Points, Polytopes, Polynomials

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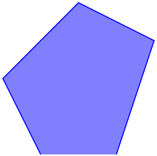
Lattice polytopes

A set $P \subset \mathbb{R}^d$ is a **polytope** if there are $x_1, \ldots, x_m \in \mathbb{R}^d$ with

$$P = \text{conv}\{x_1, \ldots, x_m\}.$$



Lattice polytopes

A set $P \subset \mathbb{R}^d$ is a **lattice polytope** if there are $x_1, \ldots, x_m \in \mathbb{Z}^d$ with

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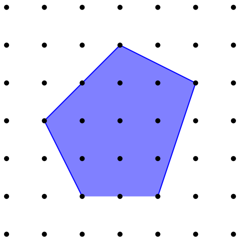
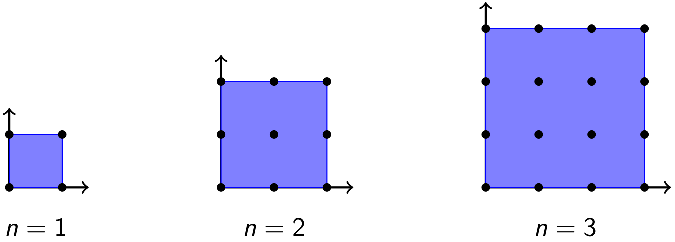


Discrete volume

The **discrete volume** of P is

$$\mathrm{E}(P) \ := \ \left| P \cap \mathbb{Z}^d \right|.$$



$$\mathrm{E}(nP) = (n+1)^2.$$

Ehrhart theory

Theorem (Ehrhart'62)

For every lattice polytope P in $\mathbb{R}^d$

$$\mathrm{E}_P(n) := |nP \cap \mathbb{Z}^d|$$

agrees with a polynomial of degree $\dim P$ *for $n \geq 1$.*

$\mathrm{E}_P(n)$ is called the **Ehrhart polynomial** of P.

Overview

Geometric Modeling

Structural Results

Valuations

Geometric Modeling

Order preserving maps

$[n] := \{0, 1, \ldots, n\}$

A map $\varphi\colon [m] \to [n]$ is **order preserving** if

$$\varphi(k) \leq \varphi(\ell) \quad \text{whenever} \quad k \leq \ell$$

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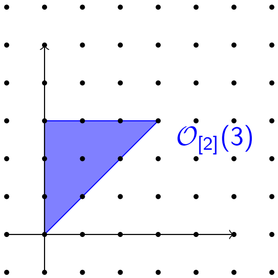
and thus to a lattice point in

$$\mathcal{O}_{[m]}(n) = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \colon 0 \leq x_1 \leq \cdots \leq x_m \leq n\}\,.$$

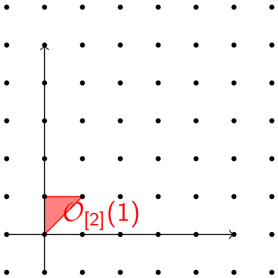
Example

$$\mathcal{O}_{[2]}(3) = \{(x_1, x_2) \in \mathbb{R}^2 \colon 0 \le x_1 \le x_2 \le 3\}\,.$$

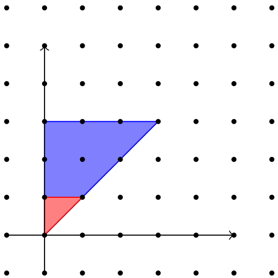


Example

$$\mathcal{O}_{[2]}(1) = \{(x_1, x_2) \in \mathbb{R}^2 \colon 0 \le x_1 \le x_2 \le 1\}.$$



Example



Order polytopes

$(\Pi, \prec)$ finite partially ordered set (poset)

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$$\mathcal{O}_\Pi(n) = \{\{x_p\}_{p\in\Pi} \in \mathbb{R}^\Pi \colon 0 \leq x_p \leq n, x_p \leq x_q \text{ for all } p \prec q\} \subset \mathbb{R}^P .$$

Order polynomials

$$\Omega_{\Pi}(n) = \#\{\varphi\colon \Pi \to [n] \textbf{ order preserving}\}$$

Q: How many order preserving preserving maps $\Pi \to [n]$ are there?

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We observe

$$\begin{aligned} \Omega_\Pi(n) &= \#\left\{\mathcal{O}_\Pi(n) \cap \mathbb{Z}^P\right\} \\ &= \#\{n\mathcal{O}_\Pi \cap \mathbb{Z}^P\} \end{aligned}$$

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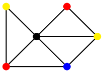
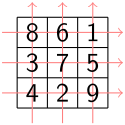
$$\Rightarrow \quad \Omega_\Pi(n) = \mathrm{E}_{\mathcal{O}_\Pi}(n) \quad \text{is a polynomial!}$$

order polynomial

Polyhedral Models

Object	**Polytope**	
Graph colorings	Inside-Out Polytopes	
Contingency Tables/ Magic Squares	Transportation Polytopes/ Birkhoff Polytope	
Extensions of graph/ colorings/Alternating sign matrices	Marked Order Polytopes **J., Sanyal '14**	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
⋮	⋮	

Structural Results

Characterizing Ehrhart polynomials

Q: Can we characterize Ehrhart polynomials? Give an interpretation of their coefficients?

It is known that...

- ...the leading coefficient equals the volume
- ...the second coefficient is the normalized surface area
- ...the constant coefficient is always one.

Pick's Theorem

For every lattice polygon $P \subset \mathbb{R}^2$

$$\mathrm{Ehr}_P(n) = \mathrm{area}(P)n^2 + \frac{|\partial P \cap \mathbb{R}^2|}{2}n + 1\,.$$

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In general, coefficients can be negative!!

Ehrhart series and h^*-polynomial

Ehrhart series

The **Ehrhart series** of an r-dimensional lattice polytope $P \subset \mathbb{R}^d$ is defined by

$$\sum_{n \geq 0} \mathrm{E}_P(n) t^n \quad = \quad \frac{h_0^*(P) + h_1^*(P)t + \cdots + h_r^*(P)t^r}{(1-t)^{r+1}} \, .$$

The polynomial $h_P^*(t) = \sum_i h_i^*(P)^i$ is the h^*-**polynomial** of P.

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Expansion into a binomial basis:

$$\mathrm{E}_P(n) = h_0^*(P)\binom{n+r}{r} + h_1^*(P)\binom{n+r-1}{r} + \cdots + h_r^*(P)\binom{n}{r}\,.$$

Inequalities for the h^*-vector

Positivity:

Theorem (Stanley '80)

For every lattice polytope P in $\mathbb{R}^d$

$$h_i^*(P) \geq 0$$

for all $0 \leq i \leq d$.

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Unimodality:

$$h_0^*(P) \leq h_1^*(P) \leq \cdots \leq h_k^*(P) \geq \cdots \geq h_d^*(P) \text{ for some } k$$

Question

Which combinatorial and geometric properties imply unimodality of $h^*(P)$?

IDP polytopes

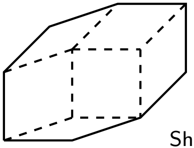
A lattice polytope $P \subset \mathbb{R}^d$ has the **integer decomposition property (IDP)** if for all integers $n \geq 1$ and all $p \in nP \cap \mathbb{Z}^d$

$$p = p_1 + \cdots + p_n$$

for some $p_1, \ldots, p_n \in P \cap \mathbb{Z}^d$.

Examples

- unimodular simplex
- lattice parallelepiped
- lattice zonotope



Shephard '74

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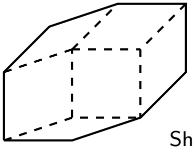


Shephard '74

Conjecture (Stanley '98; Hibi, Ohsugi '06; Schepers, Van Langenhoven '13)

If P is IDP then the h^*-polynomial of P has unimodal coefficients.

Polytopes with unimodal h^*-polynomials

- **Birkhoff polytope** (Athanasiadis '05)
- **Gorenstein polytopes admitting a regular unimodular triangulation** (Bruns, Römer '07)
- **Lattice parallelepipeds** (Schepers, Van Langenhoven '13)
- **Dilated polytopes** (J. '18)
- **Simplices for numeral systems** (Solus '19)
- **Zonotopes** (Beck, J., McCullough '19)
- ...

Q: Polytopes with unimodular triangulations? Alcoved polytopes? Hypersimplices? Order polytopes?...

Unimodality and real-rootedness

Let $a_0, \ldots, a_d \geq 0$ be real numbers.

Real-rootedness of
$$a_d t^d \; + \; a_{d-1} t^{d-1} \; + \; \cdots \; + \; a_1 t \; + \; a_0$$

$$\Downarrow$$

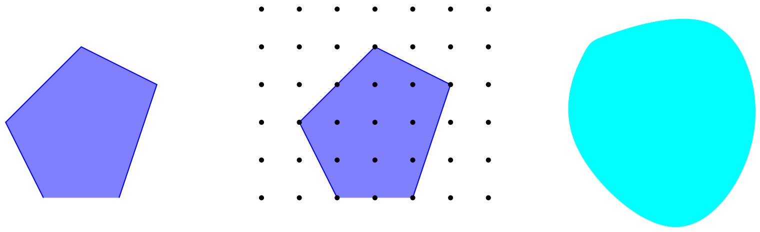
Log-concavity
$$a_i^2 \; \geq \; a_{i-1} a_{i+1} \text{ for all } 0 \leq i \leq d$$

$$\Downarrow$$

Unimodality
$$a_0 \; \leq \; \cdots \; \leq \; a_i \; \geq \; \cdots \; \geq \; a_d \text{ for some } 0 \leq i \leq d$$

Valuations

Polytopes and convex bodies



Λ : $\mathbb{R}^d$ or $\mathbb{Z}^d$

$\mathcal{P}(\Lambda)$: set of all polytopes with vertices in Λ, called Λ-**polytopes**

$\mathcal{K}$: set of all convex bodies in $\mathbb{R}^d$

Volume

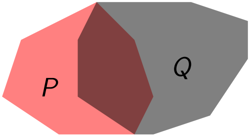
$\mathrm{Vol}_d(P) = \int_P 1 d\mu$ d-dimensional **volume** of $P \in \mathcal{K}$

Properties:

- homogeneous: $\mathrm{Vol}_d(\lambda P) = \lambda^d \,\mathrm{Vol}_d(P)$ for all $\lambda \geq 0$
- positive: $\mathrm{Vol}_d(P) \geq 0$
- (continuous)
- rigid-motion invariant
- **valuation** property:
  1. $\mathrm{Vol}_d(\emptyset) = 0$,
  2. for $P, P' \in \mathcal{K}$ such that $P \cup P' \in \mathcal{K}$

$$\mathrm{Vol}_d(P \cup P') = \mathrm{Vol}_d(P) + \mathrm{Vol}_d(P') - \mathrm{Vol}_d(P \cap P').$$

Hadwiger's Characterization Theorem



$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q)$$

Theorem (Hadwiger '57)

The family of continuous, real-valued, ridig-motion invariant valuations on convex bodies is a $(d+1)$-dimensional vector space spanned by the quermassintegrals $W_0, W_1, \ldots, W_d$.

Hadwiger's Characterization Theorem

Steiner polynomial:

$$\mathrm{Vol}_d(P + n\mathcal{B}_d) \;=\; \sum_{i=0}^{d} \binom{d}{i} W_i(P) n^i$$

where $\mathcal{B}_d$ is the unit ball and $W_i(P)$ is the so-called i-th quermassintegral.

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Minkowski sum: For $P, Q \in \mathcal{K}$

$$P + Q \;:=\; \{p + q \colon p \in P, q \in Q\}$$

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Properties:

- rigid-motion invariant
- valuation
- positive

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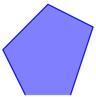
Theorem (Hadwiger '57)

A continuous, rigid-motion invariant valuation $\varphi\colon \mathcal{K} \to \mathbb{R}$ is positive if and only if there are $c_0, c_1, \ldots, c_d \geq 0$ with

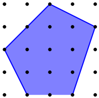
$$\varphi = c_0 W_0 + c_1 W_1 + \cdots + c_d W_d.$$

Discrete volume

$\mathrm{E}(P) = \left|P \cap \mathbb{Z}^d\right|$ **discrete volume** of a lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$



volume
valuation
positive
rigid-motion invariant
homogeneous



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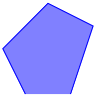

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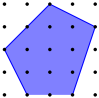

discrete volume
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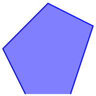

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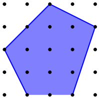

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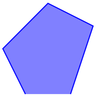

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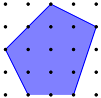


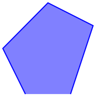
discrete volume
valuation
positive
lattice invariant

A valuation $\varphi \colon \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ is **lattice invariant** if for all $T \in \mathrm{GL}_d(\mathbb{Z}^d)$ and $t \in \mathbb{Z}^d$

$$\varphi(T(P) + t) = \varphi(P).$$

Discrete volume

$\mathrm{E}(P) = \left|P \cap \mathbb{Z}^d\right|$ **discrete volume** of a lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$



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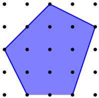

discrete volume
valuation
positive
lattice invariant
polynomial

Betke-Kneser Theorem

Ehrhart polynomial:

$$\mathrm{E}_P(n) = \mathrm{E}_0(P) + \mathrm{E}_1(P)n + \cdots + \mathrm{E}_d(P)n^d$$

Each coefficient $\mathrm{E}_i \colon \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ is a...

- valuation
- lattice invariant

Theorem (Betke-Kneser '85)

The family of lattice invariant valuations form a $(d+1)$-dimensional vector space spanned by the coefficients $E_0, E_1, \ldots, E_d$ of the Ehrhart polynomial.

Betke-Kneser Theorem

Example: $d = 2$

$$\mathrm{E}_2 = \mathsf{Vol}_2, \quad \mathrm{E}_0 = \chi, \quad 2\mathrm{E}_1 = |\partial P \cap \mathbb{Z}^2|$$

Each coefficient $\mathrm{E}_i \colon \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ is a...

- valuation
- lattice invariant
- in general **not positive**

Question: Is there a classification for positive lattice invariant valuations?

Reminder: Stanley's Nonnegativity Theorem

Theorem (Stanley '80)

Let $P \in \mathcal{P}(\mathbb{Z}^d)$ *be lattice polytopes. Then*

$$h_i^*(P) \geq 0 \qquad \textbf{(positivity)}$$

for all $0 \leq i \leq d$.

Translation-invariant valuations

A map $\varphi\colon \mathcal{P}(\Lambda) \to \mathbb{R}$ is a **translation-invariant valuation** if

- $\varphi(\emptyset) = 0$,
- for $P, P' \in \mathcal{P}(\Lambda)$ such that $P \cup P', P \cap P' \in \mathcal{P}(\Lambda)$
$$\varphi(P \cup P') = \varphi(P) + \varphi(P') - \varphi(P \cap P'),$$
- for all $P \in \mathcal{P}(\Lambda)$ and all $t \in \Lambda$
$$\varphi(P + t) = \varphi(P).$$

Theorem (McMullen '77)

Let φ be a translation-invariant valuation and P be a Λ-polytope. Then $\varphi(nP)$ agrees with a polynomial $\varphi_P(n)$ of degree at most $\dim(P)$ for integers $n \geq 0$.

Combinatorial positivity

For an r-dimensional polytope $P \in \mathcal{P}(\Lambda)$ and a translation-invariant valuation φ let

$$\sum_{n \geq 0} \varphi_P(n) t^n = \frac{h_0^{\varphi}(P) + h_1^{\varphi}(P) t + \cdots + h_r^{\varphi}(P) t^r}{(1-t)^{r+1}} .$$

$h_P^{\varphi}(t) = h_0^{\varphi}(P) + h_1^{\varphi}(P) t + \cdots + h_r^{\varphi}(P) t^r$ is the h^*-**polynomial** of P with respect to φ.

We define

φ **combinatorially positive** $\quad : \quad 0 \leq h_i^{\varphi}(P) \quad \forall P \in \mathcal{P}(\Lambda), \forall i.$

Discrete Hadwiger-type theorem

$$\mathrm{E}_P(n) \;=\; f_0^*(P)\binom{n-1}{0} + f_1^*(P)\binom{n-1}{1} + \cdots + f_d^*(P)\binom{n-1}{d}$$

Properties:
The coefficients f_i^* are

- lattice invariant valuations
- nonnegative (Breuer '12)
- combinatorially positive (J. , Sanyal '18)

Theorem (J., Sanyal '18)

A lattice invariant valuation $\varphi\colon \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ is combinatorially positive if and only if there are $c_0, c_1, \ldots, c_d \geq 0$ such that

$$\varphi \;=\; c_0 f_0^* + c_1 f_1^* + \cdots + c_d f_d^*$$

Research directions

- What results in Ehrhart theory can be generalized to valuations?
  - unimodality/real-rootedness for zonotopes
  - real-rootedness for dilated polytopes
- Discrete Mixed Valuations
  - combinatorially mixed valuations
  - inequalities
  - discrete Brunn-Minkowski theory
- Polytopes from root systems
  - alcoved polytopes
  - generalized permutahedra