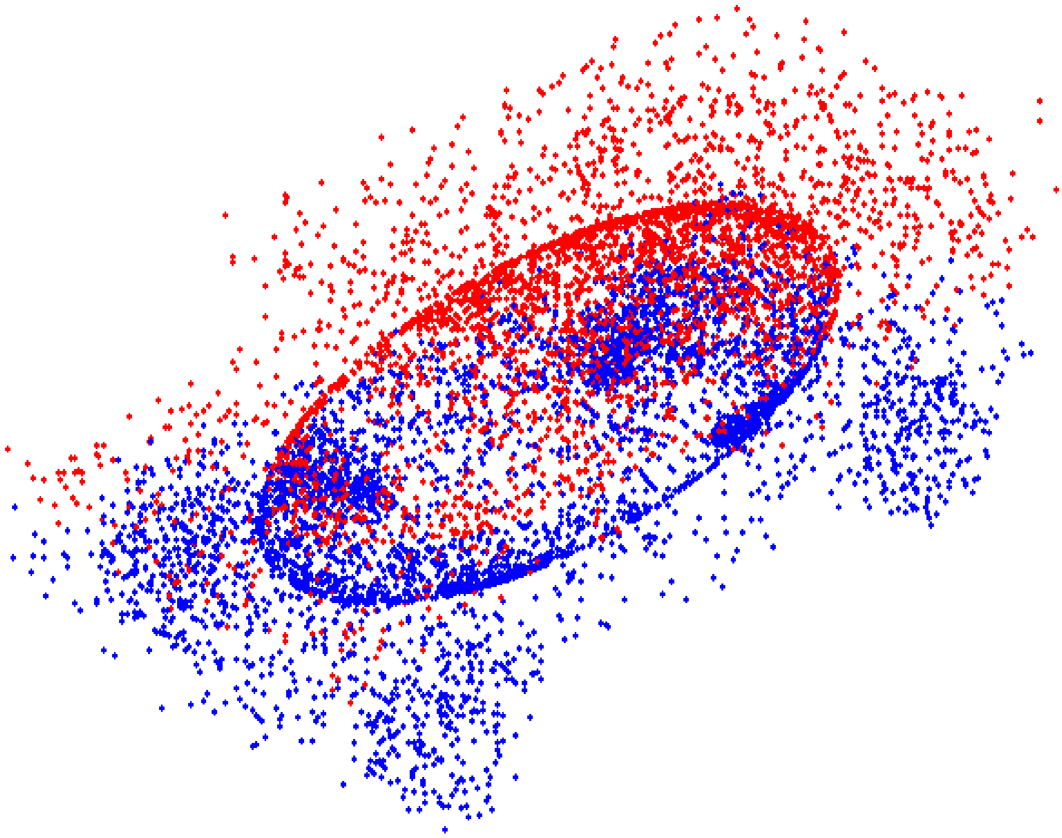
Understanding Basic Principles
Optimization
Regulation and Adjustment

The Beginners Guide



In the beginning there is balance

Preface

This booklet is designed to serve as an easy to understand and digest guideline to adjustment and optimization technique. The basic (mechanical) principles of balance or generalized equilibration are described starting with simple means.

It is our target to achieve transparency and ease of understanding. We shall not concentrate on sophisticated scientific explanations

In the beginning there is balance (equilibrium)

This almost philosophical statement shall be found in some basic attitudes of modern adjustment and optimization technology. Its application is reflected within the inverse balancing that comes along as a strategy to judge and evaluate parameter estimation with so called indirect observation

We shall explain and investigate

- the calculation of means (balanced position)
- the generalized calculation of means and their different targets
- the inverse balanced position and the necessary re-weighting
- visible and hidden restrictions or crisp restrictions

Within adjustment and optimization there is a strong impact of statistical concepts for parameter estimation and determination

We shall provide a variety of tools that supplement each other in order to serve most appropriate and reliable results that are significant for the use of data evaluation within Manufacturing Excellence Control (MEC).

Georg Gottfried Kampmann

Stockholm - February 13, 2009

Content

1 Basics about different means and targets

1.1 About the mechanical background (balance) of the simple arithmetic mean

Consider some mechanical guys taking time to determine the actual width of a vise.

Initially they use a simple ruler as seen on the picture above. And – as a matter of fact – these guys start a discussion how **to make it best**.

Idea: Get an impression about the mechanical properties of the ruler. A screw driver is attached to the vise to build an apex. Its top is taken to balance the ruler and – no surprise – there is a single location (almost exactly in the middle of the ruler body) that comes along with **balance**.

Idea: How does this peak point on the ruler change if some equal weights are placed arbitrarily on the ruler surface?

The mechanics immediately recognize a real change in the balancing position of the ruler. And pretty soon they realize they do not have to test out this individual point of balance, they may determine the screwdrivers location numerically by just adding the weights positions (location on the ruler) and dividing the result by the number of the weights.

This is the so called **arithmetic mean AM** which is well known. It is simple to calculate, for example

$$AM = (l_1 + l_2 + l_3 + l_4) / 4$$

In this simple formula the l_1 , l_2 , l_3 , l_4 denote the individual positions of the sphere weights on the ruler and 4 denotes their number (you may recognize 4 sphere weights on the picture).

The mechanics repeat their experiments using the arithmetic mean several times and they realize it is correct unless some "mistakes" are made namely

a) Miscalculating of the AM
b) Misreading the spheres position

From all the different experiments they even recognize another interesting fact, namely the **repetition number**. This term comes from placing multiple sphere weights at the same ruler location as can bee seen from the picture below (three identical sphere weights almost at the same position l_2)

From the computation of the AM there is no difference. Consider that the multiple position l_2 can be expressed as three different portions l_{2A}, l_{2B} and l_{2C} then the AM for the peak balance position can be expressed as

$$AM = (l_1 + l_{2A} + l_{2B} + l_{2C} + l_3 + l_4) / 6$$

The mechanics quickly realize the ordinary arithmetic mean might be extended to the arithmetic mean with repetitions, or generally the **weighted arithmetic mean WAM**.

Consider the individual 4 positions (l) attached to their repetition number (p), say

$$(l_1 \text{ and } p_1 = 1) ,\ (l_2 \text{ and } p_2 = 3) ,\ (l_3 \text{ and } p_3 = 1) , (l_4 \text{ and } p_4 = 1)$$

In this case the weighted arithmetic mean **WAM** can be expressed as

$$WAM = (l_1\, p_1 + l_2\, p_2 + l_3\, p_3 + l_4\, p_4) / (p_1 + p_2 + p_3 + p_4)$$

This is nothing more than a numerical extension of the simple arithmetic mean. Nevertheless, this formula can still be used in the case of non integer weights (or repetition numbers). Repetition numbers in terms of equilibration or balance may be regarded as weights – hence may take any real positive number. They have to be positive since negative weights are not defined.

Summary: We have recognized the close relationship between the balanced position (center of gravity) and the arithmetic mean (weighted or equally weighted). When data evaluation comes to apply arithmetic means (or its generalization) there is a straight correspondence and relationship towards mechanics, a so called **duality**. We shall not loose this attitude when working within data evaluation.

1.2 The data evaluation strategy from different means

After having some discussion about center of gravity and its relationship towards the arithmetic mean the mechanic guys start to determine the width of the vise.

What they already feel from their previous experiments

a) to determine the width of the vise there should be repeated measurements to account for the possibility that one was incorrect (wrong determination)
b) involve different persons (independent measurements) to avoid repeated errors – this comes along with the statistical term "no correlations" or independent observations

Five different persons now provide a single measurement result of the vises width using a caliper rule as the measurement device.

No.	1	2	3	4	5
Size	8.12	8.13	8.14	8.14	8.15

Figure: Series of five (direct) observations from with caliper rule

The result from the simple arithmetic mean is

AM = (8.12+8.13+8.14+8.14+8.15) / 5 = **8.136**

This value of the AM being derived from a sample of size 5 (number of observations) is called **parameter estimation.** The parameter to be estimated is the width of the vise and the estimation technique is the (weighted) arithmetic mean.

Each measurements (observations) deviation from the AM is called **correction** and the numerical computation is quite easy to provide

Correction = AM - measurement

Generally

Correction = Parameter (Estimation) Result - Observation

For example: first correction to the first observation yields 8.136 – 8.12 = **0.016**. Hence the following results for the corrections are computed:

No.	1	2	3	4	5
Size	0.016	0.006	-0.004	-0.004	-0.014

Figure: Corrections from the AM caliper rule

Remark: Summing up all computed corrections from the simple arithmetic mean **MUST be 0.0**. This fact derived from theory may be consequently applied to check the proper computation of AM and the related corrections.

Remark: Often the corrections are called **residuals,** derived from the Latin word for remnant or leftover.

Now the mechanic guys chose another device to determine the width of the vise. Instead of the calliper rule they apply a micrometer to “enhance the accuracy”.

We shall see what accuracy is meant to be. It is plain to see: accuracy is related to the choice of the measurement device **and** the number of measurements being executed.

Both facts have to be taken into account when achieving pre formulated accuracies of the parameter estimation (width of the vise).

The picture shows the calliper rule, the micrometer and two different rulers. Here are the results from the measurements with the micrometer:

No.	1	2	3	4	5
Size	8.134	8.135	8.135	8.137	8.138

Figure: Series of five (direct) observations micrometer

The result from the simple arithmetic mean of the data is

AM = (8.134+8.135+8.135+8.137+8.138) / 5 = **8.1358**

And the corrections are

No.	1	2	3	4	5
Size	0.0018	0.0008	0.0008	-0.0012	-0.0022

Figure: Corrections from the AM micrometer

The maximum size of the corrections obviously decreases with the application of the "more accurate" measurement device (micrometer instead of calliper rule)

To express this enhancement of accuracy in terms of the arithmetic mean as the parameter estimation we shall introduce the well known statistical term **Standard Deviation (mean square error, RMS = root means square and so on).**

Here is the recipe: Square each single correction and sum them up. Instead dividing the result by the number of observations divide it by **f = degree of freedom** = (number of observations – 1). The result is the so called **variance**. Compute the square root of the variance **V** and you obtain the standard deviation **SD**.

Here is the formula: Let e_i denote the single correction and n their number, then

$$V = \text{variance} = (e_1\, e_1 + e_2\, e_2 + e_3\, e_3 + \ldots + e_n\, e_n) / (n\text{-}1)$$

$$SD = \text{Standard Deviation} = \pm \sqrt{V}$$

If you compare both measurements (calliper rule = less accurate and micrometer = more accurate) you recognize from the values of the Standard deviations

SD (caliper rule) = √ (0.00052 / (5-1)) = ± 0.011402…

SD (micrometer) = √ (0.0000108 / (5-1)) = ± 0.001643…

Higher accuracy comes along with a smaller value of SD and less accuracy comes along with larger numerical value for SD.

Remark: There is a brilliant book on the History of Statistics by Professor Stephen **Stigler**. Anyone who enjoys learning more about the beginning of statistics – this is my recommendation. Another outstanding book with historical background comes from Stephen **Skinner**: Sacred Geometry.

1.3 The multiple method approach (for means)

Beside the arithmetic mean there are quite a number of means that serve purposes other than the AM mechanical approach of centre of gravity or balance.

We shall focus on two other ones, the **Median** and the **MinMax** because these are somewhat polarities in the whole concept. At a first glance for both there is no easy way to recognise the relationship to the mechanical centre of gravitation (balance) as there is with the arithmetic mean. We shall see about the benefits of **Median** and **MinMax** (like our mechanic guys did during their computations with the AM).

The **Median** is determined by ordering the observations by their size (magnitude) and selecting the middle one. In case of an even number of observations take the arithmetic mean of both neighbors.

The **MinMax** mean is determined by building the arithmetic mean of the smallest and the largest value of the observations.

Let us return to the calliper rule measurements of the vise.

No.	1	2	3	4	5
Size	8.12	8.13	8.14	8.14	8.15

Figure: Series of five (direct) observations from caliper rule

Arithmetic Mean = **8.136**

Median (sorting the observations by its magnitude, select the middle (3)) = **8.14**

MinMax (arithmetic mean of smallest and largest = (8.12+8.15)/2) = **8.135**

The following figure shows the resulting corrections for each mean. The computation follows: correction = mean – observation.

Correction No.	1	2	3	4	5
Median	0.020	0.010	0.000	0.000	-0.010
AM	0.016	0.006	-0.004	-0.004	-0.014
MinMax	0.015	0.005	-0.005	-0.005	-0.015

Figure: Correction for different means caliper rule data

Remark: Remember the SD (from arithmetic mean) = ± 0.011402

If you compare the results from these 3 different means and compute the differences in between, you will recognize they are all within a certain frame of the Standard Deviation SD (or at least a factor multiplied with SD).

Remark: For the corrections of the Median the maximum (absolute) size correction is 0.020, for the AM there is 0.016 and for MinMax there is 0.015. Hence the largest (maximum) absolute correction becomes the smallest (minimum) in magnitude for the MinMax mean. MinMax means: **Min**imization of the **Max**imum correction.

Remark: What about the idea to choose the MinMax maximum correction to describe the range of the corrections in addition to arithmetic mean Standard Deviation SD? The MinMax correction denotes the upper level that restricts all other corrections to be smaller in magnitude. Hence this value is a limitation of measurement accuracy, somewhat better than the SD is.

Remark: Median and MinMax as means may be interpreted and calculated from sorting. There is a standard lecture book from Robert **Sedgewick** – Algorithms in C. The sorting devices are so fast that even millions of items may be sorted in real time. From computational point of view Median and MinMax might be computed even faster than the arithmetic mean.

Remark: Besides Sorting the Median and MinMax mean from the observations, another mathematical tool called Linear Programming or Linear Optimization may be applied. This tool – the Simplex Algorithm- may operate these "target functions" in general.

Now an important question is posed: Why use 3 different means instead of the simple arithmetic mean? From the discussion of the mechanical guys we know: No trouble in measurement or observation, no problems with the AM. But in case there is unexpected deviation: Better be equipped with a nice toolbox.

Now this is what happened to the mechanic guys in the garage. When they did the first computation of the calliper rule data, they typed 815 instead of 8.15 to the calculator.

This is the spoiled data (containing the spoiled measurement No. 5)

No.	1	2	3	4	5
Size	8.12	8.13	8.14	8.14	815

Figure: Spoiled Series of five (direct) observations from caliper rule

And the arithmetic mean is then AM = (8.12+8.13+8.14+8.14+815) / 5 = **169.506**

What an impact towards the "correct result" 8.136. What we learn:

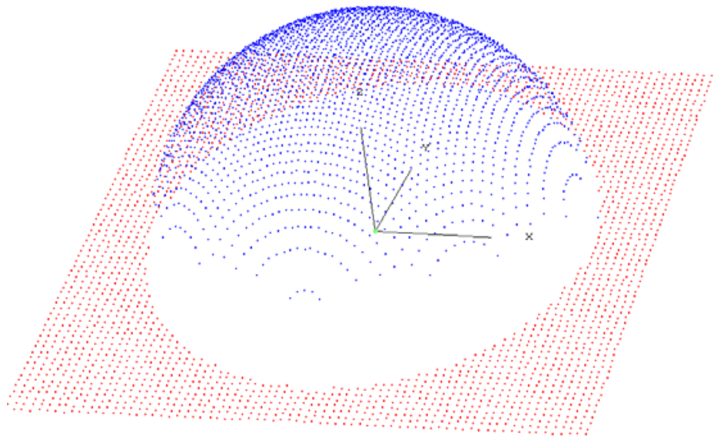
Practical technical Advice: Never apply arithmetic mean and corresponding statistics like SD without making sure that there are no unexpected deviations **UD** left in the data (outlier, gross error, blunder …) or else you will obtain spoiled results.

Remark: We decide to call these blunders **unexpected deviations** or **UD**.

Here are some reasons to explain why.

a) Please see the example: If someone would have done "error" 8.16 instead of 8.15 the actual value, the arithmetic mean result sure would not indicate a gross data deviation within the whole data set. Hence the term blunder is related to the amount of "good data" and its accuracy. Unexpected deviations are clearly related to the mass data properties.

b) An error is something done wrong. It does not sound appropriate or even gentle to assume 30% gross error in the data. See the following picture taken from a scan of a hemisphere in front of a wall (positioning purpose). The blue points represent the hemisphere, the red ones the unexpected deviations. Modern machine engineering has to deal with scanning devices, and it is almost impossible to hit the target (hemisphere) without deviations (outliers, blunders…). Hence the unexpected deviation is part of the whole measurement process using scanners.



Return to example: Because the mechanics apply the three different means, they obtain

Median = **8.14**
Arithmetic Mean = **169.506**
MinMax = **411.56**

And it is obvious the median result is not harmed by the UD. However, MinMax suffers from an utmost spoiled result. Keep in mind – in the case of no unexpected deviations the three different means obtained quite similar results.

In fact: The mechanic guys recognized their mistake immediately and excluded measurement No. 5 from the data. Then they did a re computation of AM, Median and MinMax and were satisfied with the result. **We shall explain the necessity and the benefits of this strategy for data evaluation with means in the coming chapter.**

1.4 Different target functions for corrections

Consider the following 5 observations with a single spoiled one.

No.	1	2	3	4	5
Size	3.0	3.0	3.0	3.0	30.0

Figure: Series of five (direct) artificially spoiled observations

1) (MEDIAN) = 3.0
2) (ARITHMETIC MEAN) = 8.4
3) (MINIMAX) = 16.5

See the corrections below:

Nr.	1	2	3	4	5
Median	0.0	0.0	0.0	0.0	-27.0
AM	5.4	5.4	5.4	5.4	-21.6
MinMax	13.5	13.5	13.5	13.5	-13.5

Figure: Corrections for Median, AM, MinMax of artificial spoiled observations

What we might recognize at first sight: With the Median the **UD** = unexpected deviation (blunder) is represented by the size of the corresponding correction or residual. The arithmetic mean corrections **start to smear** this peak towards the other corrections, and this **smearing is extreme with MinMax** mean.

Obviously the Median is best for the detection of unexpected deviation (within the different means). **UDD = Unexpected Deviation Detection (error, blunder outlier, gross error detection)** should be executed with the Median because the mean is harmed minimally and the UD correction is maximal.

Now let us have a close look on what we shall call a **breakpoint.** This is the percentage of measurements we may spoil without affecting the median mean too much, that is a breakdown (change in parameter estimation when introducing blunders) as we suffer from the arithmetic mean and the MinMax mean.

Consider the table below and try to verify it by your self (keep in mind the sorting of magnitude for median mean).

No. of Values	No. of BLUNDERS	%
3	1	33.3
4	1	25.0
5	2	40.0
6	2	33.3
7	3	42.9
8	3	37.5
9	4	44.4
10	4	40.0
50	24	48.0
100	49	49.0
1000	499	49.9

Figure: Number of gross errors (in % of number of observations) the simple median is able to resist

This is the message: In terms of robustness (a statistical term that wants to describe the resistance of a parameter estimation technology against blunders) the simple median can deal with a maximum possible 50 % breakdown if the number of observations comes to an infinite number.

Remark: This table demonstrates the maximum percentage of spoiled data we might introduce without harming the median mean. Robustness – this is a statistical strategy to resist against spoiled data. Caution: For the one-dimensional case (the means) we apply the median technology and will succeed within the demonstrated limits. It will become more competitive, when we try to generalize these properties from direct observations –one parameter- to indirect observations – more than one parameter. What we shall see is necessary for this attitude – the basic feeling and understanding of mechanical balance.

Now we shall execute some computations for the obtained corrections (residuals) of the above example.

a) **We compute all the corrections from the AM**

$$5.4 + 5.4 + 5.4 + 5.4 + (-21.6) = \mathbf{0.0}$$

The (arithmetic) sum of correction of the arithmetic mean has to be 0.0 – a nice numerical control for your computations. Because this is a must – it is a so called **restriction**. Hence the standard deviation for the arithmetic mean comes along with a division of n-1, where n denotes the number of measurements.

Remark: There is a very thorough description of these facts within Least Squares. Les **Kirkup** and Bob **Frenkel** provide “An introduction to uncertainty in Measurement”

b) **We compute the sum of absolute (just positive pre sign) corrections** for Median, AM and MinMax.

Sum of absolute corrections Median = **27.0**
Sum of absolute corrections arithmetic mean = **43.2**
Sum of absolute corrections MinMax = **67.5**

What we see: The absolute sum of corrections from the Median is the smallest among the means. Hence this parameter estimation technique is called **L**east **A**bsolute **V**alue **E**stimation or **LAVE** as an abbreviation found in literature.

Remark: Now we are introducing another mathematical tool, the so called Lp-Norm estimation technology within linear parameter estimation. Within this mathematical concept, the **LAVE** is called **L1-Norm-Estimation**. It minimizes the absolute sum of corrections from its target function (minimization of the sum of absolute corrections or residuals).

c) **We compute the sum of the squared corrections** for Median, AM and MinMax.

Sum of squared corrections for Median = **729.00**
Sum of squared corrections for AM = **583.20**
Sum of squared corrections for MinMax = **911.25**

What you see: The sum of the squared corrections for the AM is the smallest one among the means. Hence the determination of the arithmetic mean comes along with the **Method of least Squares (LS)** that is the minimization principle. Sometimes the Least Squares is called **L2-Norm Estimation** or (more statistically related) **Best Fit**.

Remark: Remember the simple formula from triangles, the Pythagoras formula. Within a more generalized mathematical concept the target function to minimize a squared sum of values is a generalized attitude of this.

d) **We determine the maximum (absolute) correction**

Maximum (absolute) correction for Median = **27.0**
Maximum (absolute) correction for AM = **21.6**
Maximum (absolute) correction for MinMax = **13.5**

The MinMax yields the smallest (minimal) maximal absolute correction.

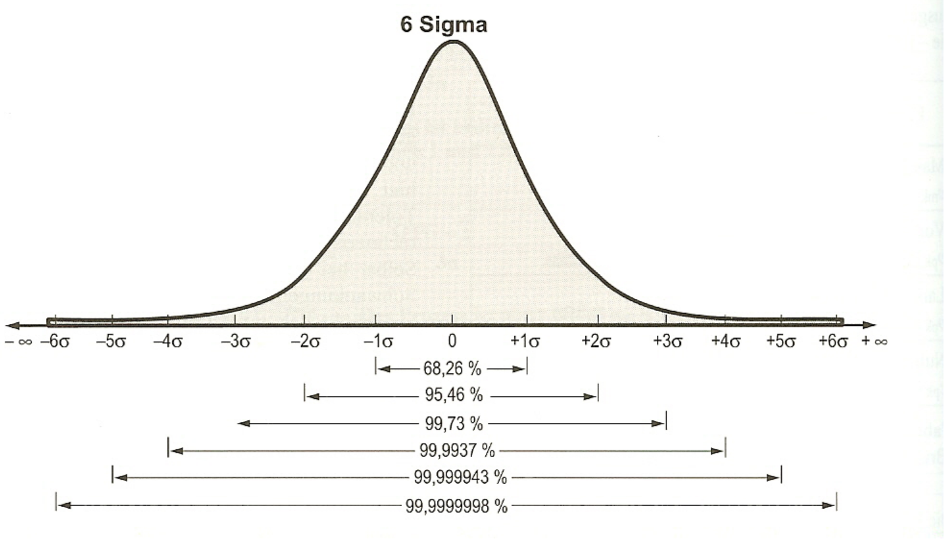
What we see: In mathematical terms (minimization and maximization of functions) with these three different means, we achieve three different minimization attitudes or target functions, namely:

1.) Minimization of the sum of absolute corrections (residuals) = Least Absolute Value Estimation (LAVE) = L_1

2.) Minimization of the sum of squared corrections (residuals) = Method of Least Squares (LS) = L_2

3.) Minimization of the maximum (absolute) correction (residual) MinMax-Method = LT (sometimes called Tschebyscheff or L_∞)

Remark: Remember curve discussion from school, where you had to find minima and maxima = extremes, reflection points and so on? Now we shall discuss a popular wide spread plane curve.



This curve is called the bell shaped curve. Mathematics sometimes regards this curve to hold a physical background, physics regard it as a mathematical truth – in reality it is nothing other than an assumption when you apply this curve to "probabilities" – like statistics do (normal distribution).

Now let us (mathematically) discuss this curve. There is a single peak directly in the middle (zero) of the curve, a maximum. Assuming that you know the mathematical formulation of this curve, finding extremes (maximum) comes along with differentiation (deriving the gradient, the slope). In general the differentiation yields a formula for the maximum (or minimum) of the curve.

This formula (of finding extremes of a function) in general is the **formula for the arithmetic mean** (method of least squares), which we have been working with above.

Further, we are able to recognize two refection points from the sketch (change of curvature within the curve). From the sketch you see them named as σ (Grecian alphabet), pre sign – for the left one and pre sign + for the right one).

Now this σ is what we called the standard deviation above (the theoretical standard deviation). Our Standard Deviation SD being computed from real data is an empirical one (division by n-1). Hence, to distinguish between theoretical and empirical value we do not use the Grecian alphabet.

Again, do you remember curve discussion from school? To determine reflection points you use the second derivate of the function and you obtain a new formula for the reflection points (if there are any). Now this formula from the bell shaped curve is the formula for the standard deviation.

Remark: This technology to derive an estimation function from distribution curve is called Maximum Likelihood Estimation (MLE). As it is true for the simple arithmetic mean, there are curves for the Median and the MinMax, called double exponential or Laplace distribution for Median and rectangular or equal distribution for the MinMax.

Here are the different curves (probability distribution) in comparison to each other. See the differences in curvature (if there is any) and in location (this is the middle of the symmetric curves)

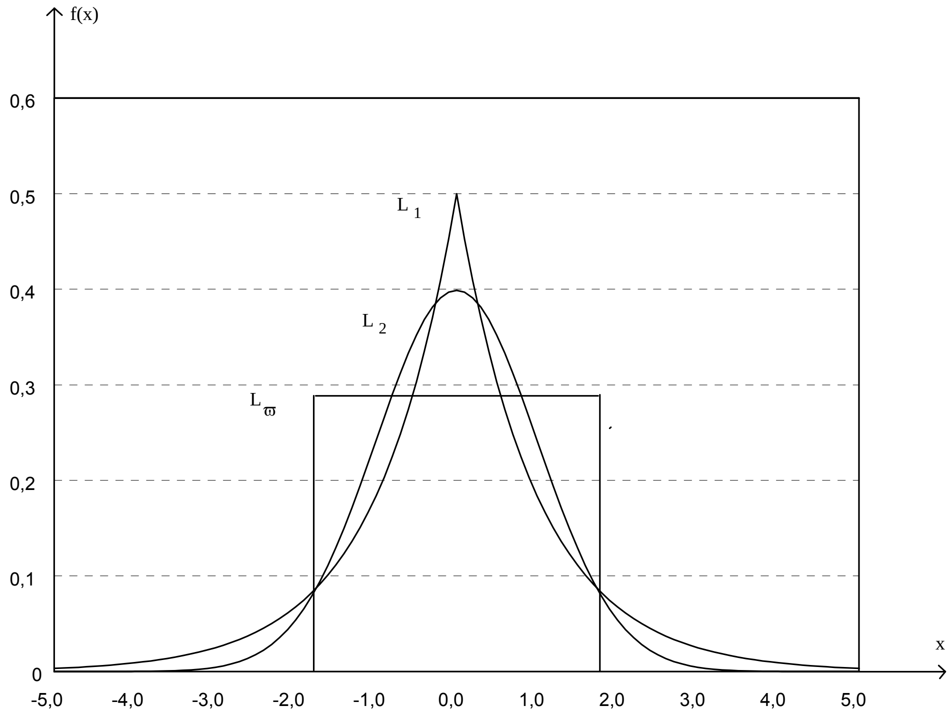


Figure: Rectangular curve, bell shaped curve and Laplace distribution curve

What we realize from this sketch: If the measurements do not contain UD unexpected deviations, the means (Median, AM, MinMax) almost **achieve the same result for the means** (center of the curves), especially if the number of measurements or observations becomes very high (infinite).

Remark: The bell shaped curve is well known as the **normal distribution curve** or **Gauss probability density curve**. However there is an academic disagreement whether Gauss or Legendre or Adrian is to honor for initially introducing the method of least squares. This parameter estimation procedure is obtained as maximum likelihood estimate MLE from the bell shaped curve.

Let us return to the actual measurements of the mechanical guys, the ones with the single spoiled observation.

Here is the data again:

No.	1	2	3	4	5
Size	8.12	8.13	8.14	8.14	815

Figure: Spoiled Series of five (direct) observations from caliper rule

Corrections from the Median mean = **8.14**.

No.	1	2	3	4	5
Correction	0.02	0.01	0.00	0.00	-806.86

Figure: Corrections from the Median

We already recognized (with eagle eye from comparison between each other) the fifth correction from Median to be an unexpected deviation, a blunder – just by its size compared to the other ones.

Now we like to do this “blunder detection” automatically or better yet: numerically. Here is the proceeding as a recipe.

a) Compute the Median of observations (ordering, sorting, whatever)
b) Compute the corrections from the Median
c) Compute a single numerical value c from the absolute corrections from the median
d) Compute the absolute values d = abs (c – correction) as a so called second stage corrections
e) Compute their median w, the so called second stage median.
f) Compute the value K = 1.483 w ((n-1)/n).

Remark: n equals the number of measurements and the term ((n-1)/n) shall be named **condition density CD**

Remark: If K becomes zero, numerical value chose a very small number instead to avoid any division by zero. Try to understand, what happens with the data if K becomes 0.0 (A little hint: No deviation at all for at least 50% of the data).

g) Create n values T for each original correction of the median by taking the absolute value of the correction and divide it by K.
h) Each T that is larger than 3 indicates an unexpected deviation (blunder, gross error)
i) Compute the level of detection of unexpected deviation **UDL** = 3 K. This numerical value denotes the threshold (level) of the corrections from initial median. Exceeding it means the corresponding measurement is suspected to be an unexpected deviation.

Remark: The value 3 is taken from the bell shaped curve. It comes along with a three sigma level. You may take 4 or even six (the corresponding significance levels a given in the sketch of the bell shaped curve). See the corresponding %part from the sketch of the bell shaped curve, for example σ = 3 = 99.73%

In machine engineering 3 is quite appropriate, you may even choose 6 – there is a lot of discussion left over for the sophisticated ones (refer to the six σ method)

Remark: Since the condition density is introduced to the T value we should use the t-distribution instead of the N-distribution. Nevertheless, if n comes to large values, both different distributions t and N yield the same statistical fraction.

Let us UDD practically for the example of the caliper rule with one spoiled measurement

Step1 (a): The median becomes **8.14**

Step2 (b): The corrections become **0.02, 0.01, 0.00, 0.00, -806.86**, set in magnitude order of absolute (no pre sign) correction **0.00, 0.00, 0.01, 0.02, 806.86**

Step3 (c): The single numerical value c, from the Median, becomes **0.01**.

Step4 (d): Compute the second stage corrections d (five observations, hence n=5):

abs (0.01-0.00) = 0.01.
abs (0.01-0.00) = 0.01.
abs (0.01-0.01) = 0.00
abs (0.01-0.02) = 0.01
abs (0.01-806.86) = 806.85

These values set in order of its magnitude **0.00, 0.01, 0.01, 0.01, 806.85**

Step5 (e): The second order median becomes **0.01**

Step6 (f): Computation of K = 1.483 w ((5-1)/5) = **0.011864**

Step7 (h): Compute each single T for the original median corrections.

T1 = 0.02 / 0.011864 = 1.6858 < 3: NO UD (blunder)
T2 = 0.01 / 0.011864 = 0.8429 < 3: NO UD (blunder)
T3 = 0.00 / 0.011864 = 0.0000 < 3: NO UD (blunder)
T4 = 0.00 / 0.011864 = 0.0000 < 3: NO UD (blunder)

T5 = 806.86 / 0.011864 = 68009,1… > 3 : UD (blunder)

Step8 (i): UDL = 3 K = 3 * 0.011864 = **0.035592**

Now this is a straight advice to you: **Verify this approach in praxis** – ***just do it***.

Please do no artificial simulation, no Monte Carlo – if you want to throw the dice you may visit a casino, or as Frank Zappa wrote: And you will do as you are told until the rights to you are sold!

Please go to your garage, and choose an arbitrary device to measure an arbitrary wrench.

Then ask friends or family members (9 different ones – think of uncorrelated observations) to take measurements.

Determine Median – perform UDD (unexpected deviation detection) and re compute Median, AM and MinMax for the remaining measurements after extinction of UD. Compare their results with the computed SD Standard Deviation of AM.

Please repeat this task several times to get acquainted to the method and approach to determine means.

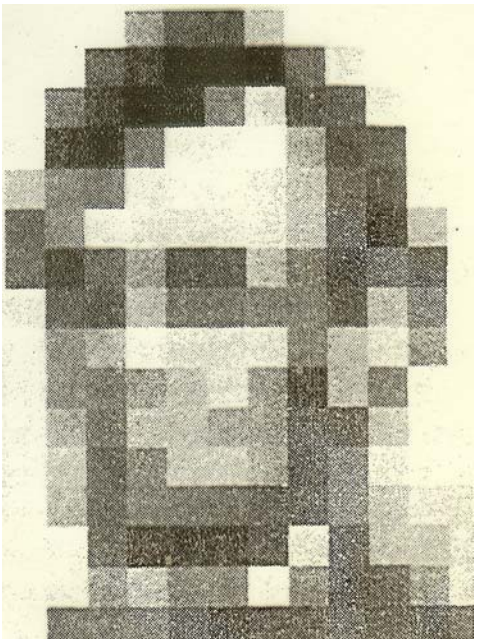
Now you are on your way understanding why the basic principles and their generalization work so well in practical applications.

Remark: This little mathematical tool for UDD (unexpected deviation detection) may be verified in general with the concept of so called “appropriate scaling in robust estimation”. Refer to Profs. **Heij**, **de Boer**, **Franses**, **Kloek** and **van Dijk** from Erasmus University Rotterdam, Netherlands with “Econometric Methods with applications in Business and Economics”.

At this point we have seen the benefits of the Median (blunder detection), and we have seen the origin of the arithmetic mean (principle of balance, MLE for bell shaped curve), but what about the MinMax?

As we already know from our computation examples, there is maximum smearing in the corrections. This feature might be quite valuable, as the next picture will demonstrate.

Remember that MinMax is only valuable, in the case that you extinguished blunders or UD.



Watch this picture. Remember MinMax **smearing**: Start to blink your eyes very fast – the positive result of smearing becomes obvious.

In case blinking your eyes fast does not lead to the intended effect – stand up and increase the distance from this picture until you recognize the person. You might catch the idea that smearing sometimes might be interesting and obtaining interesting results.

Summary:

We have learned about the means, that there are three different ones with three different objectives (target functions) and three different approaches. We are conscious, that repetition numbers are similar to weights, and we know that the arithmetic mean is closely related to mechanical interpretation of balance (centre of gravity).

Now here is the general recipe for Optimization, Regulation and Adjustment means:

a) Compute the Median and extinguish the unexpected deviations
b) Compute Median, AM and MinMax for the reduced data and compute their differences
c) To describe the accuracy of your result (parameter estimation) compute Standard Deviation SD, MinMax correction and UDL

It is a challenging task to generalize these basics to other applications in order to enjoy the benefits.

And keep in mind an (technician) attitude:

A vision without an action is a daydream
An action without a vision is a nightmare

2. Generalisation of common application of different means

2.1 Extension of the means

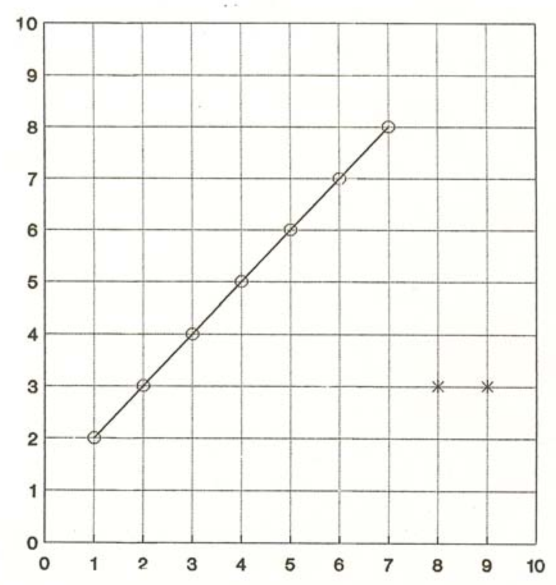
We learned about one dimensional data evaluation (means): **a successful strategy to operate 3 different means competitively**. We shall be protected against unexpected deviations and we provide an efficient toolbox to describe the accuracy of our result, the parameter estimation.

We should be comfortable with this concept before trying to generalize it, for example towards a **determination of a straight line**. This task involves not just one parameter (one dimensional, the mean), but at least two parameters, namely the axis cut of the straight line and its slope.

Further, with the Spatial Reference System SRS, there are spheres that determine the end of the bars. A sphere even has four parameters, namely its centre in Cartesian co-ordinates x,y,z and its diameter or radius.

Now when we extend the concept of means to more than one parameter, we do not observe the parameter (width of the vise) directly (**parameter estimation of direct observations**). Usually we have to estimate **parameters from indirect observations** (as is true with the parameter of the spheres of the SRS that are determined from measured points at the surface of the spheres, hence indirect observations or measurements for centres and diameters of spheres)

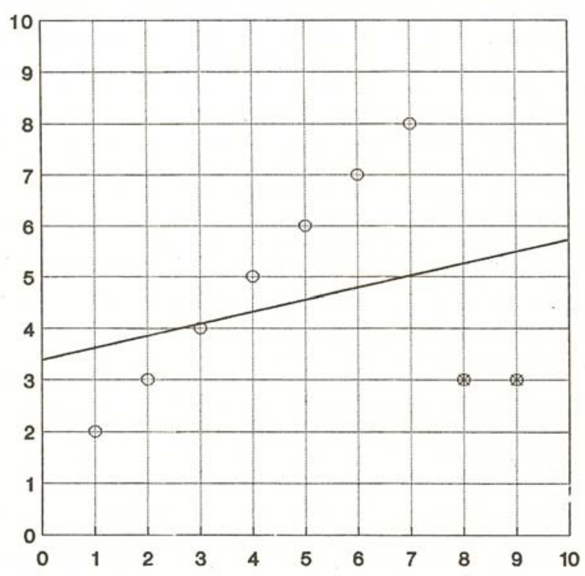
The determination of the two parameters of a straight line is therefore a **parameter determination within indirect observations**. See the picture below of some points forming a straight line.



This picture shows some points in 2 dimensional cartesian co-ordinates. These kinds of co-ordinates are quite familiar to the machine engineering and mechanical people. Cartesian co-ordinates are described by straight axis, which are perpendicular (orthogonal) to each other.

In addition, what you see from the sketch, are seven points that define a straight line and some unexpected deviations are located on the right end of the sketch.

If you calculate this data using a so called linear regression from **Least Squares (LS, Best Fit),** in a wide spread software (for example Excel), then you will get a parameter for the axis cut and the slope. If you then graph the straight line, then you will receive the following (disappointing) result.



From our former investigations of properties of the different means we did not expect the generalized arithmetic mean (minimisation of the squared sum of corrections) = **Least Squares** to operate correctly, leaving out the unexpected deviations. Well, we are a little disappointed.

This is because the ***Least Squares is called Best Fit*** , and what we achieve is ***anything other than best!***

Remark: ***Best*** is a statistical term and **denotes minimal variances**. Because the definition of the variances includes the squared sum of corrections, the variance from least squares becomes minimal, hence **statistically best**. Another issue comes from the theory of statistics: These properties (to be statistically best) only are true in case of real normal distributed data, no blunders or unexpected deviations allowed.

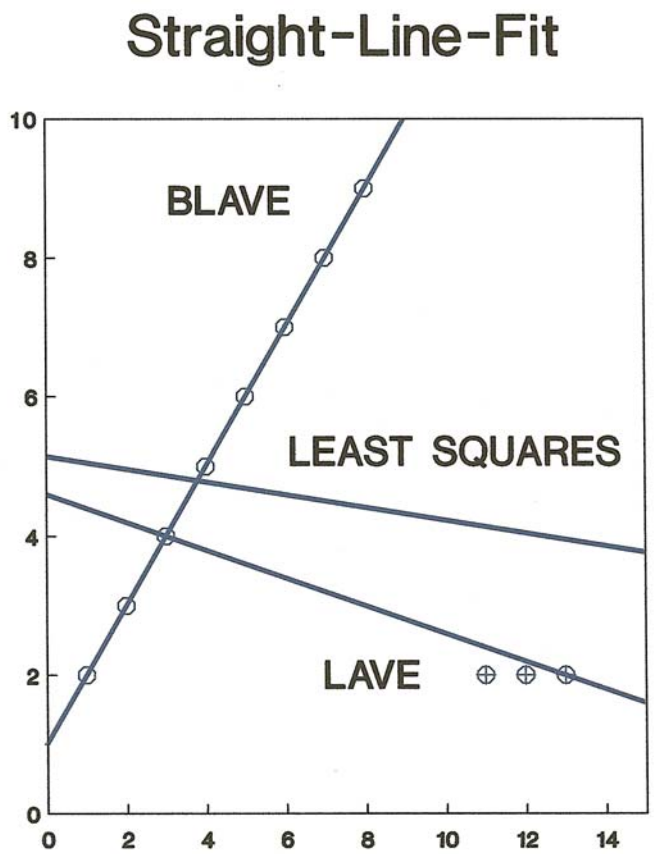
Moreover what makes this best fit result even more devastating: the least squares adjustment statistics can not detect the (obviously included) blunders.

Remark: A lot of efforts have been done to solve this serious problem for mechanical engineering data evaluation. There is a large chapter within theoretical statistics that deals with that problem, the so-called **robust statistics**.

Remark: Within robust statistics the Median enjoys maximum robustness against gross errors as a simple mean (parameter estimation of direct observations)

We shall not apply the MinMax: The result will be even worse. And here is even more disappointment: ***The Median target function is even the worst!***

2.2 Introduction to leverages, restrictions and balancing



This straight line shows an even more disastrous result for the minimization of the absolute sum of corrections (**L**east **A**bsolute **V**alue **E**stimation), because one of the blunders becomes a point, that determines the straight line.

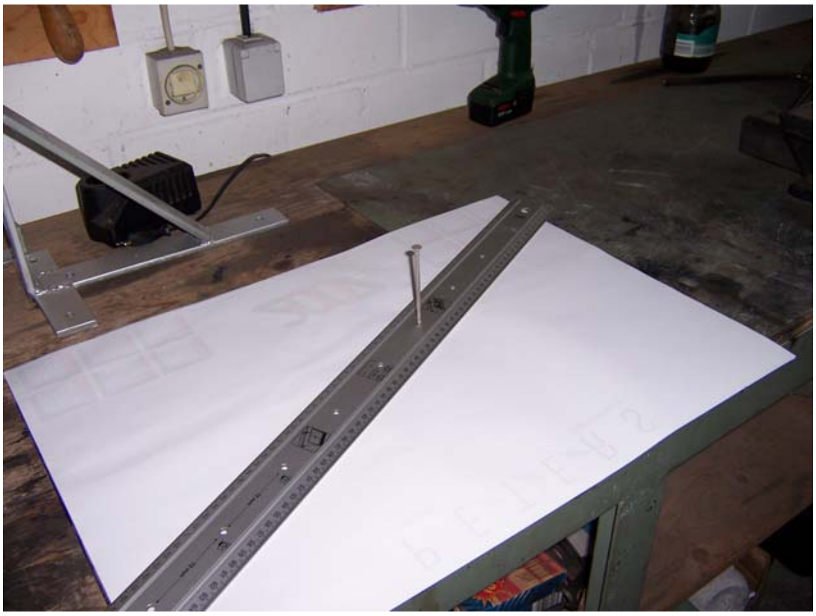
Remember the median was built by sorting. Within this procedure each and every single observation deserves the same influence in the result. Same with the arithmetic mean, where each observation shares an equal influence in the result because you have to add it all up.

Now, obviously the target function of L1 (Median) is not sufficient enough to obtain equal influence of each single observation within indirect parameter estimation as it is in direct parameter estimation.

It seems as if there are different influences. This is just a gut feeling at this time. (This is the sincere suspicion: **Perhaps we have to face *unintentional weights* with indirect observations**)

We already know from mathematics and mechanics, that we even can exceed the influence of a single measurement up to infinity. Remember the ruler with the 3 spheres at one single location (repetition numbers for the weighted arithmetic mean WAM).

Or, in case of fitting a straight line, we can even restrict the resulting line running through a certain – predefined point.



View the picture of the ruler being fixed to a certain point with a nail. Just a rotation of the ruler is left (leaving one parameter instead of two parameters of a free straight line). If we would attach two points of the ruler with a nail, the ruler cannot even move anymore. Just a rotation is left over. (We see, if the number of restrictions equals the number of parameters, the result is determined).

Remark: If there is a parameter estimation containing crisp restrictions (the nail within the ruler), the number of parameters reduces itself by the number of restrictions. A mathematical **restriction** is the utmost expression of a **must**. There is a slighter (fuzzy) version of it in terms of mechanical engineering. It is a so-called **leverage**.

From the picture of the straight lines above we might obtain the impression, that the (robust) target function of the median is harmed by some leverages (unintended weights) namely the straight line points far out of the bulk of the remaining co-ordinates or points.

Leverages: mechanicals are well acquainted to this device. In case you need them and know them, they are very helpful. In case you do not know leverages and you operate devices or material anyway, the results might be damaged. See the illustration of a leverage taken from ancient famous Galileo.

Now please digest a picture from construction engineering. This engineering structure finds its mechanical expression from the position and the thickness of the bars towards its edges.

What do you think, is there any opportunity to get this building into a certain harmony (balance) by just changing the diameter of the bars?

Remark: To create a balanced structure, Hooks law may be applied to distribute the degree of static uncertainty (degree of freedom) equally towards all bars and knots.

What we know up here:

a) the target function for median is not sufficient to generalize the properties of robustness from direct observations to indirect observations

b) within indirect parameter evaluation we might introduce or obey so-called restrictions, that restrict the degree of freedom

c) We have a gut feeling of unintended influences of observations depending on the "location" towards the bulk of the others

Remark: There is another (theoretically) quite inconvenient influence towards adjustment and optimization. Sometimes (especially with structures, networks and geometric dimensioning and tolerance) the functional model, the so-called **DESIGN** suffers from or obeys restrictions, and we do not even know! We call them hidden or latent restrictions. Within all parameter estimation procedures it is of utmost important to find and eliminate these influences (regardless towards the target function)

Remark: The very brilliant Prof. **Grafarend** provides the necessary technology in his recent textbook "Linear and nonlinear Models", 2006.

This is the question: Where does this burden (leverages, latent restriction) come from? Is there any location to see and judge the individual influence of observations towards parameter estimation result?

Well, we have to start from the scratch. We shall do it briefly to avoid this humorous booklet becoming boring.

Through all the years we have become addicted to the cartesian co-ordinates and we have lost sight of other, more appropriate tools.

If you choose the so-called **Pluecker-Grassmann** co-ordinates from the 19th century, you can detect all these restrictions, latent restrictions, different influences from different observation and deal with them. This is to extinguish some effects and, more importantly to synchronize (enhance to the same level) influences. This idea has something to do with sensitivity analysis and necessary repair.

What has to be done (in order to generalize the properties of the simple median for robustness in unexpected deviation detection):

- Calculate the individual influence of an observation after extinction of latent and crisp restrictions

- Standardize (invert) this unintended influence like 3 ⅓ = 1 or 8 ⅛ = 1 (where 3 is the influence and ⅓ is the balance factor). The result reflects equal influence of all observations and in terms of mechanics it is balanced.

- Apply these factors (inverted unintentional weights or balancing factors) to your generalized median target function (LAVE) and you receive a generalized UDD (blunder detection strategy)

- Remember, we apply straight mechanical understanding (from mechanical nature principle of balancing) towards statistical procedures. Hence, these balancing factors have to be expressed statistically.

Remark: When you follow this mechanical idea with natural structures like bee hives, minerals or other nicely constructions (nice in human eyes), often you realize there is no change necessary – the structures are balanced initially by nature.

Back to our example of the straight line fit with the disastrous result for LAVE = L1. The result **BLAVE** that stands for **Balanced LAVE** should satisfy what we expected from the properties of the simple median.

Balancing in this context comes along with the extinction of different influences of the measurement on the parameter determination.

Summary of chapter 2

This is what we could recognize

- Never rely on a Least Squares result solely without protection against deviations, use the combination of different tools.

- There are different target functions to thoroughly check the evaluation (convergence of results).

- Within parameter estimation of indirect observations we are suffering from unintended weights (repetition numbers). There is a remedy from mechanical engineering called balancing (enhancing to the same level).

- Restrictions of a crisp and latent kind have to be taken into account before parameter evaluation (elimination).

- Generalizing the properties of the means comes along with an extension of the target functions towards geometrical equilibration (balance).

- There is a close relationship between data evaluation and mechanical engineering and structural engineering in terms of balance.

Within this booklet we concentrated on the means and its generalisation – this is fitting.

When applying the mechanical principles of balancing to any other functionality (design), this approach offers more than simple data fitting, namely

- sensitivity analysis
- optimization structuring
- regulation
- decision making (numerical control)
- others

3 Inverse balancing and its shape, the Inner Reference

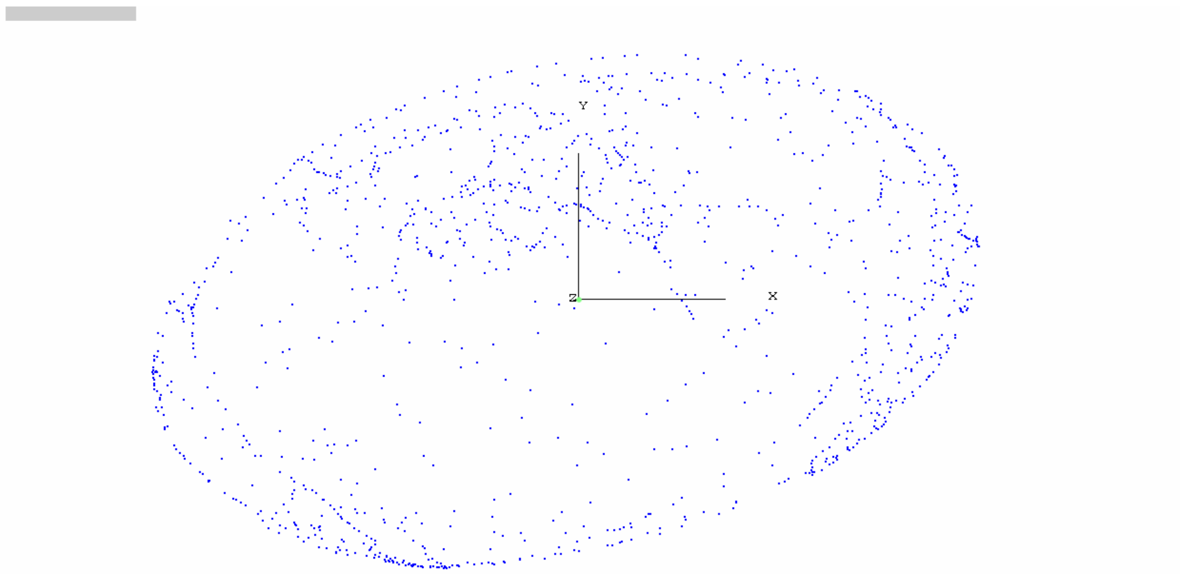
From the former discussion on arithmetic mean and corresponding balance you should have obtained an idea about the importance finding and creating “equal influence”. This target and attitude is called ***inverse balancing***.

Now we may start with an amazing fact about the result of a balanced structure (what ever dimension or (mixed) unit it is. We shall leave out more sophisticated topics from full or latent restrictions)

Do you remember this amazing numerical value π = 3.141592… or the golden section Φ = 1.618… that can be explained mathematically from the *Fibunacci series*?

Here is another nice circumstance that should be regarded as a generalized natural impact (in fact, it is a figure). At least this technology from balancing is a so called projection towards a unique figure.

Consider the below sketch of an (almost) arbitrary bulk of points below



Picture 1: 1200 (almost) arbitrary points in a plane

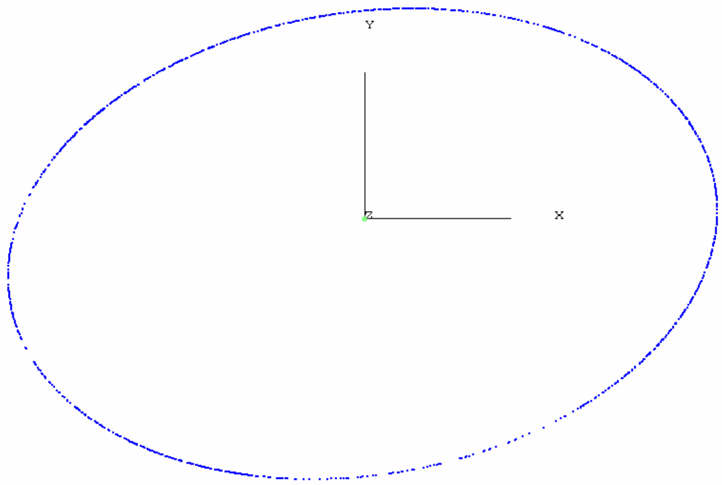
Now consider these 1200 plane points (X1/Y1, X2/Y2, … X1200/Y1200) being ordered arbitrarily towards a box or tablet (or in mathematical terms called a matrix **A** with 1200 rows and two columns) like

X1	Y1
X2	Y2
X3	Y3
….	….
X1200	Y1200

Now apply the generalized balance towards these points (Computation of balancing factors for each row of this matrix and multiply the corresponding row with the positive square root of the balancing factor). Quite a variety of numerical procedures may be applied, like the variety of algorithms to create an inverse matrix.

The resulting orthogonal projection $\mathbf{C} = (\mathbf{A}\ (\mathbf{A}^T\ \mathbf{A})^{-1}\ \mathbf{A}^T)$ becomes diagonal equally sized like an **A** for the application of the arithmetic mean always is.

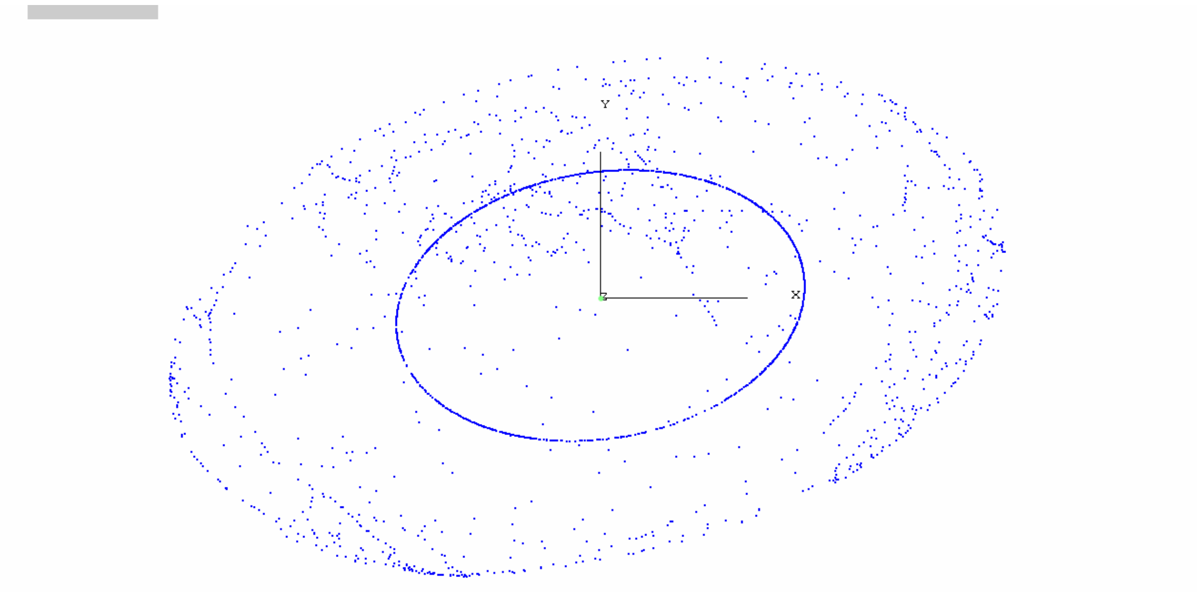
Now figure out the next sketch, that shows the balance of these points and the corresponding figure.



Picture 2: Inner Reference IR of 1200 points from picture1

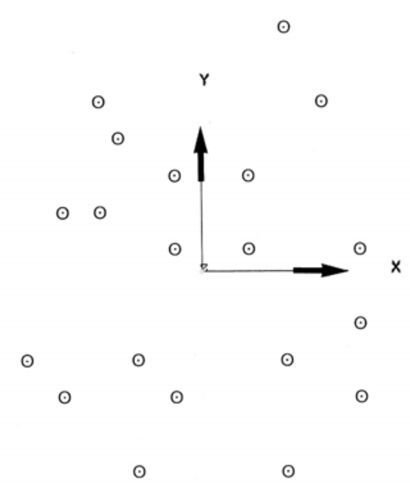
This is the guess: ***the (mechanical) balance of the points always comes along with a certain figure and it’s an ellipse (in a plane).***

The picture below illustrates the correspondence.

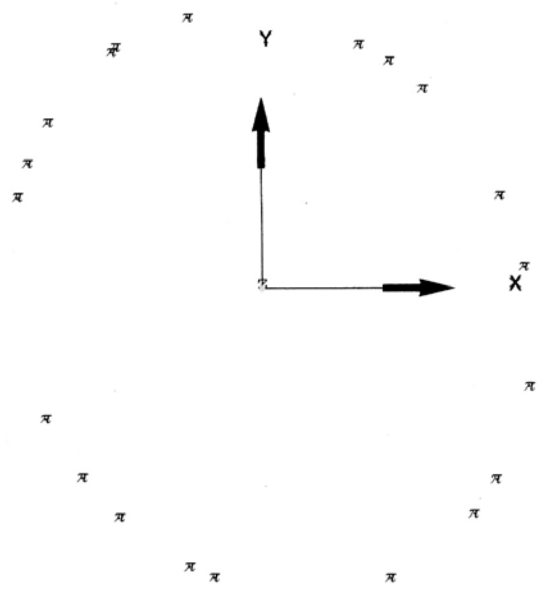


Picture 3: Original 1200 points and their elliptical IR Inner Reference

The next pictures will give you an idea, what happens geometrically when you apply the balancing factors towards the original co-ordinates. The points are not as dense as above to reveal the idea.



Picture 4: Some 20 (arbitrarily) plane points



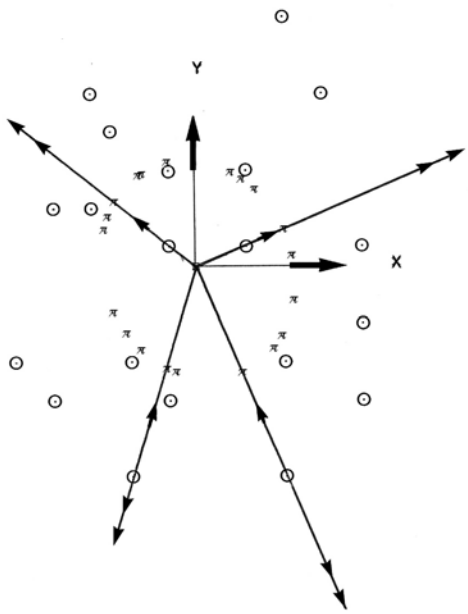
Picture 5: The inner reference of Picture 4, π denote the points of an ellipse

Next picture below reveals what happens geometrically when inverse balancing is applied. The π points in picture 6 indicate the inner reference IR – the ellipse (2D).

From the original 20 points there is a drawing line (straight line) towards the co-ordinate origin of the system that indicates the direction and length (balancing factor) of point change through balancing.

And the result of the balancing (stretching of points towards their co-ordinate origin) results towards the inner reference IR, fortunately a well known figure – the ellipse.

Keep in mind: *We did not expect the balancing resulting towards a certain shape of the co-ordinates. We just recognize this fact from balancing.*



Picture 6: Origin System Points and Inner Reference points explanation

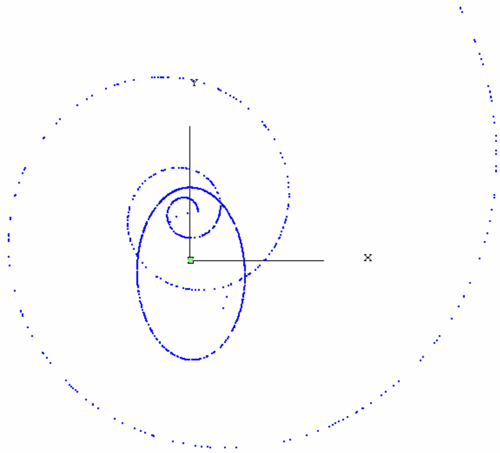
From these illustrations we recognize

- the balance comes with a certain (mathematical and topologically easy) figure in any dimension and any unit
- this figure may serve as a reference (like π) to relate and compute with
- we have to investigate this figure, digest and evaluate it's properties and operate differences to it as **INVERSE BALANCE**

Remark (to the advanced reader): Starting the explanation of IR we focus on homogenous properties of the columns of a matrix. In fact you may mix up different type dimensions and properties because of the properties of the orthogonal projection matrix that is idempotent. As a matter of fact you might mix up ordinal, nominal and cardinal parameters in case of statistical inference.

Before we start scrutinizing these tools (within adjustment and optimization, physics, mechanics) we may take a look on the inner reference of a spiral (Spiral-galaxy).

The following picture shows some 1000 spiral points and the related inner reference IR in a plane. Not surprisingly an ellipse, but nice to see the axis of the ellipse holding the proportion of the golden section = 1.618…



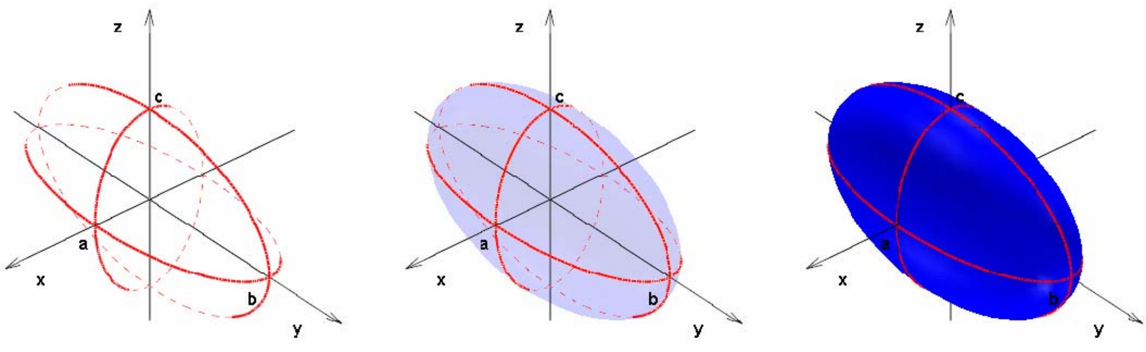
We should feel (academic and mental) relief for detecting a certain figure that reveals the BALANCE instead of a pure mathematical of physical formula.

And moreover – it is so close to a circle, that one being related to π. In fact the constant π is always included from areas (plane) and volumes of ellipses and ellipsoids. Nevertheless- we have to invest some work towards the impact of IR

Remark: Refer to Stephen Hawking: God created the integers (ISBN-13: 978-0-7624-1922-7). He provides comments about the background of scientific investigation and the impact.

From IR we start discussing the ellipse (two dimensions), the ellipsoid (three dimensions) and hyper ellipsoid (more than three dimensions).

See the picture of an ellipsoid in normal position (axis being parallel to the co-ordinate axis) below.



Other than a sphere an (hyper-) ellipsoid may have a direction or better say ***orientation***. An orientation is a deviation of a direction towards the reference systems direction.

This results from the IR ellipsoid axis not necessarily being parallel towards the system (Cartesian) co-ordinates direction (mathematical normal position).

The cure is the application of a generalized principle axis transformation that rotates the arbitrary (hyper-) ellipsoid axis towards normal position (parallel with systems co-ordinate axis).

The next issue to operate the IR: compute *perpendicular distance of a point towards the surface of the (hyper-) ellipsoid.* This was quite a mathematical challenge because this task may result in the solution of higher order equations. Well, it can be shown that this task may be solved with an easy quadratic equation (p,q formula) – in any dimension of the inner reference IR.

So what is left to focus on the generalized inverse balance or the inner reference IR?

Generally there could be restrictions that spoil any balance, as it may be interpreted from a children scale that is bound towards a fixed point. Hence there is no balancing because of the restrictions impact upon the degree of freedom.

The cure is: Recognize restrictions and eliminate them for balancing. Create a non spoiled degree of freedom and proceed.

Additionally there might be hidden or latent restrictions (often you find these ones in neural networks). Same as the full restrictions these ones spoil the natural balance and hence the corresponding IR. Detect these latent restrictions and reduce the issue towards a real clear degree of freedom.

Remarks for enhanced readers (to mention the most important issues):

a) Avoid Cholesky decomposition for correlated data. This metric spoils the order of the rows within your coefficient matrix.

b) The computation of balancing factors is a quite easy numerical procedure and may be execute within a variety of algorithms. For single redundant data (4 points in three dimensional space) just take the Plücker-Grassmann co-ordinates. Once getting familiar to these still unpopular mathematical tools you will not miss it any more.

c) The ellipsoid is one of the most important figures of the engineering science *physical and mathematical Geodesy*. In fact the most important lecturer on these topics of ellipsoids comes with the book by GRAFARED, E. and KRUMM, W.: Map projections, 2006. ISBN-10: 3-540-36701-2.

Remark (personally): It was the famous **Prof. Helmut Wolf**, Bonn, Germany who called this attitude of inverse balancing a ***wonder drug*** in 1986. Once you get acquainted and used to this tool you do not want to miss this ***Inner Reference***

4 Additional: TECHNICAL TERMS AND TECHNOLOGY

(CMM, Scanner Co-Ordinate, Points and others)

N = Number of independent observations (measurements)
U = Number of parameters (for shapes, sizes or functional description)

Independent observation: Non multiple observations or measurements (hence introducing the same point on a circle multiple times become dependent observations)

Unexpected Deviation (UD)**:** gross error, blunder, outlier, deviated measurement. Generally and theoretically this event can not be avoided for any expert

Degree of freedom (DF) = N – U: If this integer value becomes

DF < 0: The unique determination of the parameters is not possible

DF = 0: Unique determination of parameters without being able to find information of their accuracy

DF > 0: Redundant measurement: hence statistical properties like standard deviations can be computed.

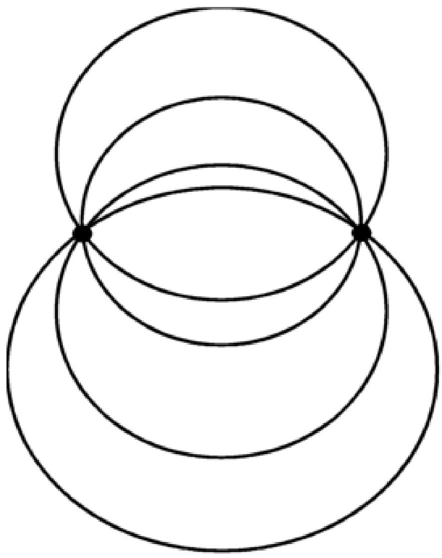
DF > U: (or N > 2*U, that is number of independent observations exceeds twice the number of parameters). This border limits the ability for the detection of unexpected deviations (gross errors).

For example: To detect and recognize a single unexpected deviation within a determination of a plane circle (3 parameters, x, y, radius) the operator has to measure at least (2 * U) + 1 = (2 * 3) + 1 = 7 different (independent) points on a circle.

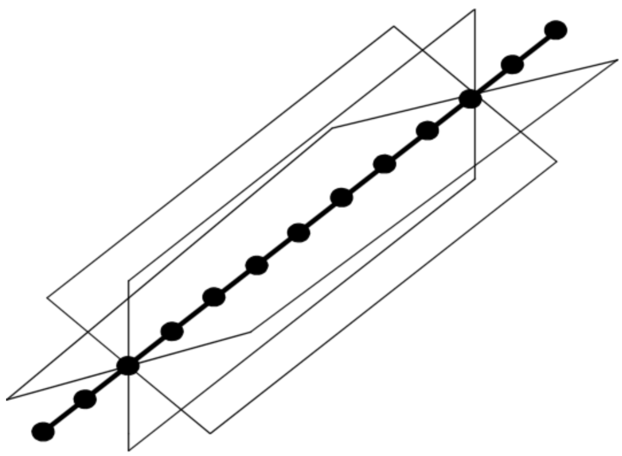
Correction: for redundant measurements corrections may be computed from adjustment and optimization that have to be added to the observations. Hence corrected observations (measurements) completely fit the parameters. (Measured points on a plane circle with deviations from noise or blunders completely fit the circle parameters x, y, radius when the corrections are added).

RANK DEFICIENY: This parameter indicates the lack of measurements to determine a unique result. The operator should add some sufficient number of measurements and observations

Rank Deficiency Typ a) A circle that has to be determined from just to points results into an infinite number of different circles running through both points (not sufficient number of observations)



Rank Deficiency Typ b) A simple plane, that has to run through a number (larger than three, because a plane is determined by three point) of points that exactly form a straight line leaves the plane in an infinite number of variations (insufficient datum definition)



In both cases typ A and typ B the analysis provides a unique solution targeting the sum of squared parameters become a minimum (among all possible solutions)

ED or UDR = Error detection or unexpected deviation recognition

Best Fit (Least Squares): Best (just in a statistical sense) determination of parameters from redundant observations (method of least squares). "Best" obtains minimal standard deviations. It is NOT suitable for the detection of unexpected deviations and errors and suffers from "smearing effects".

MINIMAX-Correction: From all adjustment and optimization technologies that might be applied to redundant data, the MINIMAX or Tschebycheff technology comes up with corrections with a very special property.

The corrections largest one is smaller than the largest correction of any other thinkable adjustment and optimization. This is its target function. Hence: **MINI**mization of the **MAX**imum correction.

The MINIMAX correction indicates the accuracy of the measurement data in case the unexpected deviations are excluded. All and every remaining measurements corrections are below this limit. Unfortunately MINIMAX suffers from maximum smearing effects, that is blunders and UD are masked maximal

Condition density (CD) = (N-U) / N: This numerical value is between

$$0 < CD < 1$$

The closer CD is to 1 the better the evaluation technology will operate to determine

- unexpected deviations
- standard deviations
- MinMax-levels (as accuracy measurements)

In practice you should chose CD larger 0.75 and avoid CD smaller 0.5. For example (determination of the plane circle parameters x, y, radius, U=3).

Hence 0.75 > (N-U) / N or N > 4*u. This comes up with 13 > 4*3 = 12

As an example: For a sphere (x, y, z, radius, U=4) the number of independent measurements becomes 17 > 4 * 4 = 16.

UDL = Unexpected Deviation Level. This numerical value (related to the dimension of the correction) denotes the numerical level from which observations are regarded to be errors, blunders or outliers. It is computed from the observations and regulation algorithms.

BP = BreakPoint. This numerical value denotes the (floor) integer value of DF / 2 (DF = degree of freedom) as the maximum number of unexpected deviations (blunders) to be detected in case N > 2*U.

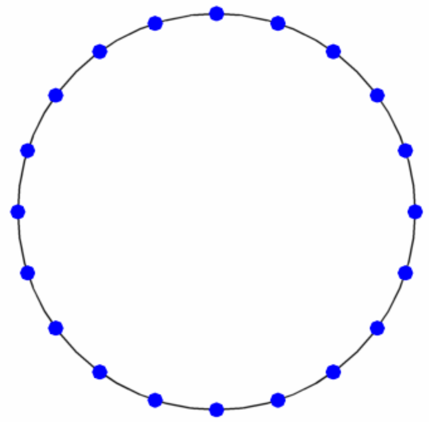
Example: Let N = 14 for the determination of a plane circle (x, y, radius). Then N > 2*U = 6. (procedure is able to detect blunders). DF = 14 – 3 = 11. BP = 11 / 2 = 5,5; (floor 5,5) = 5.

This is the maximum number of observations (measurements) that is allowed to be blundered (from the theoretical point of view). This is break point.

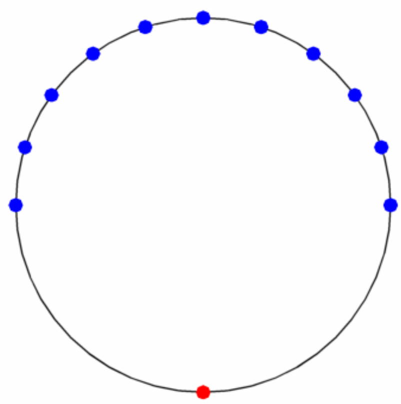
The more the CD (condition density) runs towards 1.0 this theoretical property become reality in praxis (when CD is small, the break point limit is asymptotically true).

Balancing Factor: These numerical values indicate geometrical properties of the observations (measurements). For example plane circle fit (x, y, radius)

Observations that are less critical from the geometrical point of view obtain large numerical balancing factors (in relationship to the other ones) and observations being critical from the geometrical point of view obtain small numerical values in comparison to the other ones.



Sketch of non critical (above) and critical (below) geometrical location of points for plane circle parameter determination.



Example for parameter determination from a sphere

SHAPE-FITTING from co-ordinates: SPHERE

Input data:

Point-co-ordinates x,y,z row wise in arbitrary but common dimensions (meter, inch, millimeter …)

What is necessary and how to proceed

a) At least 4 different points on the sphere (not multiple points), preferable well distributed on the spheres surface to determine the four spatial spheres parameters (center of sphere x,y,z and radius) in co-ordinate related dimensions

b) To protect against unexpected deviations (blunders) introduce at least more than ten (10) different points (redundancy) preferable well distributed on the spheres surface

c) To enhance the accuracy of sphere determination and protection towards smearing effects from blunders introduce as much as possible different points of the sphere (just as much as it is economically sufficient). Hence the redundancy is extended – the reliability and repeatability (R+R) soars.

What numerical results to obtain

a) The center of the sphere and its radius (x,y,z,radius) related to the dimension of the input data (meter, inch, millimeter …)

b) Unexpected deviations (blunders) are extinguished from the initial data automatically and pointed out as blunders within the result

c) Accuracy parameters for (x,y,z, radius) in terms of standard deviations from Least Squares Adjustment (Best FIT)

d) Accuracy parameters for the points in terms of maximum orthogonal distance towards the adjusted sphere (related to the dimension of the co-ordinates). This comes from MinMax.

e) Geometrical information that describes the point distribution on the sphere in numerical terms.

f) Corrections (orthogonal distances to the sphere) from ED, Best Fit and MinMax

Error (hint-) messages and how to proceed

a) Not sufficient number of points to determine the spheres parameters: Hence introduce some more (not multiple) points of the sphere

b) Rank deficiency: This message indicates special, not sufficient geometric constellations to determine a sphere. For example: points form a circle or a straight line.

c) If the sphere parameters from ED, Best Fit and MinMax differ from each other more than three times the standard deviation from least squares add some more points to enhance the reliability and repeatability (R+R)

d) In case you are heading towards certain pre-defined accuracies (standard deviations from Best Fit) of the spheres parameters (x,y,z,radius) add some more different points to the data and hence increase the accuracy (smaller standard deviations) of the sphere parameters from coming up redundancy

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