An application of Hopf monoids and generalized permutahedra in combinatorics

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Content

Set Species

$\mathbb{K}$ is a field of characteristic 0.

Definition (Set species)

A set species P consists of the following data:

- *for each finite set I, a set $P[I]$;*
- *for bijections $\sigma : I \to J$ and $\tau : K \to I$, maps $P[\sigma] : P[I] \to P[J]$ and $P[\tau] : P[K] \to P[I]$ such that $P[\sigma \circ \tau] = P[\sigma] \circ P[\tau]$ and $P[id_I] = id_{P[I]}$.*

Remark

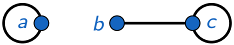
$P[\sigma]$ is invertible and its inverse is $P[\sigma^{-1}]$.

Set Species

Example

For each finite set I, let $G[I]$ be the set of all graphs with vertex set I and, for each bijection $\sigma \colon I \to J$ and graph $\mathfrak{g}$ in $G[I]$, let $G[\sigma](\mathfrak{g})$ be the graph in $G[J]$ whose vertices were relabeled by σ. G is clearly a set species.

For example, if $I = \{a, b, c\}$, $J = \{1, 2, 3\}$, $\sigma \colon I \to J$ is such that

$\sigma(a) = 1$, $\sigma(b) = 2$ and $\sigma(c) = 3$, then, for $\mathfrak{g} =$ 

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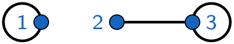
we have $G[\sigma](\mathfrak{g}) =$ 

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Hopf monoids in set species

Definition (Hopf monoid in set species)

A connected Hopf monoid in set species consists of the following:

- *a set species H such that $H[\emptyset]$ is a singleton set written as $\{1\}$;*
- *for each finite set I and each decomposition $I = S \sqcup T$, product and coproduct maps*

$$\mu_{S,T}\colon H[S] \times H[T] \to H[I] \qquad (x, y) \mapsto x \cdot y$$

$$\Delta_{S,T}\colon H[I] \to H[S] \times H[T] \qquad z \mapsto (z|_S, z/_S)$$

that satisfy the naturality, the unitality, the associativity and the compatibility axioms.

Hopf monoid in set species

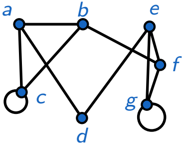
Example

We now turn G into a Hopf monoid in set species as follows:

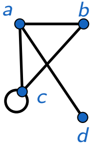
- *Let $I = S \sqcup T$ be a decomposition. The product of two graphs $\mathfrak{g}_1 \in G[S]$ and $\mathfrak{g}_2 \in G[T]$ is their disjoint union;*
- *To define the coproduct of $\mathfrak{g} \in G[I]$, we define the restriction, $\mathfrak{g}|_S \in G[S]$, as the induced subgraph on S and the contraction, $\mathfrak{g}/_S \in G[T]$, as the graph on T given by all edges incident to T, where an edge $\{t, s\}$ in $\mathfrak{g}$ joining $t \in T$ and $s \in S$ becomes a loop $\{t\}$ in $\mathfrak{g}/_S$.*

Hopf monoid in set species

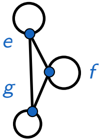
For example, let $I = \{a, b, c, d, e, f, g\}$, $S = \{a, b, c, d\}$ and $T = \{e, f, g\}$. If $\mathfrak{g} =$



, then $\mathfrak{g}|_S =$



and $\mathfrak{g}/_S =$



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Vector Species

All vector spaces and tensor products that follows are over a fixed field $\mathbb{K}$.

Definition (Vector species)

A vector species $\mathbf{P}$ *consists of the following data:*

- *for each finite set* I*, a vector space* $\mathbf{P}[I]$*;*
- *for each bijection* $\sigma \colon I \to J$*, a linear map* $\mathbf{P}[\sigma] \colon \mathbf{P}[I] \to \mathbf{P}[J]$ *such that* $\mathbf{P}[\sigma \circ \tau] = \mathbf{P}[\sigma] \circ \mathbf{P}[\tau]$ *and* $\mathbf{P}[id_I] = id_{\mathbf{P}[I]}$*.*

All maps $\mathbf{P}[\sigma]$ are invertible.

Hopf monoids in vector species

Definition (Hopf monoid in vector species)

A connected Hopf monoid in vector species is a vector species $\mathbf{H}$ *such that* $\mathbf{H}[\emptyset] = \mathbb{K}$ *and which is equipped with linear maps*

$$\mu_{S,T} \colon \mathbf{H}[S] \otimes \mathbf{H}[T] \to \mathbf{H}[I] \text{ and } \Delta_{S,T} \colon \mathbf{H}[I] \to \mathbf{H}[S] \otimes \mathbf{H}[T]$$

for each decomposition $I = S \sqcup T$*, subject to the naturality, unitality, associativity and compatibility axioms. We write*

$$\mu_{S,T}(x \otimes y) = x \cdot y \text{ and } \Delta_{S,T}(z) = \sum_i z^{(i)}|_S \otimes z^{(i)}/_S = \sum z|_S \otimes z/_S.$$

The antipode

If $I = S_1 \sqcup \ldots \sqcup S_k$ is a composition of the finite set $I \neq \emptyset$, we write

$$(S_1, \ldots, S_k) \models I.$$

Definition (Antipode)

Let $\mathbf{H}$ *be a connected Hopf monoid in vector species. We define the antipode of* $\mathbf{H}$ *as the collection of maps* $s_I \colon \mathbf{H}[I] \to \mathbf{H}[I]$ *given by* $s_\emptyset = id_{\mathbb{K}}$ *and*

$$s_I = \sum_{(S_1,\ldots,S_k)\models I} (-1)^k \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k} \qquad (1)$$

for $I \neq \emptyset$*.*

The antipode

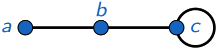
The previous formula is known as *Takeuchi's formula*.

Remark

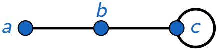
Since a composition of I can have, at most, $|I|$ parts, the sum in equation 1 is finite. The number of terms in 1 is the ordered Bell number $a(n) \approx \dfrac{n!}{2 \cdot (\ln 2)^{n+1}}$. *The first terms in this sequence are 1, 1, 3, 13, 75, 541, 4683 and 47293.*

The antipode

Example

Let $\mathfrak{g} =$ 

. Consider the linearization $\mathbf{G}$ *of* G *and the antipode* $s_I \colon \mathbf{G}[I] \to \mathbf{G}[I]$*. By Takeuchi's formula, the antipode of a graph with 3 vertices is an alternating sum of 13 graphs. But,*

because of cancellations, we have $s_I(\mathfrak{g}) = -$ 

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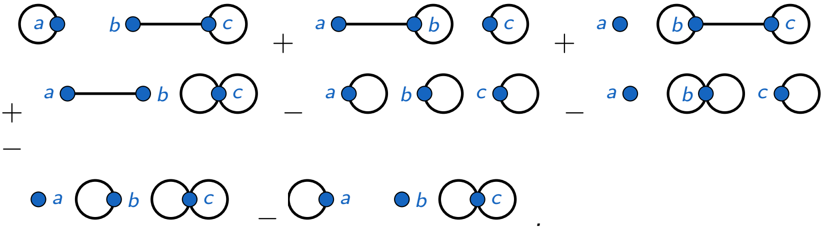

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Generalized Permutahedra

Let $n := |I|$. We define the *stantard permutahedron* $\mathbb{R}I$, π_I in $\mathbb{R}I$, as $\pi_I := \text{conv}\{(a_i)_{i\in I} : \{a_{i_1}, a_{i_2}, \dots, a_{i_n}\} = [n]\}$. We write π_n for the standard permutahedron in $\mathbb{R}[n]$.

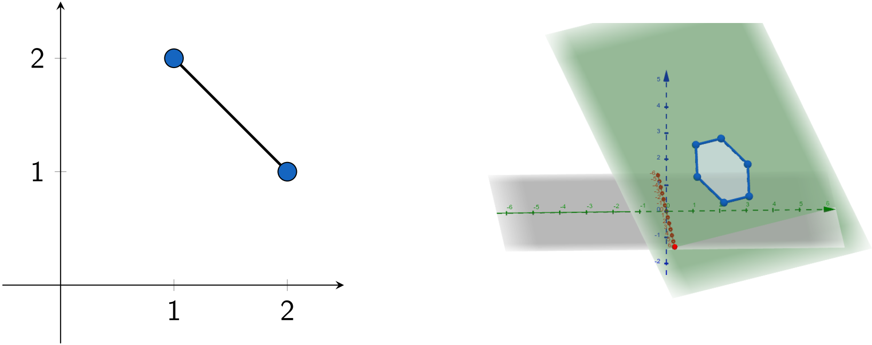


Figure: π_2 (on the left) and π_3 (on the right).

Generalized Permutahedra

Definition (Generalized permutahedron)

Let $a_1 \leq a_2 \leq \ldots \leq a_n$ be real numbers not all equal. The generalized permutahedron $\pi(a_1, \ldots, a_n)$ is the convex hull of all the vectors given by all the permutations of the multiset $\{a_1, \ldots, a_n\}$.

The Hopf monoid **GP**

The set species of generalized permutahedra is defined as follows:

- for each finite set I, we define $GP[I]$ as the set of all bounded generalized permutahedra on $\mathbb{R}I$;
- for each bijection $\sigma\colon I \to J$, we define $GP[\sigma]\colon GP[I] \to GP[J]$ as the function that sends a point $\sum_{i\in I} a_i e_i$ to the point $\sum_{i\in I} a_{\sigma(i)} e_{\sigma(i)}$.

The Hopf monoid **GP**

Now, we turn GP into a Hopf monoid in set species by defining the product and the coproduct as follows:

- for $\mathfrak{p} \in GP[S]$ and $\mathfrak{q} \in GP[T]$, let their product be $\mathfrak{p} \cdot \mathfrak{q} := \mathfrak{p} \times \mathfrak{q} \in GP[I]$;
- for $\mathfrak{p} \in GP[I]$, we define its coproduct as $\Delta(\mathfrak{p}) = (\mathfrak{p}|_S, \mathfrak{p}/_S)$, in which the restriction $\mathfrak{p}|_S$ and the contraction $\mathfrak{p}/_S$ are such that $\mathfrak{p}_{S,T} = \mathfrak{p}|_S \times \mathfrak{p}/_S$.

The antipode of **GP**

Proposition

The antipode of the Hopf monoid of **GP** *and of generalized permutahedra is given by the following cancellation-free and grouping-free formula. If $\mathfrak{p} \in (\mathbb{R}I)^*$ is a generalized permutahedron, then*

$$s_I(\mathfrak{p}) = (-1)^{|I|} \sum_{\mathfrak{q} \leq \mathfrak{p}} (-1)^{dim(\mathfrak{q})} \mathfrak{q},$$

where the sum is over all nomempty faces $\mathfrak{q}$ of $\mathfrak{p}$.

Characters

Definition

Character] Let $\mathbf{H}$ *be a Hopf monoid in vector species. A character* ζ *on* $\mathbf{H}$ *is a collection of linear maps* $\zeta_I : \mathbf{H}[I] \to \mathbb{K}$*, one for each finite set* I*, satisfying the following axioms:*

- *Naturality: For each bijection* $\sigma : I \to J$ *and* $x \in \mathbf{H}[I]$*,* $\zeta_J(\mathbf{H}[\sigma](x)) = \zeta_I(x)$*;*
- *Multiplicativity: For each* $I = S \sqcup T$*,* $x \in \mathbf{H}[S]$ *and* $y \in \mathbf{H}[T]$*, we have* $\zeta_I(x \cdot y) = \zeta_S(x)\zeta_T(y)$*;*
- *Unitality: The map* $\zeta_\emptyset : \mathbf{H}[\emptyset] \to \mathbb{K}$ *sends* $1 \in \mathbb{K} = \mathbf{H}[\emptyset]$ *to* $1 \in \mathbb{K}$*.*

Permutahedra and the multiplication of power series

Let $\overline{\Pi}$ be the Hopf submonoid of $\overline{GP}$ generated by the standard permutahedra.

Theorem

The group of characters $\mathbb{X}(\overline{\mathbf{\Pi}})$ *of the Hopf monoid of permutahedra is isomorphic to the group of exponential formal power series* $\{1 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots : a_1, a_2, \dots \in \mathbb{K}\}$ *under multiplication.*

Multiplicative inversion formulas

Theorem (Polytopal version of the multiplicative inversion of formal power series)

The multiplicative inverse of $A(x) = 1 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots$ *is* $\frac{1}{A(x)} = 1 + b_1 x + b_2 \frac{x^2}{2!} + b_3 \frac{x^3}{3!} + \dots$, *where*

$$b_n = \sum_{F \text{ face of } \pi_n} (-1)^{n - \dim(F)} a_F,$$

where $a_F = a_{f_1} \cdot \dots \cdot a_{f_k}$ *for each face* $F \cong \pi_{f_1} \times \dots \times \pi_{f_k}$ *of the permutahedron* π_n.

Multiplicative inversion formulas

Theorem (Enumerative version of the multiplicative inversion of formal power series)

The multiplicative inverse of $A(x) = 1 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots$ *is* $\frac{1}{A(x)} = 1 + b_1 x + b_2 \frac{x^2}{2!} + b_3 \frac{x^3}{3!} + \dots$*, where*

$$b_n = \sum_{\langle 1^{m_1} 2^{m_2} \cdots \rangle \vdash n} (-1)^{|m|} \binom{n}{1, 1, \dots, 2, 2, \dots} \binom{|m|}{m_1, m_2, \dots} a_1^{m_1} a_2^{m_2} \cdots ,$$

summing over all partitions $\langle 1^{m_1} 2^{m_2} \cdots \rangle \vdash n$ *of* n *such that* m_1 *sets of a given partition have size* 1*,* m_2 *sets of a given partition have size* 2*, …and* $|m| = m_1 + m_2 + \dots$*.*

Boolean functions

Definition (Boolean function)

Let I be a finite set and 2^I be the collection of subsets of I. A Boolean function is any function $z\colon 2^I \to \mathbb{R}$ such that $z(\emptyset) = 0$.

Boolean functions

Define the set species BF as follows:

- for each finite set I, the set of Boolean functions on I is denoted by $BF[I]$;
- for each bijection $\sigma\colon I \to J$, let $BF[\sigma](z)\colon 2^J \to \mathbb{R}$ be the function that sends $\sigma(A)$ to $z(A)$.

It is possible to define a product and a coproduct to turn BF into a Hopf monoid.

Submodular functions

Definition (Submodular function)

Let I be a finite set and z be a Boolean function on I. We say that I is a submodular function if

$$z(A \cup B) + z(A \cap B) \leq z(A) + z(B) \tag{2}$$

for every $A, B \subseteq I$.

Remark

The Hopf monoid of submodular functions SF is a Hopf submonoid of BF.

Submodular functions and generalized permutahedra

Definition (Base polytope)

Given a finite set I and a Boolean function $z\colon 2^I \to \mathbb{R}$, we define its base polytope as the set

$$\mathscr{P}(z) = \{x \in \mathbb{R}I : x(I) = z(I) \text{ and } x(A) \leq z(A)\ \forall A \subseteq I\}. \quad (3)$$

Remark

Given an $x \in \mathbb{R}I$ and a subset $A \subseteq I$, we denote

$$x(A) := \sum_{i \in A} x_i.$$

Submodular functions and generalized permutahedra

Theorem

The collection of maps

$$\begin{aligned} SF[I] &\to GP[I] \\ z &\mapsto \mathscr{P}(z) \end{aligned}$$

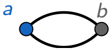
is an isomorphism of Hopf monoids in set species, that is, $SF \cong GP$.

Graphic zonotopes

Let $\mathfrak{g}$ be a graph with vertex set I. Given $A \subseteq I$ and an edge e of $\mathfrak{g}$, we say that e is *incident* to A if either endpoint of e belongs to A. We define the *incidence function*

$$\begin{aligned} \mathrm{inc}_{\mathfrak{g}} \colon 2^I &\to \mathbb{Z} \\ A &\mapsto \mathrm{inc}_{\mathfrak{g}}(A) = \text{ number of edges and loops of } \mathfrak{g} \text{ incident to } A \end{aligned}$$

Graphic zonotopes

For example, if $\mathfrak{g} =$ 

, then $\mathrm{inc}_{\mathfrak{g}}(\emptyset) = 0$ and $\mathrm{inc}_{\mathfrak{g}}(\{a\}) = \mathrm{inc}_{\mathfrak{g}}(\{b\}) = \mathrm{inc}_{\mathfrak{g}}(\{a, b\}) = 2$.

Graphs as a submonoid of generalized permutahedra

Proposition

The map

$$\mathrm{inc}\colon G^{cop} \to SF \cong GP$$
$$\mathfrak{g} \mapsto \mathrm{inc}_{\mathfrak{g}}$$

is an injective morphism of Hopf monoids.

The antipode of graphs

Proposition

The antipode of the Hopf monoids of graphs $\mathbf{G}$ *is given by the following cancellation-free and grouping-free expression: if* $\mathfrak{g}$ *is a graph on* I*, then*

$$s_I(\mathfrak{g}) = \sum_{\mathfrak{f},o} (-1)^{c(\mathfrak{f})} \mathfrak{g}(\mathfrak{f}, o),$$

summing over all pairs of a flat $\mathfrak{f}$ *of* $\mathfrak{g}$ *and an acyclic orientation* o *of* $\mathfrak{g}/\mathfrak{f}$*, where* $c(\mathfrak{f})$ *is the number of connected components of* $\mathfrak{f}$*.*

What else?

- Do analogous constructions for the families of matroids and posets;
- Obtain polytopal and enumerative formulas for the compositional inversions of power series;
- Define the polynomial invariant of a character and prove reciprocity theorems for graphs, matroids and posets.

Thanks for your atention! Any questions?

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