_{t\in(t_k^n,t_{k+1}^n]}}{\zeta_{t_{k+1}^n}-\zeta_{t_k^n}}\left(\frac{(\mathbf{x}_{t_k^n}-\zeta_{t_{k+1}^n})^+}{m-\zeta_{t_{k+1}^n}}-\frac{(\mathbf{x}_{t_k^n}-\zeta_{t_k^n})^+}{m-\zeta_{t_k^n}}\right).$$

Denote by $D_\zeta \subset (0,1)$ the subset of all discontinuous points of ζ. Now, under assumption (i), for $t \notin D_\zeta^c \cap \{t : \mathbf{x}_t = \zeta_t\}$, the $f_n(\cdot;\zeta)$ converge pointwise to $f(\cdot;\zeta)$, with

$$f(t;\zeta)=\begin{cases}\dfrac{\mathbf{x}_t-m}{(m-\zeta_t)^2}\mathbf{1}_{\{\mathbf{x}_t\geq\zeta_t\}}, & t\in D_\zeta^c,\\ \dfrac{1}{\zeta_{t+}-\zeta_t}\left(\dfrac{(\mathbf{x}_t-\zeta_{t+})^+}{m-\zeta_{t+}}-\dfrac{(\mathbf{x}_t-\zeta_t)^+}{m-\zeta_t}\right), & t\in D_\zeta.\end{cases}$$

On the other hand, by use of Fubini's theorem and assumption (i), we obtain that for $\mathbb{P} \in \mathcal{M}^d(\mu)$,

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^1 \mathbf{1}_{\{\mathbf{x}_t=\zeta_t\}}\,d\zeta_t^c\right]=\int_0^1 \mathbb{P}[\mathbf{x}_t=\zeta_t]\,d\zeta_t^c=\int_0^1 \mu_t(\{\zeta_t\})\,d\zeta_t^c=0.$$

That is to say, $\int_0^1 \mathbf{1}_{\{X_t=\xi_t\}}\,d\zeta_t^c = 0$, $\mathcal{M}^d(\mu)$-q.s. Since, for all $\varepsilon > 0$, $\zeta \mapsto \frac{(x-\zeta)^+}{m-\zeta}$ is Lipschitz on $(-\infty, m-\varepsilon]$, there is $K > 0$, such that $f_n(t) \leq K$, $t \in [0,1]$, $n \geq 0$. Hence, by use of dominated convergence we obtain $\int_0^1 f_n(t;\zeta)\,d\zeta_t \to \int_0^1 f(t;\zeta)\,d\zeta_t$, $\mathcal{M}^d(\mu)$-q.s., which implies (4.6).

Next, suppose that assumption (ii) holds. Then, for $t \in [0,1]$, the function $f_n(t;\zeta)$ converges pointwise to $f^0(t;\zeta)$, where

$$f^0(t;\zeta)=\begin{cases}0, & t\in D_\zeta^c,\\ \dfrac{1}{\zeta_{t+}-\zeta_t}\left(\dfrac{(\mathbf{x}_t-\zeta_{t+})^+}{m-\zeta_{t+}}-\dfrac{(\mathbf{x}_t-\zeta_t)^+}{m-\zeta_t}\right), & t\in D_\zeta.\end{cases}$$

By use of the same arguments as in case (i), we may then pathwise apply the dominated convergence theorem and easily conclude. □

4.2. *Proof of Theorem* 2.3. We first argue that the optimal SEP given finitely many marginals, defined in (3.1), may be reformulated similarly to problem (2.6). Concretely, for a given discrete time grid $\pi_n : 0 = t_0^n < \cdots < t_n^n = 1$, we call (μ, π_n)-embedding a term

$$\alpha = \big(\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{F}^\alpha = (\mathcal{F}_t^\alpha)_{t\geq 0}, \mathbb{P}^\alpha, (W_t^\alpha)_{t\geq 0}, (T_k^\alpha)_{k=1,\dots,n}\big), \tag{4.7}$$

such that in the filtered space $(\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{F}^\alpha, \mathbb{P}^\alpha)$, $W_\cdot^\alpha$ is a Brownian motion, $T_1^\alpha \leq \cdots \leq T_n^\alpha$ are stopping times, the stopped process $(W^\alpha_{T_n^\alpha \wedge \cdot})$ is uniformly integrable, and $W^\alpha_{T_k^\alpha} \sim^{\mathbb{P}^\alpha} \mu_{t_k^n}$ for each $k = 1, \dots, n$. We also extend the sequence of stopping times T^α to a nondecreasing process, $\tilde{T}^\alpha = (\tilde{T}_s^\alpha)_{s\in[0,1]}$, by defining $\tilde{T}_s^\alpha := 0$ for $s \in [0, t_1^n)$,

$$\tilde{T}_s^\alpha := T_k^\alpha, \text{ for } s \in [t_k^n, t_{k+1}^n), k = 1, \dots, n-1, \text{ and } \tilde{T}_1^\alpha := T_n^\alpha.$$

Let $\mathcal{A}_n(\mu)$ denote the collection of all (μ, π_n)-embeddings α. Then it is clear that every term in $\mathcal{A}_n(\mu)$ induces on the canonical space $\overline{\Omega}$ a probability measure in $\mathcal{P}_n(\mu)$, and that every probability measure $\mathbb{P} \in \mathcal{P}_n(\mu)$ together with the space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}})$ forms a (μ, π_n)-term in $\mathcal{A}_n(\mu)$. Hence, for a given reward function $\Phi_n : \Omega \times (\mathbb{R}_+)^n \to \mathbb{R}$, we have that

$$P_n(\mu) = \sup_{\alpha \in \mathcal{A}_n(\mu)} \mathbb{E}^{\mathbb{P}^\alpha}\big[\Phi_n\big(W_\cdot^\alpha, T_1^\alpha, \dots, T_n^\alpha\big)\big]. \tag{4.8}$$

Before proving Theorem 2.3, we present a lemma. Its proof is partly adapted from the proof of Theorem 11 in Monroe [34] and that of Theorem 3.10 in Jakubowski [29].

LEMMA 4.5. *Let $\alpha_n \in \mathcal{A}_n(\mu)$, $n \in \mathbb{N}$, be a sequence of terms of the form* (4.7). *Let $\mathbb{P}_n$ be the probability measure on $\overline{\Omega}$ induced by $(W_\cdot^{\alpha_n}, \tilde{T}_\cdot^{\alpha_n})$ in the probability space $(\Omega^{\alpha_n}, \mathcal{F}^{\alpha_n}, \mathbb{P}^{\alpha_n})$. Then the sequence $\{\mathbb{P}_n\}_{n\geq 1}$ is tight, and any limiting point $\mathbb{P}$ lies in $\mathcal{P}(\mu)$.*

PROOF. (i) We first argue that the sequence $\{\mathbb{P}_n\}_{n\geq 1}$ is tight. To this end, note that the projection measure $\mathbb{P}_n|_\Omega$ on Ω is the Wiener measure for every $n \geq 1$, and hence the sequence $(\mathbb{P}_n|_\Omega)_{n\geq 1}$ is trivially tight. Next, since $\tilde{T}_1^{\alpha_n}$ are all minimal stopping times in the sense of Monroe [34], it follows from Proposition 7 in [34] that

$$\mathbb{P}_n(T_1 \geq \lambda) = \mathbb{P}^{\alpha_n}\big(\tilde{T}_1^{\alpha_n} \geq \lambda\big) \leq \lambda^{-1/3}\big(\mathbb{E}^{\mathbb{P}^{\alpha_n}}\big[\big|W^{\alpha_n}_{\tilde{T}_1^{\alpha_n}}\big|\big]^2 + 1\big), \qquad \forall \lambda > 0.$$

Let A_λ be the set of functions in $\mathbb{V}_r^+([0,1], \mathbb{R}_+)$ which are bounded by $\lambda > 0$ and π the projection of $\overline{\Omega}$ onto $\mathbb{V}_r^+([0,1], \mathbb{R}_+)$. It follows that

$$\mathbb{P}_n\big(\pi^{-1}(A_\lambda)\big) = \mathbb{P}_n(T_1 \leq \lambda) \geq 1 - \lambda^{-1/3}\big(\big(\mu_1\big(|x|\big)\big)^2 + 1\big).$$

Since A_λ, $\lambda > 0$, is compact (see Remark A.1), it follows that $\{\mathbb{P}_n\}_{n\geq 1}$ is tight.

(ii) Let $\mathbb{P}$ be a limiting point of $(\mathbb{P}_n)_{n\geq 1}$; by taking subsequences if necessary, we can assume that $\mathbb{P}_n \to \mathbb{P}$. We now prove that B is an $\overline{\mathbb{F}}$-Brownian motion under the limiting measure $\mathbb{P}$. Since the measures $\mathbb{P}_n$ are induced by $(W^{\alpha_n}, \tilde{T}^{\alpha_n})$ under $\mathbb{P}^{\alpha_n}$, we know that B is an $\overline{\mathbb{F}}$-Brownian motion under each $\mathbb{P}_n$, $n \geq 1$. Let $t > s$, $0 < \varepsilon < t - s$, and $\phi : \overline{\Omega} \to \mathbb{R}$ be a bounded continuous function which is $\overline{\mathcal{F}}_{s+\varepsilon}$-measurable. Then, for every $\varphi \in C_b^2(\mathbb{R})$, we have

$$\mathbb{E}^{\mathbb{P}_n}\left[\phi(B_\cdot, T_\cdot)\left(\varphi(B_t) - \varphi(B_{s+\varepsilon}) - \int_{s+\varepsilon}^t \frac{1}{2}\varphi''(B_u)\,du\right)\right] = 0.$$

By taking the limit $n \to \infty$, it follows that

$$\mathbb{E}^{\mathbb{P}}\left[\phi(B_\cdot, T_\cdot)\left(\varphi(B_t) - \varphi(B_{s+\varepsilon}) - \int_{s+\varepsilon}^t \frac{1}{2}\varphi''(B_u)\,du\right)\right] = 0. \tag{4.9}$$

According to Lemma A.2, the equality (4.9) holds true also for every bounded random variable $\phi : \overline{\Omega} \to \mathbb{R}$ that is $\overline{\mathcal{F}}_s$-measurable. Let $\varepsilon \to 0$, it follows that for every $\phi : \overline{\Omega} \to \mathbb{R}$ bounded and $\overline{\mathcal{F}}_s$-measurable, and every $\varphi \in C_b^2(\mathbb{R})$,

$$\mathbb{E}^{\mathbb{P}}\left[\phi(B_\cdot, T_\cdot)\left(\varphi(B_t) - \varphi(B_s) - \int_s^t \frac{1}{2}\varphi''(B_u)\,du\right)\right] = 0.$$

Hence, B is an $\overline{\mathbb{F}}$-Brownian motion under $\mathbb{P}$.

(iii) We now show that the process $(B_{t\wedge T_1})_{t\geq 0}$ is uniformly integrable under $\mathbb{P}$. For every $\varepsilon > 0$, there is $K_\varepsilon > 0$ such that

$$\int_{\mathbb{R}} (|x| - K_\varepsilon)^+ \mu_1(dx) \leq \varepsilon.$$

Since $|x|\mathbf{1}_{\{|x|\geq 2K\}} \leq 2(|x| - K)^+$, it follows that

$$\mathbb{E}^{\mathbb{P}_n}\big[|B_{T_1\wedge t}|\mathbf{1}_{\{|B_{T_1\wedge t}|\geq K_\varepsilon\}}\big] \leq 2\mathbb{E}^{\mathbb{P}_n}\big[(|B_{T_1}| - K_\varepsilon)^+\big] \leq 2\varepsilon, \qquad \forall t \geq 0.$$

Then, for every bounded continuous function $p : \mathbb{R} \to \mathbb{R}$ such that $p(x) \leq |x|\mathbf{1}_{\{|x|\geq 2K_\varepsilon\}}$, it follows by the dominated convergence theorem that

$$\mathbb{E}^{\mathbb{P}}\big[p(B_{t\wedge T_1})\big] = \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_n}\big[p(B_{t\wedge T_1})\big] \leq 2\varepsilon, \qquad \forall t \geq 0,$$

which implies that $(B_{t\wedge T_1})_{t\geq 0}$ is uniformly integrable under $\mathbb{P}$.

(iv) Next, we prove that $B_{T_t} \sim^{\mathbb{P}} \mu_t$, $t \in [0, 1]$. We shall adapt an idea from the proof of Theorem 3.10 in Jakubowski [29]. For every n, let α_n be the solution of the optimal SEP (3.1) and denote by m^{α_n} the random measure on $([0, 1], \mathcal{B}([0, 1]))$ defined by

$$m^{\alpha_n}([0, t], \omega) := \frac{\tilde{T}_t^{\alpha_n}(\omega)}{1 + \tilde{T}_1^{\alpha_n}(\omega)}, \qquad \forall t \in [0, 1].$$

Notice that m_n takes values in the space $M_1^+([0, 1])$ of all positive measures on $[0, 1]$ with mass less than 1. Since $[0, 1]$ is compact, $M_1^+([0, 1])$ is tight, and relatively compact by Prokhorov's theorem. We can then easily check that $M_1^+([0,1])$ is compact under the weak convergence topology. Therefore, the sequence $(\mathbb{P}^{\alpha_n} \circ (m_n)^{-1})_{n\geq 1}$ is tight, and hence $(\mathbb{P}^{\alpha_n} \circ (W^{\alpha_n}, \tilde{T}_1^{\alpha_n}, m_n)^{-1})_{n\geq 1}$ is tight. By taking subsequences, we can assume that $\mathbb{P}^{\alpha_n} \circ (W^{\alpha_n}, \tilde{T}_1^{\alpha_n}, m_n)^{-1}$ converges. Next, using the Skorokhod representation theorem, we can further assume that there is some probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ in which there are processes $(W^{*,n}, T_1^{*,n}, m^{*,n})_{n\geq 1}$, such that $\mathbb{P}^{\alpha_n} \circ (W^{\alpha_n}, \tilde{T}_1^{\alpha_n}, m^{\alpha_n})^{-1} = \mathbb{P}^* \circ (W^{*,n}, T_1^{*,n}, m^{*,n})^{-1}$, for all $n \geq 1$, and

$$\big(W^{*,n}, T_1^{*,n}, m^{*,n}\big) \to \big(W^*, T^*, m^*\big), \qquad \mathbb{P}^*\text{-a.s.}$$

Further, the map $t \mapsto \mathbb{E}^{\mathbb{P}^*}[m^*([0,t])]$, from $[0,1]$ to $\mathbb{R}$, is nondecreasing, and hence admits at most countably many discontinuous points. It follows that there is some countable set $\mathbb{Q}_1 \subset [0,1)$ such that $\mathbb{E}^{\mathbb{P}^*}[m^*(\{t\})] = 0$, for every $t \in [0,1] \setminus \mathbb{Q}_1$. Thus, for every $t \in [0,1] \setminus \mathbb{Q}_1$, we have $\mathbb{P}^*$-a.s. that $m^{*,n}([0,t]) \to m^*([0,t])$, and hence $T_t^{*,n} \to T_t^*$, $\mathbb{P}^*$-a.s. In particular, we have

$$W^{*,n}_{T_t^{*,n}} \to W^*_{T_t^*}, \qquad \mathbb{P}^*\text{-a.s. } \forall t \in [0,1] \setminus \mathbb{Q}_1.$$

Moreover, by Hirsch and Roynette [21] Lemma 4.1, there exists a countable set $\mathbb{Q}_2 \subset [0,1]$ such that $t \mapsto \mu_t$ is continuous at any $s \in [0,1] \setminus \mathbb{Q}_2$. Hence, for every $t \in [0,1] \setminus (\mathbb{Q}_1 \cup \mathbb{Q}_2)$, we have for any limit point $\mathbb{P}$ of $(\mathbb{P}_n)_{n\geq 1}$,

$$\mathbb{P} \circ (B_{T_t})^{-1} = \mathbb{P}^* \circ \big(W^*_{T_t^*}\big)^{-1} = \mu_t. \tag{4.10}$$

By the right continuity of $t \mapsto \mu_t$, it follows that (4.10) holds true for every $t \in [0,1]$.

In summary, we have proven that in the filtered space $(\overline{\Omega}, \overline{\mathcal{F}}_\infty, \overline{\mathbb{F}}, \mathbb{P})$, B is a Brownian motion, T_1 is a minimal stopping time and $B_{T_t} \sim \mu_t$ for every $t \in [0,1]$. We easily conclude. □

PROOF OF THEOREM 2.3. By taking expectation over each side of the inequality defining $\mathcal{D}(\mu)$ in (2.8), for all $\mathbb{P} \in \mathcal{P}(\mu)$, we easily obtain the weak duality $P(\mu) \leq D(\mu)$. Next, consider an increasing sequence $(\pi_m)_{m\geq 1}$ of partitions of $[0,1]$ such that $|\pi_m| \to 0$ as $m \to \infty$. Let $(m_n)_{n\in\mathbb{N}}$, $(\Phi_n)_{n\in\mathbb{N}}$ and $(\tilde{\Phi}_n)_{n\in\mathbb{N}}$ be the sequences of functions approximating Φ as given in Lemma 4.1. Further, let $P_n(\mu)$ and $D_n(\mu)$ be the primal and dual n-marginal problems defined w.r.t. Φ_n in (3.1) and (3.2). Since $\Phi_n \geq \Phi$, it follows that

$$D_n(\mu) \geq D(\mu). \tag{4.11}$$

For each $n \in \mathbb{N}$, let $\alpha_n \in \mathcal{A}_n(\mu)$ be the solution of the optimal SEP $P_n(\mu)$ defined with respect to Φ_n in (4.8). Let $\mathbb{P}_n$ be the probability measure on $\overline{\Omega}$ induced by $(W^{\alpha_n}_\cdot, \tilde{T}^{\alpha_n}_\cdot)$ in the probability space $(\Omega^{\alpha_n}, \mathcal{F}^{\alpha_n}, \mathbb{P}^{\alpha_n})$. Then, according to Lemma 4.5, the sequence $\{\mathbb{P}_n\}_{n\geq 1}$ is tight, and $\mathbb{P} \in \mathcal{P}(\mu)$, when $\mathbb{P}$ is a limiting point of $\{\mathbb{P}_n\}_{n\geq 1}$.

Note that by taking sub-sequences if necessary, we can assume that $\mathbb{P}_n \to \mathbb{P}$. Further, notice that $(\tilde{\Phi}_n)_{n\geq 1}$ is a nonincreasing sequence of bounded continuous functions approximating Φ by Lemma 4.1. By use of the monotone convergence theorem and the optimality of $\mathbb{P}_n$ for problem (3.1), for $n \in \mathbb{N}$, it therefore follows that

$$P(\mu) \geq \mathbb{E}^{\mathbb{P}}\big[\Phi(B_\cdot, T_\cdot)\big] = \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\big[\Phi_n(B_\cdot, T_\cdot)\big] \geq \lim_{n\to\infty}\Big(\lim_{k\to\infty} \mathbb{E}^{\mathbb{P}_k}\big[\Phi_n(B_\cdot, T_\cdot)\big]\Big)$$

$$\geq \lim_{n\to\infty}\Big(\lim_{k\to\infty} \mathbb{E}^{\mathbb{P}_k}\big[\Phi_k(B_\cdot, T_\cdot)\big]\Big) = \lim_{n\to\infty} P_n(\mu).$$

In consequence, since $P_n(\mu) \geq P(\mu)$ for all $n \geq 1$, we have that

$$\lim_{n\to\infty} P_n(\mu) = P(\mu). \tag{4.12}$$

Since the Φ_n satisfy (4.2), we may apply the duality result for the optimal SEP with finitely many marginal constraints (see Proposition 3.1). Hence, it follows from (4.12) combined with (4.11) that $P(\mu) \geq D(\mu)$. Combined with the weak duality, this yields $P(\mu) = D(\mu)$. As a by-product, we also obtain that $\mathbb{P}$ is an optimal embedding for the optimal SEP (2.4). This completes the proof. □

REMARK 4.6. The proof of Theorem 2.3 is based on an approximation argument together with the duality result for the problem given finitely many marginals. Specifically, if Φ can be approximated from above by a sequence $(\Phi_n)_{n\geq 1}$, where for each $n \geq 1$, Φ_n depends only on $(t_1^n, \dots, t_n^n)$ and $\omega_{\theta_1 \wedge \cdot}$, and admits a duality result under n marginal constraints, then one can obtain the same duality result for the full marginal case.

PROOF OF PROPOSITION 3.6. Given the form of $\xi : \tilde{\Omega} \to \mathbb{R}$, Proposition 3.2 applies. Hence, $\tilde{P}_n(\mu) = P_n(\mu)$. Next, note that $P_n(\mu)$ is of the form (3.1), for all $n \in \mathbb{N}$. Let $\mathbb{P}_n$ be the optimal measure for $P_n(\mu)$. Then, according to Lemma 4.5, passing to a subsequence if necessary, $\mathbb{P}_n \to \mathbb{P}$, with $\mathbb{P} \in \mathcal{P}(\mu)$. It follows that

$$P(\mu) \geq \mathbb{E}^{\mathbb{P}}\big[\Phi(B_\cdot, T_\cdot)\big] \geq \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_n}\big[\Phi(B_\cdot, T_\cdot)\big] = \lim_{n\to\infty} P_n(\mu).$$

Since $P_n(\mu) \geq P(\mu)$, for $n \geq 1$, we easily conclude. □

4.3. *Proof of Theorems 2.7 and 2.9.*

PROOF OF THEOREM 2.7. By Lemma 4.3, the mapping

$$\big(\omega(\cdot), \theta(\cdot)\big) \longmapsto \sup_{0\leq s\leq \theta(1)} \omega(s)$$

is continuous with respect to the product topology on $\overline{\Omega}$. Hence, Theorem 2.3 applies and $\lim_{n\to\infty} P_n(\mu) = P(\mu) = D(\mu)$. Moreover, for the optimal SEP with finitely many given marginals, we have [see (3.8)]

$$P_n(\mu) = D_n(\mu) \leq \phi(0) + \int_0^\infty C_n(m)\, d\phi(m),$$

where equality holds under Assumption 2.6(ii) [see (3.10)]. It thus suffices to use Lemma 4.2 together with the monotone convergence theorem to deduce that

$$\lim_{n\to\infty}\int_0^\infty C_n(m)\,d\phi(m)=\int_0^\infty C(m)\,d\phi(m). \qquad \square$$

PROOF OF THEOREM 2.9. Due to assumption (2.17), the pair $(\hat{\lambda}, \widehat{H})$ is well-defined and $(\hat{\lambda}, \widehat{H}) \in \Lambda(\mu) \times \mathcal{H}$. According to (2.7) and (2.14), for $m > 0$ and $\zeta \in \mathbb{V}_l^+$ such that $\zeta < m$, the cost of $\lambda^{\zeta,m}$ is given by

$$\begin{aligned}\mu(\lambda^{\zeta,m}) &= \frac{c(1,\zeta_1)}{m-\zeta_1} - \sum_{t\in D}\left[\frac{c(t,\zeta_{t+})}{m-\zeta_{t+}} - \frac{c(t,\zeta_t)}{m-\zeta_t}\right] - \int_0^1 \frac{\partial}{\partial\zeta}\left\{\frac{c(t,\zeta)}{m-\zeta}\right\}\Big|_{\zeta=\zeta_t^c} d\zeta_t^c \\ &= \frac{c(0,\zeta_0)}{m-\zeta_0} + \int_0^1 \frac{\partial_t c(s,\zeta_s)}{m-\zeta_s}\,ds,\end{aligned}$$

where it was used that $\frac{\partial}{\partial\zeta}\frac{(x-\zeta)^+}{m-\zeta} = \mathbf{1}_{\{x\ge\zeta\}}\frac{x-m}{(m-\zeta)^2}$. Since $\hat{\zeta}_\cdot^m$ minimizes (2.12), it follows that $\mu(\lambda^{\hat{\zeta}^m,m}) = C(m)$. Integration w.r.t. $d\phi(m)$ and application of Theorem 2.7 and Corollary 2.5 yields $\mu(\hat{\lambda}) = D_0(\mu)$.

Next, let $\pi_n : 0 = t_0^n < \cdots < t_n^n = 1$, $n \in \mathbb{N}$, be a sequence of discrete time grids such that $|\pi_n| \to 0$, as $n \to \infty$. According to Proposition 3.3, for $(\theta,\omega) \in \overline{\Omega}$,

$$\begin{aligned}\mathbf{1}_{\{\omega^*_{\theta(1)}\ge m\}} &\le \sum_{i=1}^n\left(\frac{(\omega_{\theta(t_i)}-\hat{\zeta}^m_{t_i})^+}{m-\hat{\zeta}^m_{t_i}} - \frac{(\omega_{\theta(t_i)}-\hat{\zeta}^m_{t_{i+1}})^+}{m-\hat{\zeta}^m_{t_{i+1}}}\mathbf{1}_{\{i<n\}}\right) \\ &\quad + \sum_{i=1}^n \mathbf{1}_{\{\omega^*_{\theta(t_{i-1})}<m\le\omega^*_{\theta(t_i)}\}}\frac{m-\omega_{\theta(t_i)}}{m-\hat{\zeta}^m_{t_i}} \\ &\quad - \sum_{i=1}^{n-1}\mathbf{1}_{\{m\le\omega^*_{\theta(t_i)};\,\hat{\zeta}^m_{t_{i+1}}\le\omega_{\theta(t_i)}\}}\frac{\omega_{\theta(t_{i+1})}-\omega_{\theta(t_i)}}{m-\hat{\zeta}^m_{t_{i+1}}}.\end{aligned} \tag{4.13}$$

Note that the left-hand side does not depend on the partition. Hence, in order to verify that $(\hat{\lambda}, \widehat{H})$ satisfies (2.19), it suffices to show, for all $\mathbb{P} \in \mathcal{P}(\mu)$, that the right-hand side in (4.13) integrated w.r.t. $d\phi(m)$ converges $\mathbb{P}$-a.s. to $\int_0^1 \hat{\lambda}(B_{T_s}, ds) + \int_0^{T_1} \widehat{H}_s\,dB_s$.

Application of Lemma 4.4, with $X_t = B_{T(t)}$ and $\zeta = \hat{\zeta}^m$, yields that the static term in (4.13) converges to

$$\int_0^1 \lambda^{\hat{\zeta}^m,m}(B_{T(t)}, dt), \qquad \mathcal{P}(\mu)\text{-q.s.}$$

In consequence, integrated w.r.t. $d\phi(m)$, the static term in (4.13) converges to $\hat{\lambda}(\overline{B}) = \int_0^1 \hat{\lambda}(B_{T_s}, ds)$, $\mathcal{P}(\mu)$-q.s.

As for the first dynamic term in (4.13), the definition of τ_m yields

$$\sum_{i=1}^{n} \mathbf{1}_{\{\omega^*_{\theta(t_{i-1})} < m \le \omega^*_{\theta(t_i)}\}} \frac{m - \omega_{\theta(t_i)}}{m - \hat{\zeta}^m_{t_i}} \longrightarrow \int_0^{\theta(1)} \frac{\mathbf{1}_{[\tau_m, I^+(\tau_m)]}(s)}{m - \hat{\zeta}^m_{\theta^{-1}(\tau_m)}} \, dB_s, \qquad \mathcal{P}\text{-q.s.}$$

Integration with respect to $d\phi$ then gives the convergence of the corresponding terms.

To prove the convergence of the second dynamic term in (4.13), we first integrate with respect to $d\phi$. The integrated term may then be rewritten as follows:

$$\begin{aligned} &\sum_{i=1}^{n-1} \int_0^\infty \mathbf{1}_{\{m \le \omega^*_{\theta(t_i)}; \hat{\zeta}^m_{t_{i+1}} \le \omega_{\theta(t_i)}\}} \frac{d\phi(m)}{m - \hat{\zeta}^m_{t_{i+1}}} (\omega_{\theta(t_{i+1})} - \omega_{\theta(t_i)}) \\ &\quad = \sum_{i=1}^{n-1} \int_0^{\hat{\zeta}^{-1}_{t_{i+1}}(\omega_{\theta(t_i)}) \wedge \omega^*_{\theta(t_i)}} \frac{d\phi(m)}{m - \hat{\zeta}^m_{t_{i+1}}} \int_{\theta(t_i)}^{\theta(t_{i+1})} dB_s \\ &\quad = \int_0^{\theta(1)} \sum_{i=1}^{n-1} \int_0^{\hat{\zeta}^{-1}_{t_{i+1}}(\omega_{\theta(t_i)}) \wedge \omega^*_{\theta(t_i)}} \frac{d\phi(m)}{m - \hat{\zeta}^m_{t_{i+1}}} \mathbf{1}_{(\theta(t_i), \theta(t_{i+1})]}(s) \, dB_s, \end{aligned} \tag{4.14}$$

where $\hat{\zeta}^{-1}_{t_{i+1}}$ denotes the inverse of $\hat{\zeta}^{\cdot}_{t_{i+1}}$. We denote the integrand in (4.14) by H^n. Note that the H^n are predictable. Further, since $\hat{\zeta}$ satisfies (2.17) and ϕ is bounded, the H^n are uniformly bounded. Due to assumption (ii), we also have that $H_n \to H$ on $\Omega \times (0, \infty)$, with

$$H_s = \int_0^{\hat{\zeta}^{-1}_{\theta^{-1}(s)}(\omega_{I^-(s)}) \wedge \omega^*_{I^-(s)}} \frac{d\phi(m)}{m - \hat{\zeta}^m_{\theta^{-1}(s)}}.$$

In consequence, for all $\mathbb{P} \in \mathcal{P}$, we have that $\int_0^{\theta(1)} H^n_s \, dB_s \to \int_0^{\theta(1)} H_s \, dB_s$ in probability. Hence, convergence holds a.s. along a subsequence and we conclude. □

4.4. *Proof of Propositions* 3.8 *and* 2.11.

PROOF OF PROPOSITION 3.8. Let $m > 0$ fixed and take $\phi(x) = \mathbf{1}_{\{x \ge m\}}$; the general case follows by integration with respect to $d\phi(m)$. Recall that (3.5) holds for all càdlàg paths $\mathbf{x}$ (although in the proof of Theorem 2.9 we only made use of this result for continuous paths). For a given sequence of partitions $\pi_n : 0 = t^n_0 < \cdots < t^n_n = 1$, such that $|\pi_n| \to 0$ as $n \to \infty$, it therefore suffices to argue that the right-hand side in (3.5) converges to the right-hand side of (3.15). The static term in (3.15) coincides with the static term in (2.19). Hence, the convergence follows by use of the same arguments as in the proof of Theorem 2.9. Next, consider the first dynamic term in (3.5). For each càdlàg path $\mathbf{x} \in D([0, 1], \mathbb{R})$, let $\tau^m(\mathbf{x}) := \inf\{t \ge 0 : \mathbf{x}_t \ge m\}$. Due to the right-continuity of $\mathbf{x}$, we have $\mathbf{x}_{\tau^m(\mathbf{x})} \ge m$. In consequence,

$$\lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{1}_{\{\mathbf{x}^*_{t_{i-1}} < m \le \mathbf{x}^*_{t_i}\}} \frac{m - \mathbf{x}_{t_i}}{m - \zeta_{t_i}} = \frac{m - \mathbf{x}_{\tau^m(\mathbf{x})}}{m - \zeta_{\tau^m(\mathbf{x})}} \le 0.$$

Next, consider the second dynamic term in (3.5). First, we argue its convergence under assumption (ii). To this end, we rewrite it as follows:

$$\sum_{i=1}^{n-1} \mathbf{1}_{\{m \leq \mathbf{x}^*_{t_i}; \zeta_{t_{i+1}} \leq \mathbf{x}_{t_i}\}} \frac{\mathbf{x}_{t_{i+1}} - \mathbf{x}_{t_i}}{m - \zeta_{t_{i+1}}} = \sum_{i=1}^{n-1} \mathbf{1}_{\{m \leq \mathbf{x}^*_{t_i}; \zeta_{t_{i+1}} \leq \mathbf{x}_{t_i}\}} \int_{t_i}^{t_{i+1}} \frac{dM_t}{m - \zeta_{t_{i+1}}}$$
$$= \int_0^1 \sum_{i=1}^{n-1} \mathbf{1}_{\{m \leq \mathbf{x}^*_{t_i}; \zeta_{t_{i+1}} \leq \mathbf{x}_{t_i}\}} \frac{\mathbf{1}_{(t_i, t_{i+1}]}(t)}{m - \zeta_{t_{i+1}}} \, dM_t.$$

Denote the integrand on the right-hand side by H^n. Note that the H^n are predictable and uniformly bounded. Further, since $\zeta_s = \sum_{k=0}^{\infty} \zeta_k^m \mathbf{1}_{(t_k, t_{k+1}]}(s)$, we may choose a sequence of partitions $\pi_n : 0 = t_0^n < \cdots < t_n^n = 1$ such that $H_n \to H$ on $D([0, 1], \mathbb{R})$, with

$$H_t = \mathbf{1}_{\{m \leq \mathbf{x}^*_{t-}; \zeta_t \leq \mathbf{x}_{t-}\}} \frac{1}{m - \zeta_t}.$$

It follows that $\int_0^1 H_t^n \, dM_t \to \int_0^1 H_t \, dM_t$ in probability; cf. Theorem I.4.40 in [28]. Hence, convergence holds a.s. along a subsequence and we conclude. Under assumption (i), the result follows by first integrating the pathwise inequality w.r.t. $d\phi(m)$, and then modifying the argument along the same lines as in the proof of Theorem 2.9 [cf. (4.14)]. □

PROOF OF PROPOSITION 2.11. Let $m > 0$ fixed and take $\phi(x) = \mathbf{1}_{\{x \geq m\}}$; the general case follows by integration with respect to $d\phi(m)$. The proof is based on the path-wise inequality (3.5). Since $\zeta_.$ is nondecreasing and M is càdlàg, it implies that

$$\mathbf{1}_{\{M_1^* \geq m\}} \leq \sum_{i=1}^{n} \left(\frac{(M_{t_i} - \zeta_{t_i})^+}{m - \zeta_{t_i}} - \frac{(M_{t_i} - \zeta_{t_{i+1}})^+}{m - \zeta_{t_{i+1}}} \mathbf{1}_{\{i < n\}} \right)$$
$$- \sum_{i=1}^{n-1} \mathbf{1}_{\{m \leq M^*_{t_i}, \zeta_{t_{i+1}} \leq M_{t_i}\}} \frac{M_{t_{i+1}} - M_{t_i}}{m - \zeta_{t_{i+1}}} + \sum_{i=1}^{n} \mathbf{1}_{\{M^*_{t_{i-1}} < m \leq M^*_{t_i}\}} \frac{m - M_{t_i}}{m - \zeta_{t_i}}.$$

We proceed by taking expectation on both sides of this inequality, and then passing to the limit. To this end, note that the expected value of the dynamic terms is bounded from above by zero (cf. Proposition 3.2 in [18]). Since the left-hand side of the inequality is independent of the partition, it follows that

$$\mathbb{E}[\mathbf{1}_{\{M_1^* \geq m\}}] \leq \frac{\mathbb{E}[(M_1 - \zeta_{1-})^+]}{m - \zeta_{1-}} - \lim_{n \to \infty} \int_0^1 f_n(t) \, d\zeta_t,$$

where

$$f_n(t) = \sum_{i=1}^{n-1} \frac{\mathbf{1}_{t \in (t_i, t_{i+1}]}}{\zeta_{t_{i+1}} - \zeta_{t_i}} \left(\frac{\mathbb{E}[(M_{t_i} - \zeta_{t_{i+1}})^+]}{m - \zeta_{t_{i+1}}} - \frac{\mathbb{E}[(M_{t_i} - \zeta_{t_i})^+]}{m - \zeta_{t_i}} \right).$$

Let

$$f(t) := \begin{cases} \dfrac{\mathbb{E}[(M_t - \zeta_t)^+] - \mathbb{P}[M_t > \zeta_t](m - \zeta_t)}{(m - \zeta_t)^2}, & t \in D_\zeta^c, \\ \dfrac{1}{\zeta_{t^+} - \zeta_t}\left(\dfrac{\mathbb{E}[(M_t - \zeta_{t^+})^+]}{m - \zeta_{t^+}} - \dfrac{\mathbb{E}[(M_t - \zeta_t)^+]}{m - \zeta_t}\right), & t \in D_\zeta. \end{cases}$$

Observe that since M_t is integrable, $\zeta \mapsto \mathbb{E}[(M_t - \zeta)^+]/(m - \zeta)$ is Lipschitz on $(-\infty, m - \varepsilon]$, for all $\varepsilon > 0$. Hence, it is differentiable almost everywhere and it follows that $f_n(t)$ converges pointwise to $f(t)$, for $t \in [0, 1] \setminus D$, where $D = D_\zeta^c \cap \{t : F(\cdot, t) \text{ discontinuous at } \zeta_t\}$, with $F(x; t) := \mathbb{P}[M_t > x]$. Moreover, there is $K > 0$ such that $|f_n(t)| \le K$, $t \in [0, 1]$, $n > 0$. Hence, by use of dominated convergence, it follows that $\int_0^1 f_n(t)\, d\zeta_t \to \int_0^1 f(t)\, d\zeta_t$ and we conclude. □

APPENDIX

We here discuss the Lévy metric on $\mathbb{V}_r^+$ and provide a characterization of the filtration on the canonical space $\overline{\Omega} := \Omega \times \mathbb{V}_r^+$.

REMARK A.1. Recall that $\mathbb{V}_r^+$ denotes the class of all nondecreasing càdlàg functions defined on $[0, 1]$ and taking values in $\mathbb{R}_+$. Denote by $M^+([0, 1])$ the collection of all finite positive measures on $[0, 1]$. Then every function $\theta \in \mathbb{V}_r^+$ can be identified with a measure $m^\theta \in M^+([0, 1])$ by defining $m^\theta([0, t]) := \theta(t)$ for all $t \in [0, 1]$. Moreover, $\theta_n \to \theta$ in the Lévy metric is equivalent to $m^{\theta_n} \to m^\theta$ in the weak convergence topology (which can be deduced from Problem 14.5 of Billinsley [8] together with an easy scaling technique). Using the above equivalence, we can easily obtain the following facts:

- $\mathbb{V}_r^+$ is a Polish space under the Lévy metric.
- Let $f : [0, 1] \to \mathbb{R}$ be a bounded continuous function, then $\theta \mapsto \int_0^1 f(t)\, d\theta_t$ is continuous from $\mathbb{V}_r^+$ to $\mathbb{R}$.
- If $\theta^n \to \theta$, then $\theta_1^n = m^{\theta^n}([0, 1]) \to m^\theta([0, 1]) = \theta(1)$. In other words, the map $\theta \mapsto \theta_1$ is continuous, and hence $\theta \mapsto \int_0^1 \theta_t\, dt = \theta_1 - \int_0^1 t d\theta_t$ is also continuous.
- Let $\lambda > 0$. Notice that since $[0, 1]$ is compact, the set $\{m \in M^+([0, 1]) : m([0, 1]) \le \lambda\}$ is compact under the weak convergence topology, and hence $\{\theta \in \mathbb{V}_r^+ : \theta_1 \le \lambda\}$ is compact under the Lévy metric.

LEMMA A.2. *The Borel σ-field of the Polish space $\overline{\Omega}$ is given by $\overline{\mathcal{F}}_\infty := \bigvee_{t \ge 0} \overline{\mathcal{F}}_t$. Moreover, $\overline{\mathcal{F}}_{t-} := \bigvee_{0 \le s < t} \overline{\mathcal{F}}_s$ coincides with the σ-field generated by all bounded continuous functions $\xi : \overline{\Omega} \to \mathbb{R}$ which are $\overline{\mathcal{F}}_t$-measurable.*

PROOF. (i) We first prove that $\overline{\mathcal{F}}_\infty$ is the Borel σ-field of the Polish space $\overline{\Omega}$. Define $\mathcal{V}_t^+$ as the σ-field on $\mathbb{V}_r^+$, generated by all sets of the form $\{\theta \in \mathbb{V}_r^+, \theta_u \le$

$s\}$ for $u \in [0,1]$ and $s \le t$; and $\mathcal{V}^+_\infty := \bigvee_{t\ge 0} \mathcal{V}^+_t$. Then $\overline{\mathcal{F}}_\infty = \mathcal{F}^0_\infty \otimes \mathcal{V}^+_\infty$, where $\mathcal{F}^0_\infty := \bigvee_{t\ge 0} \mathcal{F}^0_t$ is the Borel σ-field of Ω (see, e.g., the discussion at the beginning of Section 1.3 of Stroock and Varadhan [38]). So it is enough to check that $\mathcal{V}^+_\infty$ is the Borel σ-field $\mathcal{B}(\mathbb{V}^+_r)$ of the Polish space $\mathbb{V}^+_r$. First, by the right-continuity of $\theta \in \mathbb{V}^+_r$, the Lévy metric on $\mathbb{V}^+_r$ can be defined equivalently by

$$d(\theta, \theta') := \inf\{\varepsilon > 0 : \theta_{t-\varepsilon} - \varepsilon \le \theta'_t \le \theta_{t+\varepsilon} + \varepsilon, \forall t \in \mathbb{Q} \cap [0,1]\},$$

where $\mathbb{Q}$ is the collection of all rational numbers. Then it follows that $\mathcal{B}(\mathbb{V}^+_r) \subseteq \sigma(T_u : u \in \mathbb{Q}) \subseteq \mathcal{V}^+_\infty$. On the other hand, for every $u \in [0,1)$, the map $\theta \mapsto \frac{1}{\varepsilon}\int_u^{u+\varepsilon} \theta(s)\,ds$ is continuous under the Lévy metric, and hence Borel measurable. Letting $\varepsilon \to 0$, it follows that $\theta \mapsto \theta(u)$ is also Borel measurable, and hence $\mathcal{V}^+_\infty \subseteq \mathcal{B}(\mathbb{V}^+_r)$. We then obtain that $\overline{\mathcal{F}}_\infty = \mathcal{B}(\overline{\Omega})$.

(ii) We now consider the σ-field generated by bounded continuous functions. First, it is well known that the filtration $\mathbb{F}^0$ on Ω is left-continuous and $\mathcal{F}^0_{t-} = \mathcal{F}^0_t$ is generated by all bounded continuous functions $\xi_1 : \Omega \to \mathbb{R}$ which are $\mathcal{F}^0_t$ continuous.

Next, we notice that for every $t \ge 0$,

$$\mathcal{V}^+_{t-} := \bigvee_{0 \le s < t} \mathcal{V}^+_s := \sigma(T_u \wedge t : u \in [0,1]).$$

Let $\xi_2 : \mathbb{V}^+_r \to \mathbb{R}$ be a bounded continuous function which is also $\mathcal{V}^+_t$-measurable. Then $\xi_2((\theta_u)_{u\in[0,1]}) = \xi_2((\theta_u \wedge u)_{r\in[0,1]})$, which is $\sigma(T_u \wedge t : u \in [0,1])$-measurable. On the other hand, the function $\theta \mapsto \frac{1}{\varepsilon}\int_u^{u+\varepsilon} (T_\ell(\theta) \wedge t)\,d\ell$ from $\mathbb{V}^+_r$ to $\mathbb{R}$ is continuous and $\mathcal{V}^+_t$-measurable for $\varepsilon > 0$. By taking $\varepsilon \to 0$, it follows that $T_u \wedge t$ is measurable w.r.t. the σ-field generated by all bounded continuous functions $\xi_2 : \mathbb{V}^+_r \to \mathbb{R}$ which are $\mathcal{V}^+_t$-measurable. Therefore, $\mathcal{V}^+_{t-}$ is the σ-field generated by all bounded continuous functions on $\mathbb{V}^+_r$ which are $\mathcal{V}^+_t$-measurable.

Finally, since $\overline{\mathcal{F}}_s = \mathcal{F}^0_s \otimes \mathcal{V}^+_s$, it follows that $\overline{\mathcal{F}}_{t-} = \mathcal{F}^0_{t-} \otimes \mathcal{V}^+_{t-}$. We hence conclude that $\overline{\mathcal{F}}_{t-}$ is the σ-field generated by all bounded continuous functions $\xi : \overline{\Omega} \to \mathbb{R}$ which are $\overline{\mathcal{F}}_t$-measurable. □

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S. KÄLLBLAD
INSTITUT FÜR STOCHASTIK
UND WIRTSCHAFTSMATHEMATIK
TECHNISCHE UNIVERSITÄT WIEN
WIEDNER HAUPTSTRASSE 8-10
1040 WIEN
AUSTRIA
E-MAIL: sigrid.kaellblad@tuwien.ac.at

X. TAN
UNIVERSITY OF PARIS-DAUPHINE
PSL RESEARCH UNIVERSITY
CNRS, UMR 7534
CEREMADE, 75016 PARIS
FRANCE
E-MAIL: tan@ceremade.dauphine.fr

N. TOUZI
CENTRE DE MATHÉMATIQUES APPLIQUÉES
ÉCOLE POLYTECHNIQUE
PALAISEAU, 91128
FRANCE
E-MAIL: nizar.touzi@polytechnique.edu