Basic ideas of using Balancing **factors within linear regression analysis**

Consider the adjustment of indirect observations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{l} + \boldsymbol{v},\ \boldsymbol{D}(\boldsymbol{l}) = \sigma^2\,\boldsymbol{I} \tag{1.1}$$

where (n denotes the number of observations and u the number of parameters):

$\boldsymbol{A}\ (n,u)$ = matrix of coefficients with $\operatorname{rk}\boldsymbol{A} = u$
$\boldsymbol{x}\ (u,1)$ = vector of "unknown" parameters
$\boldsymbol{l}\ (n,1)$ = vector of (reduced) observations
$\boldsymbol{v}\ (n,1)$ = vector of corrections

$\boldsymbol{D}(\boldsymbol{l})(n,n)$ = variance covariance matrix of observations a priori
σ^2 = reference variance
$\boldsymbol{I}\ (n,n)$ = unit matrix

Consider matrix $\boldsymbol{C}(n,u)$, that transforms the original observations into the adjusted observations (method of least squares)

$$\hat{\boldsymbol{l}} = \boldsymbol{C}\boldsymbol{l} = \boldsymbol{A}(\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top\boldsymbol{l} \tag{1.2}$$

The matrix $\boldsymbol{C}$ is the well-known "extraordinary unit matrix",
because for $u = n$ with $\operatorname{rk}\boldsymbol{A} = u$:

$$\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top = \boldsymbol{I} \tag{1.3}$$

No redundancy $(n = u)$ does not cause any corrections. Important facts on $\boldsymbol{C}$:

$$\operatorname{rk}\boldsymbol{C} = \operatorname{tr}\boldsymbol{C} = u \quad \text{and} \quad 0 \le \boldsymbol{C}_{ii} \le 1\,. \tag{1.4}$$

The diagonal elements of $\boldsymbol{R} = \boldsymbol{I} - \boldsymbol{C}$ are called "observation redundancies", where

$$\operatorname{rk}\boldsymbol{R} = \operatorname{tr}\boldsymbol{R} = n - u \quad \text{and} \quad 0 \le \boldsymbol{R}_{ii} \le 1\,.$$

Necessary restrictions of the matrix A for the discussion:

a) $0 < \boldsymbol{R}_{ii} < 1$
b) No "*latent*" restrictions

and: $\boldsymbol{u + 1 = n}$ **(just one redundant observation)**

Then

$$\operatorname{rk}\boldsymbol{C} = n - 1 = u \tag{2.1}$$

and corresponding

$$\operatorname{rk}(\boldsymbol{I} - \boldsymbol{C}) = n - u = 1 \tag{2.2}$$

Fact: Every symmetric matrix $\boldsymbol{D}(n,n)$ with $\mathrm{rk}\,\boldsymbol{D} = 1$ can be formulated as the *dyadic product* of a vector with itself:

$$\boldsymbol{D} = \boldsymbol{I} - \boldsymbol{C} = \boldsymbol{a}\,\boldsymbol{a}^{\mathsf{T}}$$

and from this

$$\boldsymbol{C} = \boldsymbol{I} - \boldsymbol{a}\,\boldsymbol{a}^{\mathsf{T}} \tag{2.3}$$

Consider all the diagonal elements of $\boldsymbol{C}$ should hold the same numerical value, then

$$\boldsymbol{C}_{ii} = \frac{u}{n} = \frac{n-1}{n} = 1 - \frac{1}{n}$$

because the sum of the diagonal elements has to equal the number u of columns of $\boldsymbol{A}$ from $\mathrm{tr}\,\boldsymbol{C} = u$.

If all diagonal elements of $\boldsymbol{C}$ contain the same numerical value $\frac{u}{n}$ the matrix is called to be **balanced** (in relationship to mechanical applications) $\boldsymbol{C} = \boldsymbol{C}_B$**.**

The elements of the $(n,1)$ sized vector $\boldsymbol{a}$ (remember to refer to $\boldsymbol{C}_B = \boldsymbol{I} - \boldsymbol{a}\,\boldsymbol{a}^{\mathsf{T}}$) have to become:

$$\boldsymbol{a}_i{}^2 = \frac{1}{n} \quad \text{and} \quad \boldsymbol{a}_i = \pm\frac{1}{\sqrt{n}}$$

Hence the matrix $\boldsymbol{C}_B$ holds certain properties:

a) Diagonal elements are equally sized:
$$(\boldsymbol{C}_{\boldsymbol{B}})_{ii} = 1 - \frac{1}{n}$$
b) all non-diagonal elements are (absolutely) equally sized:
$$|(\boldsymbol{C}_{\boldsymbol{B}})_{ii}| = \frac{1}{n}$$
c) all non-diagonal elements of $\boldsymbol{C}_B$ are (absolutely) smaller than the diagonal elements:
$$(\boldsymbol{C}_{\boldsymbol{B}})_{ij} < (\boldsymbol{C}_{\boldsymbol{B}})_{ii}$$
d) for large values of n the balanced matrix $\boldsymbol{C}_B$ approximates the unit matrix $\boldsymbol{I}$:
$$\boldsymbol{C}_B \longrightarrow \boldsymbol{I}$$

$$\boldsymbol{C}_B = \boldsymbol{C}_{\text{balanced}} = \begin{pmatrix} 1-\frac{1}{n} & \pm\frac{1}{n} & \dots & \pm\frac{1}{n} \\ \pm\frac{1}{n} & 1-\frac{1}{n} & \dots & \pm\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \pm\frac{1}{n} & \pm\frac{1}{n} & \dots & 1-\frac{1}{n} \end{pmatrix}$$

From this follows: If $n = u + 1$ becomes large for $\boldsymbol{n \longrightarrow \infty}$ the observation redundancies become very small, because the total degree of freedom of the adjustment remains 1.

The matrix of observation redundancies $\boldsymbol{R = I - C_B}$ has to become a zero matrix $\boldsymbol{O}(n, n)$. So, the corresponding matrix $\boldsymbol{C}_B$ has to become a unit matrix $\boldsymbol{I}(n, n)$.

Particularly the non-diagonal elements of $\boldsymbol{C}_B$ have to become zero.

This fact is **not necessarily true** for any choice of the vector $\boldsymbol{a}(n, 1)$:

Example: From $\boldsymbol{C = I - a\, a^{\top}}$ we choose the elements of $\boldsymbol{a}$ such that

$$\text{tr}\,(\boldsymbol{I - C}) = n - u = 1 = \sum_{i=1}^{n} a_i^2$$

Hence $0 < |a_i| < 1$, however not all $a_i \longrightarrow 0$ for $n \longrightarrow \infty$. Choose for example

$$a_1 = \sqrt{\frac{1}{2}};\ a_2 = a_3 = \cdots = a_n = \sqrt{\frac{1}{2(n-1)}}$$

This demonstrates the fact, that even for $n \longrightarrow \infty$ the matrix $\boldsymbol{C}$ does not necessarily approximate the unit matrix $\boldsymbol{I}$.

Consider a numerical example of a matrix $\boldsymbol{A}$, that ensures equal diagonal elements in $\boldsymbol{C = A(A^{\top}A)^{-1}A^{\top}}$:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 1 & 3 \end{pmatrix};\quad \boldsymbol{C} = \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

These **ideal properties** of $\boldsymbol{C}_B$ usually have to be achieved introducing a *diagonal matrix of positive weights* $\boldsymbol{P}_G$. The diagonal elements are named "**balancing factors**".

Then

$$(\boldsymbol{C_B})_{ii} = (\boldsymbol{P_G}^{½}\boldsymbol{A}\,(\boldsymbol{A^{\top}P_G\,A})^{-1}\boldsymbol{A^{\top}P_G}^{½})_{ii} = \frac{u}{n}\,.$$

For the computation of these "balancing factors" the formulation of the "adjustment of conditioned observations" may be introduced.

The introduction of the "balancing factors" $\boldsymbol{P}_G$ has to be interpreted in the following sense: Starting from the numerical formulation

$$\boldsymbol{Ax = l + v},\ \boldsymbol{D(l)} = \sigma^2\,\boldsymbol{I} \tag{1.1}$$

it is replaced by

$$\boldsymbol{Ax = l + v},\ \boldsymbol{D(l)} = \sigma^2\,\boldsymbol{P_G}^{-1} \tag{2.4}$$

in order to ensure the ideal properties of the matrix $\boldsymbol{C}$.

So the initial $\boldsymbol{D}(\boldsymbol{l}) = \sigma^2 \boldsymbol{I}$ is numerically replaced by the matrix of balancing factors (2.4) $\boldsymbol{D}(\boldsymbol{l}) = \sigma^2 \, \boldsymbol{P}_G{}^{-1}$ to achieve the numerical model (1.1)

What happens to the adjustment of indirect observations if balancing factors are introduced?

Using the well know transformation of the observations $\boldsymbol{l}$ and the design matrix $\boldsymbol{A}$

$$\boldsymbol{l}_B = \boldsymbol{P}_G{}^{½} \, \boldsymbol{I} \quad \text{und} \quad \boldsymbol{A}_B = \boldsymbol{P}_G{}^{½} \, \boldsymbol{A} \tag{2.5}$$

the adjusted observations within (2.4) can be obtained by

$$\widehat{\boldsymbol{l}_B} = \boldsymbol{A}_B \left(\boldsymbol{A}_B{}^{\top} \boldsymbol{A}_B \right)^{-1} \boldsymbol{A}_B{}^{\top} \boldsymbol{l}_B$$

or analogous

$$\boldsymbol{l}_B = \boldsymbol{P}_G{}^{½} \boldsymbol{A} \, (\boldsymbol{A}^{\top} \boldsymbol{P}_G \, \boldsymbol{A})^{-1} \boldsymbol{A}^{\top} \boldsymbol{P}_G{}^{½} \boldsymbol{P}_G{}^{½} \, \boldsymbol{l}$$

If the matrix

$$\boldsymbol{P}_G{}^{½} \boldsymbol{A} \, (\boldsymbol{A}^{\top} \boldsymbol{P}_G \, \boldsymbol{A})^{-1} \boldsymbol{A}^{\top} \boldsymbol{P}_G{}^{½}$$

becomes the unit matrix $\boldsymbol{I}$ for $n = u + 1$ and $n \longrightarrow \infty$ (see 1.3), it is obtained

$$\widehat{\boldsymbol{l}_B} = \boldsymbol{I} \, \boldsymbol{P}_G{}^{½} \, \boldsymbol{l} = \boldsymbol{P}_G{}^{½} \, \boldsymbol{l} = \boldsymbol{l}_B \tag{2.6}$$

and hence

$$\widehat{\boldsymbol{l}} = \boldsymbol{l}$$

This is the basic target for large n from $\boldsymbol{A}(\boldsymbol{A}^{\top} \boldsymbol{A})^{-1} \boldsymbol{A}^{\top} \longrightarrow \boldsymbol{I}$.

Summary: For $u + 1 = n$ it is demonstrated, that introducing a diagonal matrix of positive weights $\boldsymbol{P}_G$ ensures equal influence (in terms of the matrix $\boldsymbol{C}$) of all observations to the adjustment result of least squares adjustment.

In general, this fact is not provided when introducing the matrix $\boldsymbol{I}$ as the matrix of "*uncorrelated equal weighted observations*".

Computation of balancing factors for $(\boldsymbol{u + 1 = n})$

To compute "balancing factors" the adjustment of **indirect observations** has to be transformed into a **conditioned adjustment** (adjustment of conditions only).

It will be shown, that balancing factors are a function of the coefficient matrix $\boldsymbol{A}$ of the adjustment of indirect observations.

Consider a numerical example:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 8 \end{pmatrix};\ \boldsymbol{l} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix};\ \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};\ \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

This data can be written in matrix notation as

$$\boldsymbol{Ax} = \boldsymbol{l} + \boldsymbol{v} \quad \Leftrightarrow \quad -\boldsymbol{I}\,\boldsymbol{v} + \boldsymbol{Ax} = \boldsymbol{l}$$

with unit matrix $\boldsymbol{I}(n,n)$. The corresponding table is:

$\boldsymbol{v_1}$	$\boldsymbol{v_2}$	$\boldsymbol{v_3}$	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{l_i}$
-1	0	0	1	1	1
0	-1	0	1	2	2
0	0	-1	1	8	4

Let:

$$\boldsymbol{Ax} = \boldsymbol{l} + \boldsymbol{v}, \qquad \boldsymbol{D}(\boldsymbol{l}) = \sigma^2\,\boldsymbol{I}$$

or:

$$-\boldsymbol{I}\,\boldsymbol{v} + \boldsymbol{Ax} = \boldsymbol{l}\,, \qquad \boldsymbol{D}(\boldsymbol{l}) = \sigma^2\,\boldsymbol{I}$$

The second form may be subdivided (from the “tableau”) into

$$-\boldsymbol{I}_{\mathrm{E}}\,\boldsymbol{v} + \boldsymbol{A}_{\mathrm{E}}\boldsymbol{x} = \boldsymbol{l}_{\mathrm{E}} \tag{3.1}$$
$$-\boldsymbol{I}_{\mathrm{R}}\,\boldsymbol{v} + \boldsymbol{A}_{\mathrm{R}}\boldsymbol{x} = \boldsymbol{l}_{\mathrm{R}} \tag{3.2}$$

The matrix $\boldsymbol{A}_{\mathrm{E}}$ has to be a regular matrix to be inverted.

With u = number of parameters $\boldsymbol{x}$ and n = number of observations $\boldsymbol{l}$,
the corresponding matrix will have these sizes:

$$\boldsymbol{A}_{\mathrm{E}}(u,u), \boldsymbol{A}_{\mathrm{R}}(n-u,u),\ \boldsymbol{l}_{\mathrm{E}}(u,1), \boldsymbol{l}_{\mathrm{R}}(n-u,1),\ \boldsymbol{I}_{\mathrm{E}}(u,n), \boldsymbol{I}_{\mathrm{R}}(n-u,n),\ \boldsymbol{x}(u,1).$$

For the example the reduced matrices are:

$$-\boldsymbol{I}_{\mathrm{E}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix};\ -\boldsymbol{I}_{\mathrm{R}} = (0 \quad 0 \quad -1)$$

$$\boldsymbol{A}_{\mathrm{E}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix};\ \boldsymbol{l}_{\mathrm{E}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix};\ \ \boldsymbol{A}_{\mathrm{R}} = (1 \quad 8);\ \boldsymbol{l}_{\mathrm{R}} = (4)$$

From equation (3.1) follows:

$$\boldsymbol{A}_{\mathrm{E}}\boldsymbol{x} = \boldsymbol{l}_{\mathrm{E}} + \boldsymbol{I}_{\mathrm{E}}\,\boldsymbol{v},$$

hence:

$$\boldsymbol{x} = \boldsymbol{A}_{\mathrm{E}}^{-1}\,(\boldsymbol{l}_{\mathrm{E}} + \boldsymbol{I}_{\mathrm{E}}\,\boldsymbol{v}) \tag{3.3}$$

Equation (3.3) introduced to (3.2) yields:

$$-\boldsymbol{I}_{\mathrm{R}}\,\boldsymbol{v}+\boldsymbol{A}_{\mathrm{R}}\boldsymbol{x}=\boldsymbol{l}_{\mathrm{R}}$$

$$\Leftrightarrow\quad -\boldsymbol{I}_{\mathrm{R}}\,\boldsymbol{v}+\boldsymbol{A}_{\mathrm{R}}\left(\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\,(\boldsymbol{l}_{\mathrm{E}}+\boldsymbol{I}_{\mathrm{E}}\,\boldsymbol{v})\right)=\boldsymbol{l}_{\mathrm{R}}$$

$$\Leftrightarrow\quad -\boldsymbol{I}_{\mathrm{R}}\,\boldsymbol{v}+\boldsymbol{A}_{\mathrm{R}}\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\boldsymbol{l}_{\mathrm{E}}\;+\boldsymbol{A}_{\mathrm{R}}\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\boldsymbol{I}_{\mathrm{E}}\,\boldsymbol{v}=\boldsymbol{l}_{\mathrm{R}}$$

$$\Leftrightarrow\quad \underbrace{\left(\boldsymbol{A}_{\mathrm{R}}\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\boldsymbol{I}_{\mathrm{E}}-\boldsymbol{I}_{\mathrm{R}}\right)}_{\boldsymbol{A}_{\mathrm{bed}}}\boldsymbol{v}=\underbrace{\boldsymbol{l}_{\mathrm{R}}-\boldsymbol{A}_{\mathrm{R}}\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\boldsymbol{l}_{\mathrm{E}}}_{\boldsymbol{l}_{\mathrm{bed}}} \qquad (3.4)$$

With $$\boldsymbol{A}_{\mathrm{bed}}(n-u,n)=\boldsymbol{A}_{\mathrm{R}}\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\boldsymbol{I}_{\mathrm{E}}-\boldsymbol{I}_{\mathrm{R}} \qquad (3.5)$$

and $$\boldsymbol{l}_{\mathrm{bed}}(n-u,1)=\boldsymbol{l}_{\mathrm{R}}-\boldsymbol{A}_{\mathrm{R}}\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\boldsymbol{l}_{\mathrm{E}} \qquad (3.6)$$

The index $_{\mathrm{bed}}$ denotes the transformed matrices of the adjustment of conditioned observations.

Now consider the adjustment of conditioned observations following the formulation:

$$\boldsymbol{A}_{\mathrm{bed}}=\boldsymbol{B};\quad \boldsymbol{l}_{\mathrm{bed}}=-\boldsymbol{w};\quad \boldsymbol{B}\boldsymbol{v}+\boldsymbol{w}=\boldsymbol{o}$$

It is well known [WOLF 1994] that:

$$\boldsymbol{B}(\boldsymbol{l}+\boldsymbol{v})+\boldsymbol{c}=\boldsymbol{o}\ \text{ or }\ \boldsymbol{B}\boldsymbol{v}+\boldsymbol{w}=\boldsymbol{o},\ \text{ with }\ \boldsymbol{w}=\boldsymbol{B}\,\boldsymbol{l}+\boldsymbol{c}$$

Introducing a diagonal matrix of weights for the observations the equations for the determination of the parameters are taken from [WOLF 1994, pp 46]:

$$\boldsymbol{v}=-\boldsymbol{P}^{-1}\boldsymbol{B}^{\top}(\boldsymbol{B}\boldsymbol{P}^{-1}\boldsymbol{B}^{\top})^{-\boldsymbol{1}}\,\boldsymbol{w}\ ;$$
$$\boldsymbol{Q}_{\boldsymbol{kk}}=(\boldsymbol{B}\boldsymbol{P}^{-1}\boldsymbol{B}^{\top})^{-\boldsymbol{1}}\ ;$$
$$\boldsymbol{v}=-\boldsymbol{P}^{-1}\boldsymbol{B}^{\top}\boldsymbol{Q}_{\boldsymbol{kk}}\,(\boldsymbol{B}\,\boldsymbol{l}+\boldsymbol{c})$$

and the variances of the adjusted observations become:

$$(\boldsymbol{Q})=\boldsymbol{P}^{-1}-\boldsymbol{P}^{-1}\boldsymbol{B}^{\top}\boldsymbol{Q}_{\boldsymbol{kk}}\,\boldsymbol{B}\boldsymbol{P}^{-1}.$$

This includes the well-known “ANSERMET-CHECK”: the sum of the diagonal elements of the matrix

$$\boldsymbol{D}=\boldsymbol{I}-\boldsymbol{P}^{-1}\boldsymbol{B}^{\top}\boldsymbol{Q}_{\boldsymbol{kk}}\,\boldsymbol{B}$$

equals the number of observations minus the number of condition equations.
We compute:

$$\boldsymbol{B}=\boldsymbol{A}_{\mathrm{R}}\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\boldsymbol{I}_{\mathrm{E}}-\boldsymbol{I}_{\mathrm{R}}\quad\text{and}\quad -\boldsymbol{w}=\boldsymbol{l}_{\mathrm{R}}-\boldsymbol{A}_{\mathrm{R}}\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\boldsymbol{l}_{\mathrm{E}}$$

$$\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix};\quad \boldsymbol{A}_{\mathrm{R}}\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\boldsymbol{I}_{\mathrm{E}}=(-6\quad +7\quad 0)\ ;$$
$$\boldsymbol{B}=\boldsymbol{A}_{\mathrm{R}}\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\boldsymbol{I}_{\mathrm{E}}-\boldsymbol{I}_{\mathrm{R}}=(-6\quad +7\quad -1)\ ;\ \boldsymbol{l}_{\mathrm{R}}-\boldsymbol{A}_{\mathrm{R}}\boldsymbol{A}_{\mathrm{E}}^{-\boldsymbol{1}}\boldsymbol{l}_{\mathrm{E}}=(-4)=-\boldsymbol{w}$$

In order to check the numerical computations, the equation $\boldsymbol{Bl} = \boldsymbol{w}$ can be used, introducing the observation vector $\boldsymbol{l}$ from the adjustment of indirect observations.

$$\boldsymbol{Bl} = (-6 \quad +7 \quad -1)\begin{pmatrix}+1\\+2\\+4\end{pmatrix} = (+4) = \boldsymbol{w}$$

From the matrix $\boldsymbol{B}$ the balancing factors are easily obtained by:

$$(\boldsymbol{P}_G)_{ii} = (\boldsymbol{B}_{1i})^2 \tag{3.7}$$

This is because: Introducing these values to the adjustment of conditioned observations the (n, n) size matrix

$$\boldsymbol{D} = \boldsymbol{I} - \boldsymbol{P}^{-1}\boldsymbol{B}^{\top}\boldsymbol{Q}_{\boldsymbol{kk}}\,\boldsymbol{B}$$

obtains equal sized numerical values on is diagonal.

Summary: The above mathematical derivation **for a single redundant observation** demonstrates the properties of “balanced observations” and their computation.

It is shown, that the projection matrix $\boldsymbol{C}$ of the method of least squares does not suffer from smearing effects any more.

Because the effect of balanced observations is target for the adjustment it is stated to introduce “balanced observations” to achieve equal weighted observations.

This indicates an extended formulation of the adjustment model within the method of least squares.

Literature:

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