Symmetry Adapted Gram Spectrahedra

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Joint Work

Joint work with

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Outline

- Symmetric Polynomials as Sums of Squares

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- Properties of the Symmetry Adapted PSD cone

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- Symmetry Adapted Gram Spectrahedron

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- Properties of the Symmetry Adapted PSD cone
- Symmetry Adapted Gram Spectrahedron
- Symmetric Polynomial Inequalities

Polynomial Optimization and SOS relaxation

$$\text{(POP)} \qquad p^* = \inf_{x \in \mathbb{R}^n} \{p(x)\}$$

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which relaxes to

$$p^{sos} = \sup\{\gamma : p(x) - \gamma \text{ is SOS}\}$$

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Feasible Region is the Gram Spectrehedron

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$$5x^2 - 2xy + 2y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{bmatrix} 5 & \\ & \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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$$5x^2 - 2xy + 2y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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$$
\begin{aligned}
5x^2 - 2xy + 2y^2 &= \begin{pmatrix} x & y \end{pmatrix} \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} x & y \end{pmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}
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Theorem (Gatermann, Parrilo)

Given an orthogonal linear representation of the symmetric group S_n, $\sigma : S_n \to Aut(\mathcal{S}^N)$, consider a semidefinite program whose objective and feasible matrices are invariant under the group action. Then the optimal value of the SDP is equal to the optimal value of the same SDP restricted to its fixed point subspace.

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$$D(e) = I$$

$$D((12)) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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Fixed point subspace: $\mathcal{F} = \{X : XD(\sigma) = D(\sigma)X, \forall \sigma \in S_2\}$

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$$p(x) = \begin{pmatrix} x_1^2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{pmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix}$$

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$$T^T Q T = \begin{pmatrix} \tilde{Q}_1 & 0 & \dots & 0 \\ 0 & \tilde{Q}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{Q}_s \end{pmatrix}$$

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$$p(x) = [x]_d^T Q [x]_d$$

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$$\begin{aligned} p(x) &= [x]_d^T Q [x]_d \\ &= [x]_d^T (TT^T) Q (TT^T) [x]_d \end{aligned}$$

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Symmetric Polynomials as Sums of Squares

Let T block-diagonalize $D(\sigma)$, $\forall\sigma \in S_n$.

$$T^TQT = \begin{pmatrix} \tilde{Q}_1 & 0 & \dots & 0 \\ 0 & \tilde{Q}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{Q}_s \end{pmatrix}$$

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$T^T[x]_d$ is called a symmetry adapted basis.

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$$T^T Q T = \begin{bmatrix} q_{11} + q_{13} & \sqrt{2}q_{12} & \\ \sqrt{2}q_{12} & q_{22} & \\ & & q_{11} - q_{13} \end{bmatrix}$$

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where

$$\tilde{Q}_i = \begin{pmatrix} Q_i & 0 & \dots & 0 \\ 0 & Q_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_i \end{pmatrix}$$

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- Size of Q_i is the multiplicity of the irrep
- Number of copies of Q_i is the dimension

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$$T^TQT = \begin{bmatrix} q_{11}+q_{13} & \sqrt{2}q_{12} & \\ \sqrt{2}q_{12} & q_{22} & \\ & & q_{11}-q_{13} \end{bmatrix}$$

Properties of the Symmetry Adapted PSD cone

Proposition

Let m_{λ_i} be the multiplicity of the irreducible representation associated to λ_i. Then the dimension of $PSD_N^{S_n}$ is,

$$\sum_i \binom{m_{\lambda_i}+1}{2}.$$

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Theorem

The extremal rays of $PSD_N^{S_n}$ are in bijection with the set of matrices $Q \in PSD_N^{S_n}$ such that exactly one matrix Q_i has rank one, and the other Q_j, $j \neq i$ have rank zero, considered up to scaling by $\mathbb{R}_{\geqslant 0}$.

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What about the Symmetry Adapted Gram Spectrahedron?

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$$\left\{ \begin{bmatrix} q_{11} + (n-1)q_{12} & 0 & \cdots & 0 \\ 0 & q_{11} - q_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{11} - q_{12} \end{bmatrix} \succeq 0 \right\}$$

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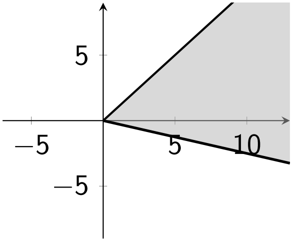

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the Symmetry Adapted PSD cone is

$$\mathrm{PSD}_6^{S_3} = \{ \begin{pmatrix} Q^{\square\square\square} & & \\ & Q^{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}} & \\ & & Q^{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}} \end{pmatrix} \succeq 0 \}$$

where

$$Q^{\square\square\square} = \begin{pmatrix} q_{11} + 2q_{12} & 2q_{14} + q_{16} \\ 2q_{14} + q_{16} & q_{44} + 2q_{45} \end{pmatrix}$$

$$Q^{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}} = \begin{pmatrix} q_{11} - q_{12} & q_{14} - q_{16} \\ q_{14} - q_{16} & q_{44} - q_{45} \end{pmatrix}$$

Symmetric Ternary Quartics

Consider

$$f(x_1, x_2, x_3) = a\sum_{i} x_i^4 + b\sum_{i\neq j} x_i^3 x_j + c\sum_{i<j} x_i^2 x_j^2 + d\sum_{i\neq j\neq k, j<k} x_i^2 x_j x_k$$

where a, b, c, d are fixed coefficients.

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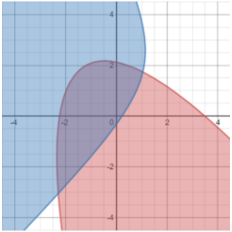
$$f(x_1, x_2, x_3) = a\sum_i x_i^4 + b\sum_{i\neq j} x_i^3 x_j + c\sum_{i<j} x_i^2 x_j^2 + d\sum_{i\neq j\neq k, j<k} x_i^2 x_j x_k$$

where a, b, c, d are fixed coefficients. *Symmetry Adapted Gram Spectrahedron* is the intersection of the two spectrahedron:

$$\{(q_{12}, q_{16}) : \begin{pmatrix} a + 2q_{12} & b + q_{16} \\ b + q_{16} & c + d - 2q_{12} - 2q_{16} \end{pmatrix} \succeq 0\}$$

$$\{(q_{12}, q_{16}) : \begin{pmatrix} a - q_{12} & \frac{b}{2} - q_{16} \\ \frac{b}{2} - q_{16} & c - \frac{d}{2} - 2q_{12} + q_{16} \end{pmatrix} \succeq 0\}$$

Symmetry Adapted Gram Spectrahedron



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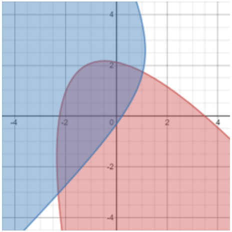

Theorem

Let f be a smooth, generic, positive symmetric ternary quartic. Of its 63 rank three matrix representations, precisely 3 are invariant under the S_3 action, 2 of which are PSD, called the vertices of the Symmetry Adapted Gram Spectrahedron. Moreover, the boundary piece given by the determinant of the trivial block give rank 5 matrices while the boundary piece given by the standard block give rank 4 matrices.

Symmetric Ternary Sextics

$$\mathrm{PSD}_{10}^{S_3} = \left\{ \begin{pmatrix} Q^{\ydiagram{3}} & & & \\ & Q^{\ydiagram{2,1}} & & \\ & & Q^{\ydiagram{2,1}} & \\ & & & Q^{\ydiagram{1,1,1}} \end{pmatrix} \succeq 0 \right\}$$

where

$$Q^{\ydiagram{3}} = \begin{pmatrix} q_{11} + 2q_{12} & \sqrt{2}(q_{14} + q_{16} + q_{18}) & \sqrt{3}q_{110} \\ \sqrt{2}(q_{14} + q_{16} + q_{18}) & q_{44} + q_{45} + q_{46} + 2q_{47} + q_{49} & \sqrt{6}q_{410} \\ \sqrt{3}q_{110} & \sqrt{6}q_{410} & q_{1010} \end{pmatrix}$$

$$Q^{\ydiagram{2,1}} = \begin{bmatrix} q_{11} - q_{12} & \frac{\sqrt{2}}{2}(2q_{14} - q_{16} - q_{18}) & \frac{\sqrt{6}}{2}(q_{16} - q_{18}) \\ \frac{\sqrt{2}}{2}(2q_{14} - q_{16} - q_{18}) & q_{44} + q_{45} - \frac{1}{2}q_{46} - q_{47} - \frac{1}{2}q_{49} & \frac{\sqrt{3}}{2}(q_{46} - q_{49}) \\ \frac{\sqrt{6}}{2}(q_{16} - q_{18}) & \frac{\sqrt{3}}{2}(q_{46} - q_{49}) & q_{44} - q_{45} + \frac{1}{2}q_{46} - q_{47} + \frac{1}{2}q_{49} \end{bmatrix}$$

$$Q^{\ydiagram{1,1,1}} = q_{44} - q_{45} - q_{46} + 2q_{47} - q_{49}$$

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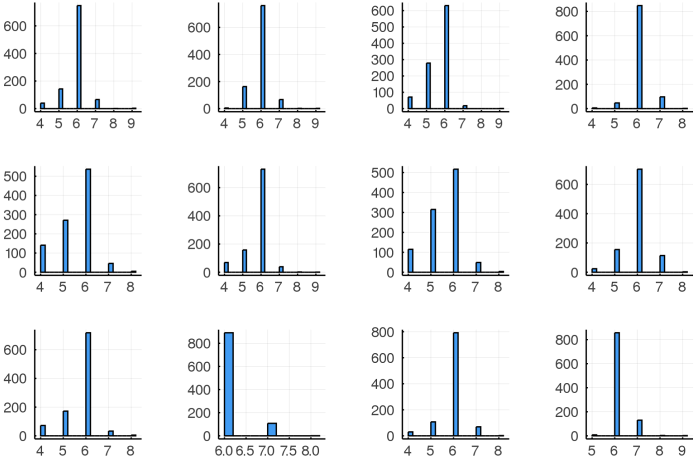

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where $g(1) = g(1, \ldots, 1)$.

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Example

$p_{3,1}(x_1, x_2) = (x_1^3 + x_2^3)(x_1 + x_2)$

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$$G(x) := \frac{g(x)}{g(1)}$$

where $g(1) = g(1, \ldots, 1)$.

Example

$p_{3,1}(x_1, x_2) = (x_1^3 + x_2^3)(x_1 + x_2)$
$P_{3,1}(x_1, x_2) = \frac{1}{4}(x_1^3 + x_2^3)(x_1 + x_2)$

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M_λ - term-normalized symmetric polynomials for monomial basis
E_λ - elementary
P_λ - power-sum
H_λ - homogeneous
S_λ - Schur

Symmetric Polynomial Inequalities

Theorem (Cuttler, Greene, Skandera, Sra)

Let λ and μ be partitions such that $|\lambda| = |\mu|$. Then

$$\begin{aligned} M_\lambda \leqslant M_\mu,\ x \geqslant 0 &\iff \mu \succeq \lambda \\ E_\lambda \leqslant E_\mu,\ x \geqslant 0 &\iff \lambda \succeq \mu \\ P_\lambda \leqslant P_\mu,\ x \geqslant 0 &\iff \mu \succeq \lambda \\ S_\lambda \leqslant S_\mu,\ x \geqslant 0 &\iff \mu \succeq \lambda \end{aligned}$$

whereas $\mu \succeq \lambda$ implies that $H_\lambda \leqslant H_\mu$, i.e.,

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Conjecture

$$H_\lambda \leqslant H_\mu,\ x \geqslant 0 \implies \mu \succeq \lambda$$

An SOS Counterexample

Theorem (Heaton, S.)

A degree-minimal counterexample exhibiting a polynomial $H_\mu - H_\lambda \geqslant 0$ *with* λ, μ *incomparable in dominance order is provided by* $H_{44} - H_{521}$.

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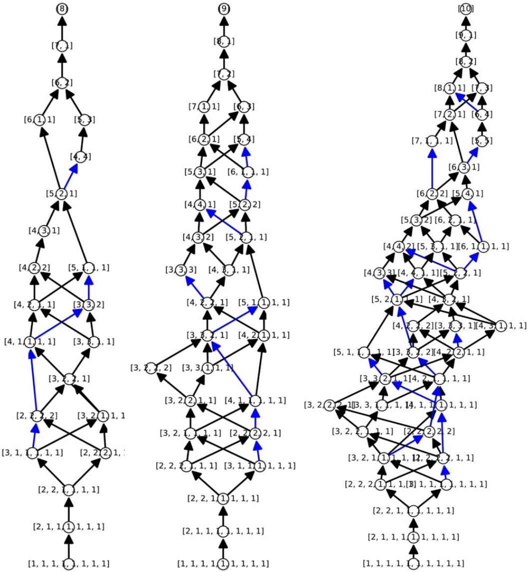
$$\big(H_{44} - H_{521}\big)(x_1^2, x_2^2, x_3^2) \qquad \square$$

An SOS Counterexample

First row of the Gram Matrix:

$$\Big(\frac{17}{9450}, 0, 0, -\frac{4}{1433}, 0, -\frac{4}{1433}, 0, 0, 0, 0, \frac{5}{6699}, 0, \frac{1}{484}, 0, \frac{5}{6699},$$
$$0, 0, 0, 0, 0, 0, \frac{1}{5516}, 0, \frac{4}{6655}, 0, \frac{4}{6655}, 0, \frac{1}{5516}, 0, 0, 0, 0, 0, 0,$$
$$0, 0, \frac{15774751961006984510532337534}{780042543477736474894875053177040 0}, 0, -\frac{1}{2243}, 0,$$
$$-\frac{490859542561433043273727488474533399}{1004640661046807224753241364163337033424}, 0, -\frac{1}{2243}, 0,$$
$$\frac{157747519610069845105323375343}{7800425434777364748948750531770400}\Big)$$

$H_\lambda - H_\mu$ Poset



Thank you

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