The mathematical conditional

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It is widely held that the material conditional does not offer a plausible interpretation of the natural language conditional "If..., then...". The material conditional $A \supset B$, which is true if and only if A is false or B is true, has logical properties—the 'paradoxes' of material implication—that does not seem to be shared by the natural language conditional.

It is also widely held that when it comes to the ordinary practice of mathematics, mathematical claims stated in the form "If..., then..." can plausibly be interpreted as material conditionals (c.f. Bennett (2003)). This would mean that the conditional behaves differently in mathematical contexts than it does in other contexts.

If the mathematical conditional is an exception this could be explained in different ways. It could be that when the subject matter is mathematics, the natural language conditional behaves like the material conditional due to the nature of mathematical content. After all mathematical claims in many cases take the form of analytical truths (or at least as claims that follow logically from some stipulated collection of axioms), and these differ from claims about contingent matters both in the nature of their contents and in the manners in which such claims are justified. This would mean that in a 'limiting case' the natural language conditional becomes the material conditional. Another possible explanation is that mathematical contexts come with a more or less implicit convention to use "If..., then..." in a special way (the material way), giving it a different meaning than the natural language conditional, a meaning that coincides with the material conditional.

But is it true? Is the mathematical conditional the material conditional? There seems to be room for doubt. Consider:

(1) If $x = 4$ then $x^2 = 16$.

This is simply true, *analytically* true. Consider instead:

(2) If $x = 4$ then x is prime.

This is simply false, *analytically* false.

As (2) is false, its negation

(3) It is not the case that if $x = 4$ then x is prime,

is true. Similarly, as (1) is true its negation

(4) It is not the case that if $x = 4$, then $x^2 = 16$,

is false.

A material conditional $A \supset B$ is false iff A is true and B is false. So as (2) is false, it follows on the material interpretation that:

(5) $x = 4$.

This is a problem. Why? Well, the following is also false:

(6) If $x = 6$ then x is prime.

On the material interpretation it then follows that:

(7) $x = 6$.

But (5) and (7) contradict each other. So we have a problem.

We can't deny that (1) and (2) are mathematical in character. Nor can we deny the apparent truth of (1) or the falsity of (2). The only plausible way of saving the material interpretation is to appeal to something like the following mathematical convention: when a sentence containing free variables is asserted it should be interpreted as containing implicit universal quantifiers. That is, what (1) and (2) *really* say is:

1^*. For all x, if $x = 4$ then $x^2 = 16$.

2^*. For all x, if $x = 4$ then x is prime.

On the material interpretation of the conditional (1^*) is true and (2^*) is false but, importantly, from the falsity of (2^*) we cannot infer that $x = 4$. So we have blocked the road to the contradiction.

What about the truth of (3)? Well, this requires another convention. What appears to be a negated conditional is really a conditional negation. What (3) *really* says is:

3^*. For all x, if $x = 4$ then x is *not* prime.

This restores the peace. The material interpretation is saved. Or is it? Did we just show that the conditionals in (1), (2) and (3) can be interpreted materially or did we show that the material conditional can be used to make claims with what appears to be the same content as (1), (2) and (3) if we say something different? Are the conventions we appeal to conventions of use or are they conventions for how to translate ordinary conditional claims into claims where the conditional can be given a material interpretation?

The myth of the 'silent' quantifier is persistent, which is strange as it is obviously false. The truth of many if not most claims in a mathematical context would not survive if a universal quantifier binding the free variables was automatically appended to the sentence asserted. For prefixing a universal quantifier to a sentence that one is warranted in asserting is to make a substantive inferential step, and is only warranted under a specific condition: when the variable thus bound does not occur free in any remaining premises or assumptions. Whether a quantifier can be justifiably added or not depends on context. The quantifier is not 'silent', it is not there at all and it is absent for a reason. We teach this elementary fact to students in natural deduction.

To be fair, maybe the myth is more complex than it is sometimes presented. Maybe we automatically add on quantifiers in some contexts but not in others. What is striking, however, is that we do not need the myth at all, as it doesn't explain anything that we don't already have to explain.

Quite generally the assertability or unassertability of a sentence with free variables depends on context. It depends on what has been assumed. A sentence with free variables is true in a context if it is true on every admissible assignment of values to the variables in that context. Assumptions serve to constrain the set of admissible assignments, making the assertability of a sentence with free variables context dependent. We need some such notion of context dependent truth to explain why assertions of sentences with free variables can be made in contexts governed by assumptions that constrain the admissible assignments (c.f. Cantwell (2018)). In some contexts $x = 4$ is true, for instance, in a context where it has been assumed that x is a positive number such that $x^2 = 16$. In other contexts it is false, for instance if it has been assumed that x is prime. In a context where there are no constraints on the admissible values of a variable a sentence $A(x)$ is true if it is true on *every possible* assignment of values to that variable. This is just a limiting case and follows the same pattern. In such a context, however, we are also allowed to infer a universally quantified claim by appending the quantifier prefix 'For all x, $A(x)$'. But it wasn't there all along. Now, on every possible assignment of values to x, either $x \neq 4$ or $x^2 = 16$. So (1)—as it stands, without 'silent' quantifiers—is true on the material interpretation. No myth required.

The problem is the falsity of (2). For on the material interpretation (2) is not determinately false. Nor is it determinately true. It is true on some assignments of variables to x (those assignments where $x \neq 4$) and false on others (when $x = 4$). On the material interpretation (2) is in a semantic limbo. It is not alone; for instance, the sentence $x = 4$ is in the same limbo. The problem is that (2) *is* false, or at least it strikes one as false, and it does so in way that $x = 4$ does not strike one as false. For the question of the truth of $x = 4$ in a context where no constraints on x have been made sticks out as strange, as it should when it is in semantic limbo. Neither it nor its negation can be judged true, or judged false. But the question of the truth of (2) does not stick out as strange. We immediately judge it false, without appending quantifiers.

If (2)—as it stands, without quantifiers—is false the conditional cannot be interpreted materially. How should it be interpreted instead? Without going into a full-blown semantic analysis here is a suggestion for what we would want the analysis to imply. A sentence 'If $A(x)$, then $B(x)$' should be true relative some set of admissible assignments G if $B(x)$ is true on every assignment in $G/A(x)$, where $G/A(x)$ is the set of admissible assignments in G that makes $A(x)$ true. Similarly, 'If $A(x)$, then $B(x)$' should be false relative to some set of admissible assingments G if $B(x)$ is false on every assignment in $G/A(x)$. The antecedent of a conditional serves to constrain the set of admissible assignments relative to which the consequent is assessed. On such an interpretation (1) is true and (2) is false, just as they seem to be. For $x^2 = 16$ is true when the only admissible value of x is 4; moreover, under the same restriction 'x is prime' is false.

This is not the material interpretation. However, when A and B do not contain free variables, 'If A, then B' reduces to the material conditional. The interpretation is backed by a more general pattern repeatedly observed in the natural language conditional. The antecedent of a conditional serves to restrict the possibilities relative to which the consequent is assessed. This becomes particularly noticeable when the consequent contains certain kinds of modalities. For instance, even though we take there to be a .25 probability that the next card drawn from a deck of cards will be diamonds, we can say 'If the next card drawn is red, there is a .5 probability that it will be diamonds'. This would make no sense on the material interpretation but makes good sense on the restrictor interpretation. The antecedent constrains the set of possible outcomes which in turn affects the semantic assessment of the probability modality in the consequent. If the antecedent of a conditional in a similar way can constrain the set of possible assignments to variables, we have a uniform semantic phenomenon to back the intuition that (2) is false.

So we have a mathematical use of a conditional that on its natural interpretation is not a material conditional. That interpretation is backed up by the fact that it instantiates a more general semantic pattern for the natural language conditional.

The content matter of mathematics is special, as is its epistemology. For these reasons one could independently expect the natural language conditional to behave in a very similar fashion to the material conditional when one makes conditional mathematical claims, almost to the point of them being indistinguishable. But there still seems to be room for straightforward mathematical uses of the natural language conditional that cannot be given a material interpretation (which is not to say that such claims cannot be translated into *materialese*).

References

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