Possibilities and limitations of semidefinite approaches to polynomial optimization

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Global optimization

Convex/linear optimization is the core of optimization theory:

- local methods give globally optimal solutions.
- local information is enough to certify optimality.

Finding a global optimum of a general non-linear problem is computationally challenging. Possible strategies:

- give it up and just find a locally optimal solution (with gradient decent, Newton etc.). That is a classical strategy.
- make case distinction and convexify (turn a non-convex problem into a convex one).

Finding a global optimum of a general non-linear problem is computationally challenging. Possible strategies:

- give it up and just find a locally optimal solution (with gradient decent, Newton etc.). That is a classical strategy.
- make case distinction and convexify (turn a non-convex problem into a convex one).

Good news:

- Real algebra can be used to develop general-purpose convexification strategies for polynomial optimization (as was pointed out by Lasserre).
- But we still need to think about different kinds of complexity issues

Algebra meets optimization: cultural differences

Algebraists	Optimizers
usually don't think about complexity	think a lot about complexity

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speak their secret language	speak their secrete language

Our topic:

- using semidefinite programming
- in polynomial optimization
- and related challenges and limitations.

Conic and semidefinite programming

Conic programming

- *Conic programming*:

$$\inf\{c^\top x \,:\, Ax = b,\ x \in \underbrace{K}_{\text{closed convex cone}}\}.$$

- $K = \mathbb{R}^n_+ \rightsquigarrow$ *linear programming*
- $K = \mathcal{S}^k_+ := \{k \times k \text{ symmetric psd matrices over } \mathbb{R}\} \rightsquigarrow$ *semidefinite programming (SDP)*:

$$\inf\{c^\top \mathrm{vec}(X) \,:\, A\,\mathrm{vec}(X) = b,\ X \in \mathcal{S}^k_+\}$$

Semidefinite programming and LMIs

- Consider a $k \times k$ symmetric matrix
$$A(x) := \Big(a_{ij}(x)\Big)_{i,j=1,\ldots,k}$$
with entries $a_{ij}(x)$ being affine functions in $x \in \mathbb{R}^n$.
- The condition
$$A(x) \in \mathcal{S}_+^k$$
is called a *linear matrix inequality (LMI)* of size k on n real-valued variables $x \in \mathbb{R}^n$.
- The set
$$\{x \in \mathbb{R}^n \ : \ A(x) \in \mathcal{S}_+^k\}$$
is called a *spectrahedron*.
- *Semidefinite programming* is optimization of a linear function subject to finitely many LMIs.

Semidefinite programming, computational aspects

- SDP is efficiently solvable using interior-point methods under mild assumptions!
- **But:** If you can avoid LMIs of large size, you should really do that!
  - running time
  - numerical stability

Connections of SDP to other computational problems

1. Linear programming: linear constraints are LMIs of size 1
2. Solving a system of linear equations over $\mathbb{R}$ $\rightsquigarrow$ minimization of a convex quadratic function $\rightsquigarrow$ SDP:

$$\begin{aligned} Ax = b \qquad & \rightsquigarrow \qquad \min\left\{\|Ax - b\|_2^2 \,:\, x \in \mathbb{R}^n\right\} \\ & \rightsquigarrow \qquad \min\Big\{t \,:\, \underbrace{\|Ax - b\|_2 \le t}_{\substack{\text{can be modelled} \\ \text{as LMI}}}\Big\} \end{aligned}$$

3. Maximum eigenvalue of a symmetric matrix $\longrightarrow$ SDP with an LMI on one variable:

$$\min\Big\{\lambda \,:\, \underbrace{\lambda I - A \in \mathcal{S}_+^n}_{\text{LMI in } \lambda}\Big\}.$$

Applications

LMIs and SDP (frequently) allow to

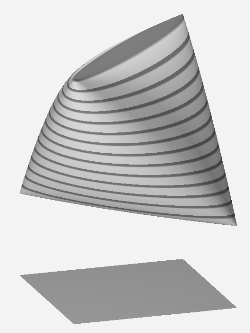
- convexify non-convex problems of algebraic nature
- solve optimization problems over semi-algebraic convex sets

Application areas

- Probability and statistics (moment problems)
- Coding theory
- Systems and control theory (Lyapunov stability)
- Combinatorial optimization (max cut and more)
- Discrete packing problems
- Global optimization (the case of polynomial optimization)
- ...

Example: eliptope in dimension three

$$\begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} \in \mathcal{S}^3_+$$



Conic and semidefinite extended formulations

Conic extended formulation

- Assume we have fixed a convex cone K and we can solve conic problems with respect to K.
- We are given a convex set C and we want to solve linear problems over C:
$$\inf \{f(x) : x \in C\},$$
where $f : \mathbb{R}^n \to \mathbb{R}$.
- We say that C has a *K-lift*, if
$$C = \pi(K \cap H),$$
where H is an affine space and π a linear map.
- Then we can use conic programming with respect to the cone K:
$$\inf \{f(x) : x \in C\} = \inf\{\underbrace{f(\pi(y))}_{\text{linear}} : \underbrace{y \in K \cap H}_{\substack{\text{conic}\\ \text{constraint}}}\}.$$

Semidefinite extended formulation

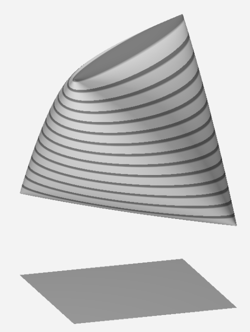
- We say that C has an *extended formulation* with m LMIs of size k if C is a linear image of a spectrahedron described by m LMIs of size k.
- Geometrically, this means that C has a K-lift for

$$K = (\mathcal{S}_+^k)^m = \underbrace{\mathcal{S}_+^k \times \ldots \times \mathcal{S}_+^k}_{m \text{ times}}.$$

- Optimizing linear functions over such C gets reduced to solving SDPs with m LMIs of size k.

Example: Square

$$\begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} \in \mathcal{S}^3_+$$



The image under $\pi(x_1, x_2, x_3) = (x_1, x_2)$ of this spectrahedron is the square

$$[-1, 1]^2$$

Example: l_4-disc

Condition

$$x_1^4 + x_2^4 \le 1$$

can be lifted to a system of three conditions

$$\begin{array}{ccc} y_1^2 + y_2^2 \le 1, & x_1^2 \le y_1, & x_2^2 \le y_2. \\ \Updownarrow & \Updownarrow & \Updownarrow \\ \begin{pmatrix} 1 - y_1 & y_2 \\ y_2 & 1 + y_1 \end{pmatrix} \in \mathcal{S}_+^2 & \begin{pmatrix} y_1 & x_1 \\ x_1 & 1 \end{pmatrix} \in \mathcal{S}_+^2 & \begin{pmatrix} y_2 & x_2 \\ x_2 & 1 \end{pmatrix} \in \mathcal{S}_+^2 \end{array}$$

Semidefinite extension complexity

Definition

sxc(C), the *semidefinite extension complexity* of C, is the smallest k such that C has a semidefinite extended formulation with one LMI of size k.

Why **one** LMI?

$$A_1(x), \ldots, A_m(x) \in \mathcal{S}_+^k \quad \iff \quad \begin{pmatrix} A_1(x) & & \\ & \ddots & \\ & & A_m(x) \end{pmatrix} \in \mathcal{S}_+^{km}$$

- However, k and m have a different impact on running time in interior-point methods: k has a stronger influence. So, we suggest...

Semidefinite extension degree

Definition

sxdeg(C), the *semidefinite extension degree* of C, is the smallest k such that C has a semidefinite extended formulation with finitely many LMIs of size k.

Remember:

- size of the LMIs is more critical than their number. When the size is too large, interior-point methods get stuck!

Semidefinite extended formulations in polynomial optimization

Polynomial optimization (POP)

- Let $\mathbb{R}[x]$ be the ring of n-variate polynomials in variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{R}$.
- *Constrained polynomial optimization (C-POP)*:

$$\inf \{ f(x) \, : \, x \in \mathbb{R}^n, \; g_1(x) \geq 0, \ldots, g_s(x) \geq 0 \} ,$$

  where $f, g_1, \ldots, g_s \in \mathbb{R}[x]$.
- *Unconstrained polynomial optimization (U-POP)*:

$$\inf \{ f(x) \, : \, x \in \mathbb{R}^n \} ,$$

  where $f \in \mathbb{R}[x]$.
- Both are hard problems in general, because nonnegativity of a polynomial is hard to decide!

Non-negativity vs. sum-of-squares property

- A polynomial $f \in \mathbb{R}[x]$ is called *sum of squares* (*SOS*) if

$$f = f_1^2 + \cdots + f_r^2$$

holds for finitely many polynomials $f_1, \ldots, f_r \in \mathbb{R}[x]$.

Convex cones in polynomial optimization

- Consider
$$\mathbb{R}[x]_d := \{f \in \mathbb{R}[x] \,:\, \deg f \leq d\},$$
where $\dim_{\mathbb{R}} \mathbb{R}[x] = \binom{n+d}{n}$.
- SOS cones:
$$\Sigma_{n,2d} := \{f \in \mathbb{R}[x]_{2d} \,:\, f \text{ is SOS}\} \subseteq \mathbb{R}[x]_{n,2d}.$$
- Cones of general non-negative polynomials:
$$P_{n,2d} := \{f \in \mathbb{R}[x]_{2d} \,:\, f \geq 0 \text{ on } \mathbb{R}^n\},$$
$$P_{n,2d}(X) := \{f \in \mathbb{R}[x]_{2d} \,:\, f \geq 0 \text{ on } X\} \qquad (X \subseteq \mathbb{R}^n).$$
- $\Sigma_{n,2d}$ is computationally much more tractable than $P_{n,2d}$ and $P_{n,2d}(X)$ in general, so frequently one uses SOS cones or some cones built up using SOS cones as substitutes for $P_{n,2d}$ and $P_{n,2d}(X)$.

SOS cones in U-POP

Let $f \in \mathbb{R}[x]_{n,2d}$.
Minimization as a search for the largest lower bound:

$$\inf\{f(x) : x \in \mathbb{R}^n\} = \sup\{\lambda \in \mathbb{R} : f - \lambda \in P_{n,2d}\}$$

This problem is too hard, but we can make it simpler:

$$\inf\{f(x) : x \in \mathbb{R}^n\} \geq \underbrace{\sup\{\lambda \in \mathbb{R} : f - \lambda \in \Sigma_{n,2d}\}}_{\text{so-called SOS relaxation}}$$

SOS cones in C-POP

Let $f, g_1, \dots, g_m \in \mathbb{R}[x]_{n,2d}$ and let

$$X := \{x \in \mathbb{R}^n \,:\, g_1(x) \geq 0, \dots, g_m(x) \geq 0\}.$$

Minimization as a search for the largest lower bound:

$$\inf\{f(x) \,:\, x \in X\} = \sup\{\lambda \in \mathbb{R} \,:\, f - \lambda \in P_{n,2d}(X)\}$$

This problem is too hard, but we can make it simpler:

$$\underbrace{\inf\{f(x) \,:\, x \in \mathbb{R}^n\} \geq \sup\{\lambda \in \mathbb{R} \,:\, f - \lambda \in \Sigma_{n,2d_0} + g_1\Sigma_{n,2d_1} + \cdots + g_m\Sigma_{n,2d_m}\}}_{\text{so-called hierarchy of SOS relaxations}}$$

Results in Real Algebra tell us that **qualitatively** this approach works (Positivstellensätze)

SOS relaxations and SDP

- **The key computational observation:** SOS relaxations can be formulated as SDPs.
- For example, the SOS relaxation for U-POP can be formulated as an SDP with 1 LMI of size $\binom{n+d}{n}$.

Semidefinite extended formulation of the SOS cone

- Consider
  - $v_{n,d}$, the vector of all monomials of degree $\leq d$ in n variables.
  - The linear bijection

    $$Y \in \mathcal{S}^k := \{k \times k \text{ symmetric matrices}\} \quad \mapsto \quad q_Y(u) := u^\top Y u$$

    between symmetric matrices and quadratic forms.
  - The linear map $\pi : \mathcal{S}^k \to \mathbb{R}[x]_{n,2d}$

    $$\pi(Y) := q_Y(v_{n,d})$$

    with

    $$k = \binom{n+d}{n}$$

- This gives a lifted representation of $\Sigma_{n,2d}$:

$$\begin{aligned}\pi(\mathcal{S}^k_+) &= \left\{q_Y(v_{n,d}) \;:\; Y \in \mathcal{S}^k_+\right\} \\ &= \Sigma_{n,2d}.\end{aligned}$$

Example: SDP formulation of $\Sigma_{1,4}$

$$f = f_0 + f_1 x + f_2 x + f_2 x^2 + f_3 x^3 + f_4 x^4 \in \mathbb{R}[x]_4$$

Identification:

$$f \in \mathbb{R}[x]_4 \longleftrightarrow \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \in \mathbb{R}^5.$$

Consider the condition

$$\underbrace{\begin{pmatrix} 1 & x & x^2 \end{pmatrix}}_{v_{1,2}^\top} \underbrace{\begin{pmatrix} y_{00} & y_{01} & y_{02} \\ y_{01} & y_{11} & y_{12} \\ y_{02} & y_{12} & y_{22} \end{pmatrix}}_{Y} \underbrace{\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}}_{v_{1,2}} = f(x)$$

Example: SDP formulation of $\Sigma_{1,4}$

This gives a formulation of $\Sigma_{1,4}$.
Linear map $\pi : Y \mapsto f$:

$$\begin{aligned} y_{00} &= f_0 \\ 2y_{01} &= f_1 \\ y_{11} + 2y_{02} &= f_2 \\ 2y_{12} &= f_3 \\ y_{22} &= f_4 \end{aligned}$$

Linear matrix inequality:

$$\begin{pmatrix} y_{00} & y_{01} & y_{02} \\ y_{01} & y_{11} & y_{12} \\ y_{02} & y_{12} & y_{22} \end{pmatrix} \in \mathcal{S}_+^3$$

Example: SOS relaxation of U-POP with $n = 1, d = 2$

Maximize λ

for $\lambda, \; y_{00}, \; y_{01}, \; y_{02}, \; y_{11}, \; y_{12}, \; y_{22} \in \mathbb{R}$

subject to:

$$\begin{aligned} y_{00} + \lambda &= f_0 \\ 2y_{01} &= f_1 \\ y_{11} + 2y_{02} &= f_2 \\ 2y_{12} &= f_3 \\ y_{22} &= f_4 \end{aligned}$$

$$\begin{pmatrix} y_{00} & y_{01} & y_{02} \\ y_{01} & y_{11} & y_{12} \\ y_{02} & y_{12} & y_{22} \end{pmatrix} \in \mathcal{S}_+^3$$

Example: SOS relaxation of U-POP with $n = 1, d = 4$

If you want to have just one LMI without linear equations:

$$\begin{aligned} &\text{Maximize } f_0 - y_{00} \\ &\text{for } y_{00}, y_{11} \in \mathbb{R} \\ &\text{subject to:} \\ &\begin{pmatrix} y_{00} & \frac{1}{2} f_0 & \frac{1}{2}(f_2 - y_{11}) \\ \frac{1}{2} f_0 & y_{11} & \frac{1}{2} f_3 \\ \frac{1}{2}(f_2 - y_{11}) & \frac{1}{2} f_3 & f_4 \end{pmatrix} \in \mathcal{S}^3_+ \end{aligned}$$

This is an SDP with 1 LMI of size 3 on two variables.

New results

Combinatorial tool to bounding sxdeg(C) from below

Theorem A (A. SIAGA 2019)

Let $X \subseteq \mathbb{R}^n$ be a set with non-empty interior. Let $C \subseteq P_{n,2d}(X)$ be a closed convex cone such that there exist finite subsets S of X of arbitrarily large cardinality with the following property:

($*$) For every k-element subset T of S, some polynomial f in the cone C is equal to zero on T and is strictly positive on $S \setminus T$.

Then $\mathrm{sxdeg}(C) > k$.

Between $\Sigma_{n,2d}$ and $P_{n,2d}(X)$

More generally, if we want to approximate the non-negativity cone $P_{n,2d}$ from inside by cone that contains $\Sigma_{n,2d}$, we would have to pay a high price:

Corollary B

Let $X \subseteq \mathbb{R}^n$ be a set with non-empty interior and C be a closed convex cone satisfying $\Sigma_{n,2d} \subseteq C \subseteq P_{n,2d}(X)$. Then

$$\operatorname{sxdeg}(C) \geq \binom{n+d}{n}.$$

- This means, whenever we use a cone that contains $\Sigma_{n,2d}$ as a subset, we are forced to use LMIs of a large size.

All about $\Sigma_{n,2d}$

Corollary C

$\mathrm{sxdeg}(\Sigma_{n,2d}) = \mathrm{sxc}(\Sigma_{n,2d}) = \binom{n+d}{n}$.

- In other words, $\Sigma_{n,2d}$ has a semidefinite extended formulation with 1 LMI of size $\binom{n+d}{n}$ ...
- **but** has no semidefinite extended formulation with LMIs of a smaller size, no matter how many LMIs are used!
- **Conclusion**: If one wants to describe $\Sigma_{n,2d}$ exactly in the context of semidefinite programming, one is forced to use LMIs of large size.

- Note that $\mathcal{S}_+^k \simeq \Sigma_{k-1,2}$
- So, as a direct consequence of Theorem A we also obtain ...

Corollary D

$\mathrm{sxdeg}(\mathcal{S}_+^k) = k$

- Growth of the expressive power. Hierarchy:

$$\mathrm{SDR}(k) := \{S \subseteq \mathbb{R}^n \ : \ n \in \mathbb{N},\ \mathrm{sxdeg}(S) \leq k\}$$

satisfies

$$\underbrace{\mathrm{SDR}(1)}_{\text{polyhedra}} \subsetneq \underbrace{\mathrm{SDR}(2)}_{\text{SOC representable sets}} \subsetneq \mathrm{SDR}(3) \subsetneq \mathrm{SDR}(4) \subsetneq \cdots$$

Known cases of above results

- The presented formulas have been known only in a few cases (Fawzi 2018):
  - $\mathrm{sxdeg}(\mathcal{S}_+^3) = 3$ (Fawzi 2018)
  - $\mathrm{sxdeg}(\Sigma_{1,4}) = 3$ (Fawzi 2018)
- $\mathrm{sxdeg}(\Sigma_{1,4}) = 3$ is also mentioned by Ahmadi & Hall & Papachristodoulou & Saunderson & Zheng 2017.

Moment cones

We introduce the *moment cones*

$$\begin{aligned} M_{n,2d} &:= \overline{\text{cone}}(\{v_{n,2d}(x) \,:\, x \in \mathbb{R}^n\}), \\ M_{n,2d}(X) &:= \overline{\text{cone}}(\{v_{n,2d}(x) \,:\, x \in X\}) \qquad\qquad (X \subseteq \mathbb{R}^n). \end{aligned}$$

($\overline{\text{cone}}$ - topological closure of the convex conic hull)

$$\begin{array}{lcl} M_{n,2d} & \text{dual to} & P_{n,2d}, \\ M_{n,2d}(X) & \text{dual to} & P_{n,2d}(X). \end{array}$$

Moment cones occur in the context of probability.

Corollary for moment cones

Corollary E

For every $X \subseteq \mathbb{R}^n$ with non-empty interior,

$$\operatorname{sxdeg}(M_{n,2d}(X)) \geq \binom{n+d}{n}.$$

We know even more

Theorem (Claus Scheiderer 2018)

For every $X \subseteq \mathbb{R}^n$ with non-empty interior and $n, d \geq 2$, $(n, d) \neq (2, 2)$,

$$\operatorname{sxdeg}(M_{n,2d}(X)) = \infty.$$

Proof strategy

- **Tool 1 (convex optimization):** Results of Gouveia & Parrilo & Thomas 2013 relating
  - the existence of K-lifts for C to
  - the existence of K-factorizations of the slack matrix of C.
- **Tool 2 (combinatorics):** Ramsey theorem for hypergraphs.

Tool from convex optimization

Some Euclidean spaces

- $\mathbb{R}^n$:

$$\langle x \, , \, y \rangle := x^\top y.$$

- $\mathcal{S}^k$:

$$\left\langle (a_{ij}) \, , \, (b_{ij}) \right\rangle = \sum_{i,j} a_{ij} b_{ij}.$$

- $(\mathcal{S}^k)^m$:

$$\langle (A_1, \ldots, A_m) \, , \, (B_1, \ldots, B_m) \rangle = \langle A_1 \, , \, B_1 \rangle + \cdots + \langle A_m \, , \, B_m \rangle$$

Conic duality and the slack matrix

- If C is a convex cone in Euclidean space V, then

$$C^* = \{u \in V \,:\, \langle u\,,\, x\rangle \geq 0 \text{ for all } x \in C\}$$

  is the *dual cone* of C. (The dual cone captures all linear inequalities valid on C.)
- Note that $\mathcal{S}^k_+$ is self-dual: $(\mathcal{S}^k_+)^* = \mathcal{S}^k_+$.
- Consequently, $(\mathcal{S}^k_+)^m$ is self-dual too.
- We define the *slack matrix* of C as the $C \times C^*$ matrix

$$\big(\langle u\,,\, x\rangle\big)_{u\in C^*,\ x\in C}$$

  The slack matrix keeps the track of evaluations of all linear functionals non-negative on C at all points of C.

Conic factorization of the slack matrix

Slack-matrix theorem (Gouveia & Parrilo & Thomas 2013)

Let C be a closed convex cone. Then the following conditions are equivalent:

(i) C has a semidefinite extended formulation with m LMIs of size k.

(ii) The slack matrix is factorisable as

$$\Big(\langle u\,,\,x\rangle\Big)_{u\in C^*,\;x\in C}=\Big(\langle A_u\,,\,B_x\rangle\Big)_{u\in C^*,x\in C},$$

where $A_u, B_x \in (\mathcal{S}_+^k)^m$.

Combinatorial tool

Hypergraphs and colorings of hyperedges

- $G = (V, E)$ with
$$E \subseteq \binom{V}{t} := \{t\text{-element subsets of } V\}.$$
is a *t-uniform hypergraph* with set V of nodes and a set E of hyperedges.
- $t = 2 \rightsquigarrow$ a graph.
- $G = (V, E)$ with $E = \binom{V}{t}$ is called *complete*.
- We can color hyperedges.

Ramsey theorem

Theodore Motzkin: “Complete disorder is impossible”

Ramsey theorem

Theodore Motzkin: "Complete disorder is impossible"

Ramsey theorem for hyperpgraphs

For $t, n, c \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that every coloring of a complete t-uniform hypergraph on N nodes contains a monochromatic copy of a complete t-uniform hypergraph on n nodes.

Ramsey theorem

Theodore Motzkin: "Complete disorder is impossible"

Ramsey theorem for hyperpgraphs

For $t, n, c \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that every coloring of a complete t-uniform hypergraph on N nodes contains a monochromatic copy of a complete t-uniform hypergraph on n nodes.

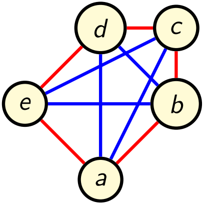
$$(t, n, c) \rightsquigarrow N$$

Definition

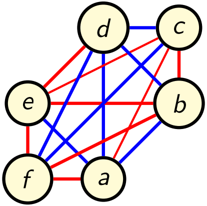
The least possible N in Ramsey theorem is called the *Ramsey number* and denote by $R_t(n; c)$.

Example: Ramsey Theorem

- Input: $t = 2$ (we are dealing with graphs), $n = 3$ (we are interested in monochromatic triangles), $c = 2$ (two colors).
- Output: $N = 6$. That is, if we color a complete graph on 6 nodes with two colors, there we will always have a monochromatic triangle.



no monochromatic triangles!



there is a monochromatic triangles!
(no matter how you 2-color)

Proof sketch

Proof ideas

- We prove
$$\mathrm{sxdeg}(\Sigma_{1,2d}) \geq d+1,$$
to illustrate the proof idea of Theorem A.
- Assuming that $\Sigma_{1,2d}$ has a semidefinite extended formulation with m LMIs of size d, we will arrive at a contradiction.
- We are going to use the slack-matrix theorem for
$$C = \Sigma_{1,2d}.$$

Considering $\Sigma^*_{1,2d}$

We can think of the elements of $\mathbb{R}[x]_{2d}$ as vectors from $\mathbb{R}^{2d+1}$. For example,

$$f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 \in \mathbb{R}[x]_4 \longleftrightarrow \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \in \mathbb{R}^5$$

With this in mind, we can introduce $\Sigma^*_{1,2d}$, the dual cone of $\Sigma_{n,2d}$.

Some elements of $\Sigma^*_{n,2d}$

Note that evaluation of $f \in \mathbb{R}[x]_{1,2d}$ at $x \in \mathbb{R}^n$ can be written as the scalar product For example,

$$f = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 = \left\langle \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}, \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \right\rangle$$

In particular, we see that $\begin{pmatrix} 1 \\ x^1 \\ \vdots \\ x^{2d} \end{pmatrix} \in \Sigma^*_{n,2d}$ for each $x \in \mathbb{R}^n$ because $f(x) \geq 0$ for $f \in \Sigma_{1,2d}$.

Sub-matrix of the slack matrix of $\Sigma_{n,2d}$

We thus conclude that the slack matrix of $\Sigma_{n,2d}$ contains the sub-matrix

$$\left(\ f(x)\ \right)_{f\in\Sigma_{1,2d},\ x\in\mathbb{R}}$$

That is the sub-matrix of the evaluations of SOS polynomials at points of $\mathbb{R}$.

Large finite sub-matrix of the slack matrix

We don't need the whole matrix but only a sufficiently large structured finite sub-matrix. Choose N distinct elements of $\mathbb{R}$, with N sufficiently large. For example, we can use the points $[N] = \{1, \ldots, N\}$. For each $T \in \binom{[N]}{d}$, we introduce the polynomial

$$f_T(x) = \left(\prod_{t \in T} (x - t) \right)^2 \in \Sigma_{1,2d}.$$

Zero pattern of the sub-matrix

We are going to use the sub-matrix

$$\left(\ f_T(s)\ \right)_{T\in\binom{[N]}{d},\ s\in[N]}$$

of the slack matrix.
Its zero pattern:

$$f_T(s)\begin{cases}=0 & s\in T,\\ >0 & s\notin T.\end{cases}$$

For example, for $N = 4$ and $d = 2$:

$$\begin{matrix}\{1,2\}\\ \{1,3\}\\ \{1,4\}\\ \{2,3\}\\ \{2,4\}\\ \{3,4\}\end{matrix}\qquad\begin{pmatrix}0 & 0 & + & +\\ 0 & + & 0 & +\\ 0 & + & + & 0\\ + & 0 & 0 & +\\ + & 0 & + & 0\\ + & + & 0 & 0\end{pmatrix}$$

Invoking slack-matrix theorem

By the slack-matrix theorem,

$$\left(\ f_T(s)\ \right)_{T \in \binom{[N]}{d},\ s \in [N]}$$

is factorizable with respect to the cone $(\mathcal{S}^d_+)^m$:

$$f_T(s) = \langle A_T\ ,\ B_s \rangle$$

holds for some

$$A_T = (A_{T,1}, \ldots, A_{T,m}) \in (\mathcal{S}^d_+)^m$$

and

$$B_s = (B_{s,1}, \ldots, B_{s,m}) \in (\mathcal{S}^d_+)^m.$$

Orthogonality of PSD matrices

Observation

For $F, G \in \mathcal{S}_+^k$:

$$\langle F\ ,\ G \rangle = 0 \qquad \Longleftrightarrow \qquad \mathrm{im}(F) \perp \mathrm{im}(G).$$

So, we do not need to know $A_{T,i}$ and $B_{s,i}$ exactly to check $\langle A_T\ ,\ B_s \rangle = 0$, it suffices to know the images.

Coloring sets T by m-tuples of dimensions

Consider

$$U_{T,i} := \sum_{t \in T} \mathrm{im}(B_{t,i}) \tag{1}$$

and

$$d_{T,i} := \dim(U_{T,i}).$$

Note that

$$\mathrm{im}(A_{T,i}) \perp U_{T,i}.$$

$\{0, \ldots, d\}^m$	our set of $(d+1)^m$ colors
$(d_{T,1}, \ldots, d_{T,m})$	the color of $T \in \binom{[N]}{d}$

Invoking Ramsey theorem

- By Ramsey theorem, if N is large enough, $\binom{W}{d}$ is monochromatic, for some $W \subseteq [N]$ with $|W| = d + 1$.
- $\Rightarrow$ All elements $T \in \binom{W}{d}$ have the same color $(d_1, \ldots, d_m) \in \{0, \ldots, d\}^m$.
- $\Rightarrow \dim(U_{T,i}) = d_i$ does not depend on $T \in \binom{W}{d}$
- By elementary linear algebra, $U_{T,i}$ does not depend of $T \in \binom{W}{d}$. This means:

Claim

For some vector spaces $U_1, \ldots, U_m$, one has $U_{T,i} = U_i$ for all $i \in [k]$ and $T \in \binom{W}{k}$.

Concluding the proof

- Since $|W| = d + 1$, we can choose an arbitrary decomposition $W = T \cup \{s\}$, where $T \in \binom{W}{d}$.
- In view of $0 = f_T(t) = \langle A_T, B_t \rangle$ for $t \in T$, we have $\langle A_{T,i}, B_{t,i} \rangle = 0$ for all $t \in T$ and $i \in [m]$.
- By the Observation, $\mathrm{im}(A_{T,i})$ is orthogonal to $\mathrm{im}(B_{t,i})$. Hence, $\mathrm{im}(A_{T,i})$ is orthogonal to $\sum_{t \in T} \mathrm{im}(B_{t,i}) = U_i$.
- By the choice of U_i, the linear space U_i contains all $\mathrm{im}(B_{w,i})$ with $w \in W$ as a subspace.
- Hence, $\mathrm{im}(A_{T,i})$ is orthogonal to $\mathrm{im}(B_{s,i})$.
- By the Observation, this means that $\langle A_{T,i}, B_{s,i} \rangle = 0$ holds for all $i \in [m]$.
- Thus, we have shown $\langle A_T, B_s \rangle = 0$. Since $s \notin T$, this contradicts $\langle A_T, B_s \rangle = f_T(s) > 0$.

Summary and outlook

Summary

Technology:

$$\underbrace{P_{n,2d}(X)}_{\text{intractable}} \xrightarrow{\text{'approximation'}} C \overset{\text{lifting}}{=\!=\!=} \pi\Big(\underbrace{H \cap K}_{\substack{\text{conic feasibility}\\ \text{set w.r.t. } K}} \Big) \xrightarrow{\text{solving a } K\text{-conic problem}}$$

Implementation of this template requires answering the questions:

- How to approximate? $\Longleftrightarrow$ How to choose C?
- What convex programming to use? Linear? Second-order cone? Semidefinite? Anything else? $\Longleftrightarrow$ How to choose K?
- How to lift? $\Longleftrightarrow$ What conic formulation of C to use?
- Which method to use for solving (large) K-conic problem?

Outlook

Lasserre hierarchies:

- C: an SOS cone or a truncated quadratic module
- $K = (\mathcal{S}_+^k)^m$ (semidefinite programming).

Alternatives:

- Other choices of C and K have also been suggested are being studied.
- More good choices of C and K?
- How to solve (large) conic and semidefinite problems?

Cones that are being studied/used:

- DSOS and SDSOS (Ahmadi, Majumdar 2019)
- SONC (De Wolff, Dressler, Iliman, Theobalad, 2016–)
- SAGE and relative entropy optimization (Chandrasekaran, Shah 2014)
- Sums of structured sparse SOS cones (A., Peters, Sager, in progress)

Enforcing small LMIs makes computations more tractable (numerical studies by A. & Peters & Sager 2019+). Optimization of a n-variate polynomials of degree $2d$ over a box. A snapshot of numerical evaluations:

	$n = 35, 2d = 4$	$n = 40, 2d = 4$
standard SOS	a few LMIs of size ≈ 700 ≈ 5.5 hours!	a few LMIs of size ≈ 800 SDP solver fails!
sparse SOS	lots of LMIs of size ≈ 40 15 seconds!	lots of LMIs of size ≈ 40 20 seconds!

Bounds of sparse SOS only slightly worse than the bounds of standard SOS.

Recent results:

- J. Saunderson (2020): A generalization of the obstruction from $\mathcal{S}_+^k$-lifts to general K-lifts.
- C. Scheiderer (2021): Every planar convex semi-algebraic set C satisfies $\mathrm{sxdeg}(C) \leq 2$.