Assertion, assumption, and deduction

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1. Content and force

Gottlob Frege characterized the assertoric quality of an utterance as an assertoric *force* ("behauptende Kraft"; Frege 1918, p. 22) of the utterance. This idea was later taken over by J L Austin (1962, pp. 99–100). Austin spoke of the *force* of an illocutionary act. Illocutionary acts are such acts as asserting, asking a question, warning, threatening, announcing a verdict or intention etc. (1962, pp. 98–102). An utterance of a sentence, by means of which a question is asked, is thus an utterance with *interrogative force*, and when an assertion is made the utterance has *assertoric force*.

Frege thought of assertion as the outer sign of an inner *judgment*. The judgment, in turn, is the mental act of moving from a thought content (a "Thought", in Frege's terminology) to its truth value. This step is a mental act of the thinker/speaker and Frege had an idea about what it consists in:

> A judgment for me is not the mere comprehension of a thought, but the acknowledgment of its truth (Frege 1892, p. 34, note 7).

In addition, he specifies the acknowledgment as a separate act of *advancing*:

> Judging can be regarded as advancing from a thought to its truth value. Naturally, this cannot be a definition. Judging is something quite of its own kind and incomparable (Frege 1892, p. 35).

Let's say that *judging* that p is a mental act in which a *belief* that p is formed, or reinforced, or at least a repetition of such a mental act even if the belief is already in place. There is clearly a difference between judging that p and merely *entertaining* or *merely thinking* the thought that p.

But there is a problem with Frege's characterization of the second step: as advancing from proposition (Thought) to truth. I think there are two inter-related problems with it. Firstly, Frege speaks of *acknowledgment* of the truth of the Thought, and secondly, of the advancing from a Thought to *its* truth value.[1] For there is no difference internal to the *judging* between judging a true Thought (to be true) and judging a false Thought (to be true). The judging itself is uniform across truth and falsity. So, Frege combines two features that are really incompatible. But we can modify Frege's view and still preserve a central part.

Instead of distinguishing between a proposition and its truth value simpliciter, we can distinguish between a proposition and its evaluation in the relevant part of reality, i.e. at a particular *point of evaluation*, or *index*. For Frege, there is only one index, *Reality*, or *The World*. In this case, there is no difference between being true *at* (or with respect to) The World, and simply being *true*. It is easy to overlook the intermediary between the proposition and the truth value.

On this modified view, if the relevant index simply is The World, then in judging that p, I take p to be true at The World. And Frege was of course right in insisting that this is distinct from merely *thinking* that: *p is true in the World*. This is just another Thought to be entertained, subject to judging.

In other frameworks, where truth is not an absolute (unary) property, there are different possible indices to judge a proposition with respect to. In a possible-worlds semantic framework with classical propositions (sets of worlds), the relevant index is simply *the actual world*. Judging is then advancing from the proposition to its evaluation at the actual world.

When I (actually) make an assertion that p, whether the assertion is correct or incorrect depends on how things are in the actual world. My utterance is, or I as the speaker am, *concerned with* the actual world. Whether what I say is true or false depends on whether the proposition p itself is true or false in the actual world. It's truth value at other possible worlds are irrelevant to this question. Whether the assertion is correct or incorrect also depends on what grounds I actually have for making it. If my grounds are actually insufficient, my assertion is deficient, and incorrect in at least one respect, even if I could have had better grounds.

These dependencies carry over to other types of evaluation points. Thus, if propositions are sets of world-time pairs, judging is advancing from a proposition to its evaluation at the actual world *and current time*. And if propositions are sets of *centered worlds*, i.e. sets of world-time-subject triples, then an assertion by a speaker X at time t in world w is evaluated as true iff it is true at w, at time t, and centered on X as the speaker (cf.

[1] Both the German original (thanks to Kathrin Glüer here) and the English translation above has a strong ring of factivity: you cannot *acknowledge* the truth of a proposition that is not true, and similarly, by advancing from a Thought to *its* truth value, you advance from a true Thought to the True, and from a false Thought to the False.

Lewis 1979, Pagin 2016).

The general pattern is that when a proposition is a function from indices to truth values, or equivalently a set of indices, *judging* is evaluating the proposition at whatever has the status of the *actual* index. Analogously, with respect to speech acts, we can think of this pattern as the general content-force connection:

(CFC) If the content of a sentence *s* is a function from indices to truth values, then the *force* of an utterance of *s* applies that content to the *actual index*.

This means that if the semantic content is a proposition, modeled as a function from indices to truth values, the outcome of an assertion that *p* is the truth value that *p* has at the actual index.

The view extends to other types of speech act. If the speech act is a *question*, and the semantic value of a question is a partition of logical space into propositions that each provides an answer to the question, then interrogative force selects the proposition that provides *the true answer at the actual index* (cf. Groenendijk and Stokhof 1994). And if instead the speech act is *imperative*, such as a request or an order, and the semantic value of an imperative sentence is a function from worlds to *compliance values* (complied with / not complied with) at indeces, then an order selects the compliance value at the actual index (cf. Charlow 2014).

This view of the relation between content and force is thus very general, both independent of what we take the indices to contain, and independent of type of speech act. I shall refer to it as the *index application model* of force. It does have intuitive support (cf. Pagin 2020). But it remains to extend it to the variety of acts performed in *reasoning*.

2. Reasoning and deduction

Assuming that indices are worlds, on the present view, a speaker who asserts that *p*, implicitly *ascribes* the proposition that *p* to the actual world (of the speaker). In so doing, the actual world cannot be identified as the world where all proposition in some particular set are true, for we simply don't know what set that is. Rather, the world is identified demonstratively as *this world*.

However, there is more to take account of. When we are involved in *reasoning*, providing *arguments*, we not only make assertions, but other kinds of acts as well. If the abstract account of assertoric force given above is correct, it must apply to assertions as they occur in reasoning. But if it does, the account must extend to all types of act performed in reasoning. Can this be done?

We shall here use Natural Deduction (ND), as developed by Gerhard Gentzen (1934-

35) and Dag Prawitz (1965) as our model of deduction. For the sake of accommodating empirical assertions and empirical based inferences, we shall apply ND also outside pure logic and mathematics.

We shall not here be directly concerned with the question of the *justification* of assertions, by themselves or in reasoning. This is the topic of Prawitz 2009 and Prawitz 2012. There, Prawitz considers the question of whether and how a valid inference provides *grounds* for an assertion of the conclusion of a valid inference, given grounds for the premises. Nonetheless, questions of the varieties of force in reasoning to some extent run parallel to questions of the varieties of grounds in reasoning, as we will have occasion to note.

We shall start with an example of an elementary proof in propositional logic:

(1)
$$\dfrac{\dfrac{[A]^2}{A \vee B}\ 1, \vee\mathrm{I}}{A \supset (A \vee B)}\ 2, \supset\mathrm{I}$$

As usual, the right-hand labels provide a number to the accompanied inference step, and names the inference rule that is instantiated. Again as usual, the rectangular brackets around the first occurrence of A indicates that the assumption A has been *discharged*, and the superscript to the brackets indicate the inference at which the assumption is discharged.

I shall also use the further convention to refer by means of a pair of a formula and a number n to the n:th occurrence of that formula in the derivation in question. Thus '$(A \vee B, 2)$' refers to the occurrence of $A \vee B$ in the consequent of the conclusion. When there is no risk of ambiguity, I'll abbreviate this to just the formula followed by the numeral. Thus, $A1$ is the first occurrence of A in (1).

I shall run together statements about formal derivations and statements about real assertions, assumptions, inferences, and conclusions. Real assertions and assumption are made by speakers by means of sentences. We are also accustomed to speak of formula occurrences in derivations as assumed or asserted. I shall in this paper be talking about formulas in derivations *as if* the derivation were actually used to set out an argument. This is merely a convenient shorthand. We can say that the conclusion of the derivation (1) "is" an assertion, in the sense that had the entire derivation (1) been used for setting out a real argument, the conclusion of that argument would have asserted.

With these conventions in place, we can say that in (1), $A1$ is an *assumption.* We shall discuss the force of assumptions in section 3.

Further, $(A \vee B, 1)$ is a formula *inferred* from the assumption. It is the conclusion of inference 1, and this conclusion *depends* on the assumption. Such formula occurrence is often called a *hypothetical assertion*. This is perfectly apt from the present point of view, given how the force itself is best characterized. I shall discuss this in section 4. Closely connected with the idea of a hypothetical assertion is the idea of the kind of force of the *inference itself* that leads to the hypothetical assertion. This, too, will be discussed in section 4.

Finally, in (1), $(A \supset (A \vee B), 1)$ is the conclusion of the argument. This conclusion does not depend on the assumption, and hence it is straightforwardly asserted. Discharging all assumptions is one way of making an assertion in the ND format. Another way, within first-order logic, is that of binding a free variable, or replacing a parameter by a bound variable, in a step of universal introduction:

(2)
$$\cfrac{\cfrac{\Sigma}{Ax}\ 1}{\forall x A x}\ 2, \forall\mathrm{I}$$

Here we follow the notation of Prawitz 1965, p. 26, where 'Σ' denotes a sequence of formula trees $\Pi_1, \ldots, \Pi_n$ (including the empty one; in case $i = 0$, Σ / A is defined to be A).

Yet a third way is acknowledged for systems that allow non-logical *axioms*. This will be represented here in the customary was as an inference where the conclusion depends on no premises:

(3)
$$\frac{}{A}$$

This model of making assertions within ND will be important in the present context. We shall apply it not only to mathematical axioms but also to *empirical* assertions where the ground for the assertion is empirical in nature, for instance an observation. Assertions based on empirical grounds will be discussed in section 7.

These matters will be discussed in section 5. There I shall also discuss the character of the inference themselves, in this case implication introduction, and of universal introduction, in which the assumption is discharged or a variable bound, respectively.

In section 6, I shall relate the current force classification scheme to the *Curry-Howard correspondence*.

3. Assumptions

Assumptions, for the sake of discussion, or for the sake of arguments, are put forward *without* being asserted. An example is

(4) (i) *Assume* that there is life on Mars.

(ii) *Then* there is water on Mars.

(iii) *Hence*, if there is life on Mars, then there is water on Mars.

Here, the italicized expressions are discourse particles that indicate the role of the sentence they respectively prefix; for instance, that of being a conclusion, in the case of (4iii).

(4i) is the assumption of a particular proposition, the proposition that there life on Mars. (4i) *seems* to be concerned with the actual world, i.e. appears to be used to make the assumption *about the actual world* that in this world there is life on Mars. An utterance of (4i) is certainly more than merely uttering ‘There is life on Mars’ just out of the blue. This suggests that there is a special *assumption force*.

However, the idea that assumptions have their own specific force does not fit the general index application model. What is assumed is the truth of an ordinary proposition, just as what is asserted in an assertion is the truth of a proposition. But according to the model, the force of the act consists in applying the proposition to the index, which results in a truth value, just as in the case of assertion. Therefore, on the index application model of force, the force of an assumption will simply be the same as that of an assertion. The distinction collapses. This indicates that assumptions *lack* force.

It can be objected here that the entire argument (4), including both the assumption and the conclusion, may well depend on implicit *background* beliefs ***B***, e.g. that life forms likely to exist on Mars are carbon based, or that the star the planet is orbiting is the size of the Sun. These background beliefs concern the actual world. If there are background beliefs about the actual world that the assumption depends on, then it seems the assumption itself must be concerned with the actual world.

This is too quick, however. We can look at the situation in two ways. On the one hand, the background beliefs ***B*** can be thought of being implicit parts of the total propositional assumption. That is, what is assumed is not just (4i), but all the propositions in ***B*** as well. That the conclusion of the argument depends on the truth of the ***B*** propositions will be preserved if they are added as part of the total assumption. On this way of looking at things, there is no particular connection at all with the actual world.

The other way of regarding the situation is that the ***B*** propositions are implicitly asserted. But this does not force the stated assumption, (4i), to be concerned with the actual index as well. An argument

(5)
$$\dfrac{\mathrm{A} \qquad \overline{B}\ \ 1}{A \wedge B}\ \ 2, \wedge\mathrm{I}$$

has the assumption A and the conclusion $A \wedge B$. B is asserted. Since step 2 is deductively valid, $A \wedge B$ is true under all interpretations and with respect to all indices where both A and B are true. Moreover, the argument from A to $A \wedge B$ leads from a true premise to a true conclusion with respect to all indices where B is true. This may and may not include the actual index. If step 1 is correct, it does. But for this it is not required that A is in any way specifically *concerned with* or applied to the actual index.[2]

We therefore have strong reasons to regard assumptions as *not* having force. Still, there is a difference between simply entertaining the thought that p and *assuming* that p. This difference needs to be accounted for. In assuming the proposition that p, where are interested in truth and falsity, not in just any character or aspect of the proposition. We shall come back to this issue in section 6.

4. Hypothetical and parametric assertions

Step 1 in (1) leads to the conclusion $A \vee B$. What is the nature of this conclusion? It is clearly not itself a new assumption, since it derives from an assumption already made; no *new* assumption is needed to infer the conclusion. But it is also not an assertion, since it depends for its status as conclusion on the assumption; the basis for claiming or putting forward $A \vee B$ is the assumption of A. How should we then characterize its status?

Given the situation, the natural suggestion is to assign the conclusion a kind of *conditional* force, and this is what I propose. The conclusion of an inference that depends on an assumption is a *hypothetical assertion*. It does not have full, i.e. *categorical* force, but assertoric force that depends the premise. More precisely, the force of the conclusion depends on the *force* of the premise. This means that if a premise is not asserted, neither is the conclusion, but if all the premises *are* asserted, so is the conclusion.[3]

To illustrate, in (6)

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[2]The objection can be pushed further by supposing that A contains demonstrative expressions which get definite values in some particular context of utterance. The result will depend on general views about the nature of demonstratives and contexts. If a context does not contain a fully determined index, then the demonstrated object is available for demonstration at more than one index. If contexts are fully determinate with respect to all index elements, then the interpretation of A fixes the index of the context. But that still allows that A is evaluated for truth value at another index, for instance at another possible world.

In David Kaplan's theory of demonstratives (Kaplan 1989, p. 552), an assertion of a sentence A is evaluated for truth value at the index i_c of the context of use c of A. This is the actual index from the point of view of the context. But an evaluation of $\Diamond A$ at i_c leads to an evaluation of A at distinct possible worlds, even if A contains demonstratives. And, of course, in an utterance of $\Diamond A$, A is not asserted; the assertion of $\Diamond A$ may be true even if A is false at c. All sentences will be evaluated at non-actual indices when embedded under modal operators, or temporal operators, or other index-shifting operator, regardless of containing demonstratives or not. Hence, any sentence can be so evaluated. There is thus no problem in principle in allowing an *assumption* of A to be evaluated at other indices than the actual one.

[3]I have suggested this basic idea before, in Pagin 2007.

$$(6)\qquad \cfrac{\cfrac{A}{A\vee C}\qquad \cfrac{}{B}}{(A\vee C)\wedge B}\ 1,\wedge\mathrm{I} \quad + \quad \cfrac{}{A} \quad \rhd \quad \cfrac{\cfrac{\cfrac{}{A}}{A\vee C}\qquad \cfrac{}{B}}{(A\vee C)\wedge B}\ 2,\wedge\mathrm{I}$$

the leftmost derivation has the assumption A and the conclusion $(A\vee C)\wedge B$, which also depends on the assertion of B. This is combined with an assertion of A, in the middle derivation. Combining these, in the rightmost derivation, yields the same conclusion, which depends on the assertion, not the assumption, of A. Since the conclusion now depends on no open assumptions, it is itself asserted. More precisely, the immediate premises of inference 2 are themselves both asserted: the right premise B is directly asserted, while the left premise $A\vee C$ is asserted because its force depends on the force of premise A, which is directly asserted.

Clearly, the move from from assuming to asserting A depicted in (6) directly mirrors the Prawitz reduction step of implication maximum formulas (Prawitz 1965, p. 37)

$$(7)\qquad \begin{array}{c} \cfrac{\cfrac{\Sigma_1}{A}\qquad \cfrac{\begin{array}{c}[A]\\ \Sigma_2\end{array}}{A\supset B}}{B} \\ \Pi_3 \end{array} \qquad \rhd \qquad \begin{array}{c} \cfrac{\cfrac{\Sigma_1}{A}}{} \\[-1ex] \cfrac{\Sigma_2}{B} \\ \Pi_3 \end{array}$$

(Here we deviate from the notation of Prawitz 1965, p. 26, of representing the set of occurrences of a formula by means round and square brackets). The change of force represented in (6) could also be represented in the reduction format of (7), where the assertion of A is added as a second premise which together with the left derivation in (6) yields an assertoric conclusion of $(A\vee C)\wedge B$. This extended left derivation would then *reduce* to the right derivation of (6). But the inference step that would be added to the left derivation of (6) does not instantiate any standard basic rule of inference.

As shown in (6), a hypothetical assertion has conditional force in three respects. Firstly, it depends on the force of its premises. Secondly, it is subject to acquiring categorical force by way asserting the open assumptions it depends on. This idea is different from what has been called *conditional assertion*, an idea first introduced by Quine (1952). A conditional assertion by means of a sentence

(8) If A, then B.

is taken to be an assertion of (the proposition expressed by) B, on condition that A is *true*. Whether A is itself asserted or held true by the speaker is irrelevant; what matters is only the actual truth value.[4]

Thirdly, conditional force should not be seen as reducing to the alternatives of either having categorical force, if not depending on assumptions, and of just not having categorical force, if so depending. The conditional force of the conclusion of the inference corresponds to *accepting the inference*. The speaker who infers $A \vee C$ from A is prepared to accept $A \vee C$ *if* prepared to accept A. This corresponds exactly to accepting the inference itself, that is, as implicitly treating it as correct or valid. I shall say that the speaker who expresses acceptance of an inference by linguistic means, such as in ND, *endorses* the inference.

I view endorsement of an inference by means of utterance of an inference expression as an utterance with *inferential force*. Exactly what this should amount to in the index application model requires further investigations, but a natural conjecture is that applied to an index i, the value is a pair $\langle P, C\rangle$ of the sequence P of truth values of the premises at i and the truth value C of the conclusion at i. Thus, the conditional force of a hypothetical assertion is not something that simply reduces to the same forceless status as that of an assumption, in case the premises are not all asserted. It is a status in its own right that depends on the endorsement of the inference.[5]

As noted above, when considering the language of first-order logic, we also need to take into account the step of inferring a conclusion based on a *parameter* or a *free variable*. Thus, we may have a derivation of the following kind:

(9)
$$\dfrac{\dfrac{\dfrac{Ax}{Ax \vee C}\ 1, \vee\mathrm{I}}{Ax \supset (Ax \vee C)}\ 2, \supset\mathrm{I}}{\forall x(Ax \supset (Ax \vee C))}\ 3, \forall\mathrm{I}$$

Here, $(Ax, 1)$ is an open formula assumption and step 1 an inference to an open formula conclusion. In step 2, the assumption discharged. From an assertion theoretic point of view, what is the status of the conclusion? The three inferences in (9) are logically valid and hence properly endorsable. The conclusion of step 3 is a closed formula, proved in (9), and hence its status in the derivation is that of an assertion.

The conclusion of the second step, however, is an open formula, but it does not depend on any assumption where its free variables are free. It is based on an endorsable argument, and so it its conclusion does not seem lack the basis for assertoric force. Yet, it is not a proper assertion, because it does not have full propositional content. Still, the deficiency belongs to the content side, not to the force side. I shall call an assertion of

[4] This idea has been developed by Nuel Belnap (1973). I have criticized it in Pagin 2007.

[5] We come back to inferential force in section 5.

(what is expressed) by $Ax \supset (Ax \vee C)$ a *parametric assertion*.

A parametric assertion lacks full propositional content but has full assertoric force. A parametric assertion of an open formula A is *correct* just in case any proper closed substitution instance of A (where all free variables of A are uniformly replaced by well-formed referring closed singular terms) is assertible. This means that no inference

$$\text{(10)} \qquad \frac{}{Ax}$$

where some substitution instance $A[t/x]$ is *false* (or some value given to x would make it false) could be a correct parametric assertion. Strictly speaking, such an assertion does not even make sense, (except relative some values given to the free variables, but then the content would be a full proposition). However, if Ax is very complex, a speaker could be mistaken about its substitution instances.

On the index application model of force, what does a parametric assertion of an open formula at an index i amount to? This depends on what counts as the semantic value of an open formula at an index. A natural view would be that this value is a set G of assignments to variables, i.e. a set of functions $g \in G$ such that Ax is true with respect to g at i. A correct parametric assertion of an open formula A at index i is therefore at a minimum one where G is the set of all assignments g such that for any free variable v in A, $g(v)$ is in the domain of i. Exactly what more should be required of correctness is a topic that goes beyond the present paper.

I shall now leave the issue of parametric assertions and return to main topic of assertoric force. The most immediate alternative to the view of conditional force presented here is provided by the Sequent Calculus (SC) version of ND. A sequent is an expression

$$\Gamma \longrightarrow \Delta$$

where Γ and Δ are finite sets of formulas. Gentzen (1934-35) devised both ND and SC, and SC was intended as a system of deduction *without* explicit assumptions. The sequents themselves are the units of a derivation, occupying the positions of premises and conclusions in inferences.

SC and ND are in other respects very close to each other. Prawitz (1965, p. 90) has suggested that SC can be regarded as a *meta-calculus* for the derivability relation $\vdash$ in ND: in a valid sequent, the formulas in the succedent, after the arrow, are derivable from the formulas in antecedent, before the arrow.

But if SC is regarded not as mere calculus for representing derivability, but as a model of real reasoning, then what is asserted or assumed are not formulas or propositions, but sequents or the conditional contents that sequents can be taken to express.

Instead of an assumption A, SC uses the identity sequent

$$A \longrightarrow A$$

(set brackets omitted). As long as the conclusion of a derivation depends on open assumptions, these assumption are elements of the antecedent. And instead of an assertion of the conclusion A of a proof, SC has a sequent with empty antecedent:

$$\longrightarrow A.$$

In a sense, SC, as an object-level system of reasoning, eliminates both ordinary assumptions and ordinary assertions, and replaces both with endorsements of sequents. This also eliminates the problem of giving a suitable characterization of the *force* of assumptions and conditional assertions. On the other hand, SC, as an object-level system of reasoning, must explain the meaning of the arrow that does not interpret it as a metalogical relation symbol. That is not an easy task. Here, I have taken on the challenge on the other side, the theory of force for Natural Deduction.[6]

5. Discharging assumptions

In step 2 of derivation (1), the assumption A is *discharged*. This means that the conclusion of the step does not depend on the assumption, and since it does not depend on any assumption, the derivation is a proof of $A \supset (A \vee B)$. The conclusion is asserted.

Inferences in which assumptions are discharged differ from others in that the premises are not, or not only, formulas, but entire derivations.[7] In the simple example of (1), it is the derivation from A to $A \vee B$ that is the premise in inference 2. How should this be understood in the present assertion theoretic context?

As before, we uphold the idea of conditional force that asserting the conclusion con-

[6] Peter Schröder-Heister (2003) notes that standard Natural Deduction is heavily assertion oriented and gives assumptions a subordinate role. Schröder-Heister undertakes the project of making their roles more equal, and uses the SC format for the task. The first basic idea is that assertions are governed by right-hand rules of inference, modifying the succedent, while assumptions are governed by left-hand rules, modifying the antecedent. The second main idea is that this requires a change of the specificity of assumptions. In standard ND, there are no restriction at all on what formula can be used as an assumption, while in the new system, there are restrictions in particular contexts of derivation what assumptions can be made there, requiring more *specific* assumptions. The cost of the change is that the derivability relation no longer is transitive by default. Rather, transitivity needs to be guaranteed by certain more general restrictions on derivations. Schröder-Heister discusses several alternatives.

By contrast, here I have accepted the assertion oriented nature of deduction.

[7] This use of 'premise' as covering entire derivations goes beyond orthodoxy. Prawitz (1965, p. 23) employs scare quotes when extending the use that way.

ditional on the force of the premise involves endorsing the inference itself. In taking step 2 of derivation (1), the inference is endorsed.

But in this case, something more is going on. Since the conclusion is based on the entire immediate sub-derivation, and the sub-derivation is not merely *assumed* to be correct, the entire sub-derivation itself is endorsed. In the case of (1), the sub-derivation consists simply of inference 1, the disjunction introduction step. According to the present theory, this inference has already been endorsed by the reasoner, so the condition is already met.

Things can be more complex, however. Consider the derivation

$$\dfrac{\dfrac{[A]^3 \qquad \dfrac{B \qquad C}{B \wedge C}\ 1, \wedge\mathrm{I}}{A \wedge (B \wedge C)}\ 2, \wedge\mathrm{I}}{A \supset (A \wedge (B \wedge C))}\ 3, \supset\mathrm{I} \tag{11}$$

In step 3 of derivation (11), the assumption A is discharged, but the conclusion of the inference is not categorically asserted, since it still depends on assumptions B and C. Hence, categorical assertion of the conclusion and discharge of an assumption only sometimes coincide. In a sense, the conclusion of step 3 is closer to a categorical assertion then its premise is, the conclusion of step 2, since the premise depends on three assumptions and the conclusion on just two. Nevertheless, both are hypothetical assertions.

Still, in step 3, the reasoner endorses the entire sub-derivation, including step 1, where the assumption A does not occur. It must be endorsed, since the conclusion of step 3 relies on it.

A third feature to notice is that endorsing the sub-derivation in step 3 involves endorsing two inference steps, 1 and 2. The reasoner has indeed endorsed inference 1 performing the step, and inference 2 in performing that step. Thus, if endorsing a derivation consists in endorsing all its individual inference steps, the condition of endorsing the sub-definition in step 3 is already met, just as in the simpler derivation (1).

The derivation relation is *transitive* iff $\Gamma \vdash A$ and $\Delta \cup \{A\} \vdash B$ together entail $\Gamma \cup \Delta \vdash B$. This indeed corresponds to the *Cut* rule in SC. In case $\vdash$ is transitive, it holds that if each individual step of a derivation is correct, so is the entire derivation.[8] If this holds, then having endorsed each single step of a derivation, the reasoner has indirectly endorsed the entire derivation itself.

In standard ND, the derivation relation *is* transitive, but transitivity can fail for various reasons in other systems. We shall look at such a reason in section 7, when dealing

[8] Easily shown by induction.

with inconclusive evidence.

Either way, in case of inferences such as ⊃I, ∨E, and ∃E, where assumptions are discharged and premises include entire derivations, the reasoner must be taken to have endorsed the entire sub-derivation that the step relies on. If indirect endorsement counts as acceptable, then in ND, the endorsement is automatically achieved in virtue of the reasoner's having endorsed each individual step.

Summing up, in section 13, we have introduced the idea of force in in terms of the index application model and applied it to assertion, and in sections 3, 4, and 5, we have characterized the force of assumptions and hypothetical assertions, including the case of inferences that discharge assumptions. This covers the kind of steps that occur in Natural Deduction derivations. But the job is not yet quite done.

6. Proof terms

Let's return to the task of distinguishing using a formula A as an *assumption* from using or uttering it in other *forceless* ways. A way of achieving this is offered by the *Curry-Howard Correspondence*.

The Curry-Howard Correspondence (or Curry-Howard *Isomorphism*), is primarily a family of correspondences between systems of deduction and simply typed lambda calculi. Especially, there is a close correspondence between minimal or intuitionistic logic under the Brouwer-Heyting-Kolmogorov (BHK) interpretation and the lambda calculi. To each formula in derivation corresponds a term in the proof term calculus (lambda calculus) that corresponds to the sub-derivation of the formula.[9]

According to the BHK interpretation, a proof of an implication $A \supset B$ is a function which, when applied to a proof of A yields as value a proof of B. Suppose we have terms s and t belong to the types A and B, respectively, which is marked by superscripts: s^A and t^B. Then the term corresponding to the proof of $A \supset B$ is $\lambda u^A.t^B$, of type $A \supset B$. This is a function term generated by lambda abstraction, binding the variable u. Applied to a term s^A, the result is

$$\lambda u^A.t^B(s^A) = t^B[s^A/u^A].$$

where free occurrences of u^A in t^B are replaced by s^A. Because of this typing, the approach is also known as the *formulas-as-types*, (or *propositions-as-types*; see below) conception.

We can represent the type A of a term t with the notation

[9] For a concise introduction, see chapters 1 and 2 of Troelstra and Schwichtenberg 2000.

$$t : A.$$

Using this notation, and omitting the superscripts, we can display derivations with in ND with proof term prefixes. We can this way restyle derivation (11) as follows:

(12)
$$\dfrac{[u:A]^3 \qquad \dfrac{v:B \qquad w:C}{\langle v,w\rangle : (B\wedge C)}\ 1, \wedge\mathrm{I}}{\dfrac{\langle u,\langle v,w\rangle\rangle : (A\wedge(B\wedge C))}{\lambda u.\langle u,\langle v,w\rangle\rangle : (A\supset(A\wedge(B\wedge C)))}\ 3, \supset\mathrm{I}}\ 2, \wedge\mathrm{I}$$

Here, the variables u, v, w are initially *free* proof term variables; they are *assumption variables*, as opposed to *individual variables* (cf. Troelstra and Schwichtenberg 2000, p. 46). The variable u gets *bound* in step 3, by the lambda abstraction operation that creates the function term. The pair operator $\langle\cdot,\cdot\rangle$ creates a complex proof term $\langle v, w\rangle$ from the terms v and w in the step of conjunction introduction, again in full correspondence with the BHK interpretation.

If we reintroduce superscript type-marking to the final line, we get

(13) $\lambda u^A.\langle u^A, \langle v^B, w^C\rangle\rangle : (A \supset (A \wedge (B \wedge C)))$

The proof term in (13) corresponds exactly to the derivation in (11): the derivation can be reconstructed from the term.

This example illustrates a further correspondence between proof terms the force-type of formula occurrences:

(F-T) *The force-term correspondence*:
If t is a proof term and A a formula occurrence such that $t : A$ in a formula context C (that is itself unembedded), then

i) A is an assumption iff t is a free variable,
ii) A is a hypothetical assertion iff t is a complex term that contains a free variable as a proper subterm,
iii) A is a (categorical) assertion iff t does not contain any free variable subterm (proper or not).

An assumption has *no* force, a hypothetical assertion *conditional* force, and a categorical assertion *full* or *categorical* force. What sets an assumption off from a mere utterance of a sentence is not force, since neither have it, but its role in reasoning: for hypothetical reasoning and for concluding hypothetical reasoning by discharging.[10]

The notation $\ulcorner t : A \urcorner$ is here used in a meta-logical sense: it represents the derivational support of an occurrence of A in a derivational context C. It is not meant as part of a logical calculus in its own right. Derivations like (12) can instead be regarded as belonging to a meta-calculus in a sense close to that of Prawitz above concerning SC.[11] The meta-logical representation $\ulcorner u : A \urcorner$ therefore adequately represents both the reasoning role of the assumption A — by including a term for its assertoric basis — as well as its lack of force — by letting that term be a free variable, and hence a term that does not refer and does not depend on any subterm for reference either.

The final aspect to consider is the issue of inconclusive evidence.

7. Inconclusive evidence

We have here decided to represent empirical assertions as inferences from the empty set of premises:

(3)
$$\frac{}{A}$$

We can add a notation from proof terms that represent such inferences:

(14)
$$\frac{}{e_i : A}$$

Here the term e (with index) indicates that the inference relies on empirical evidence that has not been articulated as a sentence.[12]

[10] Prawitz 2012, pp. 191–4 uses a parallel notation for grounds of assumptions, grounds for assertions under assumptions (*unsaturated grounds*), and grounds for categorical assertions (*closed grounds*).

[11] By contrast, in Martin-Löf's *Intuitionistic Type Theory* (ITT), the format $\ulcorner t : A \urcorner$ is used for expressing *judgments*. ITT employs the *propositions-as-types* interpretation: propositions are the types of their proofs. A judgment $t : A$ says that t is of type A and again that t is a proof of A. The *locus classicus* of ITT is Martin-Löf 1984. For a recent introduction, see Dybjer and Palmgren 2020.

Sundholm (2019) uses the free variable notation for propositional assumptions, which he also characterizes as an *alethic* or *ontological* assumption, and writes it "x : Proof(A)" (2019, p. 557). He contrasts this with *epistemic assumptions*, which are assumptions that a proposition *has been proved*. This is written "a : Proof(A)" with a closed proof term a. Here it is the judgment that is assumed rather than the proposition. What it amounts that a judgment is assumed is taken in the paper as informally understood.

It may be noted that Sundholm characterizes ND derivations (the Gentzen 1932 format) under assumptions as *dependent proof-objects* Π of the form $\Pi : C(x_{1:A_1,\ldots,x_k:A_k})$, saying that Π is a proof of C under the assumptions that $x_1, \ldots, x_k$ are proofs of $A_1, \ldots, A_k$, respectively. This has a clear affinity with the characterization of hypothetical assertion in terms of proof terms given here.

[12] To make the picture complete, we also have consider empirical — typically inductive — inferences from non-empty sets of premises, exemplified by "The streets are wet. Hence, it has been raining." In these cases the inference step adds uncertainty even in a single premise inference. The main point of the present section does not require us to take this into account.

Now, empirical evidence differs from mathematical and purely logical in being *inconclusive*: that empirical evidence e obtains for the truth of A does not entail the truth of A. This fact, combined with the fact that we have practice of treating empirical evidence as good enough for *assertion* has crucial consequences for deduction. This comes out clearly if we represent the strength of evidence by probabilities. Assume that a certain probability $d < 1$ for a sentence is good enough for asserting it. This leads to the problem of accumulating uncertainties.

The most striking example of this problem is that of the *lottery ticket paradox*, due Kyburg (1965). In a lottery of 1000 tickets there is one winning ticket. The chance that any particular ticket, such as #1, will *lose* is .999. Since the probability is very high, we can assume it is high enough for assertion (by increasing the number of tickets in the lottery, the probability can be made as high as one wants). Since the chance is the same for every ticket, it holds for each ticket #i that it is assertible that ticket #i will lose. By iterated conjunction introduction, we have the assertion

(15) Ticket #1 will lose *and* ticket #2 will lose *and* ... *and* ticket #1000 will lose.

At the same time, the fact that the there is a winning ticket can be expressed by the disjunction

(16) Ticket #1 will win *or* ticket #2 will win *or* ... *or* ticket #1000 will win.

which is therefore also assertible. But of course, (15) and (16) jointly inconsistent. Hence, there is something wrong with the assertion of (15).

The lottery ticket paradox is an extreme case, since we also know that the complete conjunction is false. But even in non-extreme cases, the probability of the conjunction is typically lower than the probabilities of the conjuncts. For instance, if the propositions A and B are probabilistically independent, then the probability $p(A \wedge B)$ of the conjunction $A \wedge B$ is the product $p(A) \times p(B)$ of the probabilities of the conjuncts. Assume that the assertibility threshold d is 0.9, and that $p(A) = p(B) = .9$. The conjuncts are therefor assertible. Assume that A and B are probabilistically independent. Then $p(A \wedge B) = .81$, and hence the conjunction $A \wedge B$ is not assertible. That is, we can have the following derivation

(17)
$$\dfrac{\dfrac{}{e_1 : A}\,1 \qquad \dfrac{}{e_2 : B}\,2}{\langle e_1, e_2 \rangle : A \wedge B}\,3, \wedge\mathrm{I}$$

where the premises of inference 3 are assertible and the conclusion is not.

A logically valid inference, such as conjunction introduction, of course must preserve truth. And truth is indeed preserved in derivation (17). But it should also preserve *assertibility*: if the premises are assertible, so is the conclusion. And this condition seems not to be met by derivation (17).[13]

The condition is indeed not met by derivation (17) if it is understood as the condition that the conclusion be assertible if the premises are *separately* assertible. By contrast, it *is* met if understood as the condition that the conclusion be assertible if the premises are *jointly assertible*.

That the premises are jointly assertible means in this context that the probability that all the premises are true (together) is at least equal to the threshold probability d. On the assumptions above of probabilistic independence and separate probabilities of 0.9, the joint probability is again 0.81, and so the premises are not jointly assertible. So, in this case, there is no violation of the preservation of assertibility.

In fact, there are limits to how low the probability of the conclusion can be in a valid inference, given the probabilities of the premises. Ernest Adams (1998) uses the concept of the *uncertainty* $u(A)$ of a proposition A, where $u(A) = 1 - p(A)$, where $p(A)$ is the probability of A. From his version of the Kolmogorov axioms for the probability calculus (Adams 1998, p. 21), by classical reasoning, Adams (1998, p. 38) proves a theorem of the upper bound of uncertainty accumulation:

(UA) *Adams on uncertainty accumulation*:
The uncertainty of the conclusion of a valid inference cannot exceed the sum of the uncertainties of the premises.

An immediate consequence of this theorem is that in a single premise valid inference, the probability of the conclusion is at least as high as the probability of the premise. This is in fact also one of the Kolmogorov axioms in Adams's presentation.

In addition, Adams (1998, p. 25) proves in particular for conjunction probabilities, again with classical reasoning, the following:

(CP) *Adams on the probabilities of conjunction:*
$p(A \wedge B) \geq p(A) + p(B) - 1$.

Thus, using (CP), it is safe to infer an assertible conclusion $A \wedge B$ provided $p(A)+p(B)-1 \geq d$, where d is the assertibility threshold.

Within the present context, the upshot is that in order for the conclusion of a valid inference from several premises to be assertible, the reasoner needs to ascertain that the

[13] Prawitz (2009) and (2012) restricts his attention to *conclusive* grounds, and therefore does not consider the issues discussed in this section. At one point, Prawitz (2009, p. 194) briefly considers observation statements, but there also treats observations as providing conclusive grounds.

joint assertibility of the premises is high enough. As a result, reasoning with empirical premises, the preservation of assertibility is not *transitive*. We shall here show this by example.

We shall let hypothetical assertions have assertibility thresholds. That is, the hypothetical assertion of a sentence C under an assumption A has a probability. We also claim that this probability is d or higher provided the joint probabilities of the other premises that C relies on is at least d. An assumption is just an assumption of truth, not an assumption that the premise has been asserted, and does not add uncertainty. Only assertions made on inconclusive evidence contribute to uncertainty.

Then let us look again at derivation (17). We can picture it as resulting from a combination of inferring A from the empty set of premises on the one hand, and the inference from the *assumption* of A and the assertion of B to the conclusion $A \wedge B$. This conclusion is then hypothetically asserted on the assumption A (in fact derivation (5)).

Let Γ be the set of premises for inference 1 in derivation (17), and let Δ be the set of premises for inference 2. Both are the empty set. Let $X \vdash_z C$ mean that C has probability z on the basis of being derived from sentences in X. Then, by our probability hypothesis, it holds that $\Gamma \vdash_d A$. We also have that $\Delta \cup \{A\} \vdash_d A \wedge B$, because the assumption A does not contribute to uncertainty, and therefore the probability is the same as that for $\Delta \vdash_d B$. Finally, we combine these two derivability relations to get $\Gamma \cup \Delta \vdash_{d \times d} A \wedge B$. This is because here we are considering both inference 1 and inference 2, with combined uncertainties. Since d is the probability threshold for assertion, and $d \times d < d$, the conclusion is not assertible. Hence, there is a failure of transitivity of the preservation of assertibility.

This means that when the reasoner *endorses* inference 3 of derivation (17), she must do more than accepting the individual inferences 1, 2, and 3. She must also endorse that the inferences are fit for being used together. Actually ensuring that this is the case is not an easy task. Fortunately, our assertions typically have high probability, well over the threshold. This allows us to carry out short arguments, with a small number of empirical assertions, without running a high risk of falling below the threshold. On the other hand, this easily happens when we are a little less careful.

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