Allocation Strategy on the Nordic Commercial Real Estate Market
- A quantitative approach

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Stockholm 2010
Master of Science thesis

Title: Allocation Strategy on the Nordic Commercial Real Estate Market.
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Master Thesis number: 493
Supervisor: Han-Suck Song
Keywords: Modern Portfolio Theory, Nordic Commercial Real Estate Market, Econometric analysis.

Abstract

“Watching your portfolio soldiers marching shoulder-to-shoulder in an up market is a lovely thing, but that sets you up for them to march together in a down market and that’s an ugly thing.” [11]

After a tumultuous market environment through 2009, individual and institutional commercial real estate investors worldwide are closely examining their portfolios- assessing losses, identifying opportunities, and looking for signals from the marketplace as to what may be next. The aim of this thesis is to find an optimal strategy of how to allocate capital in a Nordic commercial real estate property portfolio based on diversification by country and types of assets. Our work should be used as a methodology giving broad indications on how to think when creating an optimal commercial real estate portfolio based on Modern Portfolio Theory. We approach the problem by an econometric analysis to determine what influence real estate return, we compare the different markets and we finally suggest an optimal portfolio allocation. The results of this study shows that when choosing assets for a commercial real estate portfolio an investor should not see geographical boarders as obstacles. This thesis demonstrates the benefits of diversification in a commercial real estate property portfolio and the approaches developed can definitely be useful for portfolio managers in solving their asset allocation problems.
Acknowledgement

Before moving on, we want to express our gratitude to all those who’ve given their support to
the thesis, in one way or another. We would especially like to thank two people for their
significant contribution throughout the process. Han-Suck Song, our thesis supervisor, at The
Royal Institute of Technology (KTH), for his feedback and advice. Emmi Wahlström, our
supervisor at the commissioner, for her servings as a sounding board and her valuable
comments and advices. We would also like to thank everyone at the commissioner’s office
and all our interview respondents; the study couldn’t have been done without you.

Stockholm, January 26, 2010

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Nomenclature

*MPT* - Modern Portfolio Theory
*IPD* - Investment Property Bank
*GDP* - Gross Domestic Product
*UE* - Unemployment
*LIR* - Long Term Interest Rate
*SIR* - Short Term Interest Rate
*CPI* - Consumer Price Index
*PC* - Private Consumption
*RS* - Retail Sales
*IMP* - Import
*EXP* - Export
*IO* - Industrial Output
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1 Introduction

This part explains the background and the nature of the problem. The aim of the thesis is described, as well as necessary delimitations to the problem statement.

1.1 Background

As investors review their asset allocations in light of the current difficult economic environment, real estate will continue to provoke attention. Many investors have taken time to consider how their portfolios should be repositioned based on lessons learned from the credit crisis, including how to tactically capture new opportunities owing to financial market dislocations.

In the beginning of the 1970s, the real estate investments industry welcomed Modern Portfolio Theory for planning capital allocations, in a mixed-asset portfolio, at the level of the board. By the 1980s this enthusiasm had carried down to the level of the real estate component within the overall portfolio. This component is also the focus of this thesis. It is argued that Modern Portfolio Theory (MPT) can and should be applied on real estate allocation, for example, to find the optimal mixture among different property types (e.g., office, retail, industrial, etc.) and/or geographic regions (e.g., Sweden, Finland, Denmark, etc.). A key tenet of MPT is that the investor cares about the risk and return of the entire portfolio, rather than any component of the portfolio in isolation. For example, a property segment with little correlation with other property segments in the portfolio would be very appealing in an MPT framework.

It is very difficult to come up with the kind of highly refined, yet reliable, risk and return expectations by property segment that are necessary for the rigorous application of MPT within the real estate portfolio. One should need good estimates of expected returns, volatility, and correlations among all pairs of property segments. Historical periodic data in the property market is just not good enough, nor is our knowledge of the determinants of future real estate returns, to permit very useful analysis at this level of detail.

For these reasons, rigorous application of MPT on real estate portfolios is not very widespread in terms of detailed investment decision making. However, it is useful for conceptual purposes, for example, to understand the potential role a new type of real estate investment
could play within the real estate portfolio. Also, less formal, common-sense-based
diversification of the real estate portfolio is a wise strategy for large institutions. Portfolio
theory builds our intuition about how to think logically concerning such diversification. For
example, portfolio theory suggests that property types whose uses tend to follow different
economic cycles, or economic regions whose economic bases are distinct and relatively
uncorrelated, would be more valuable for diversification purposes.

This thesis disregards the difficulties of using MPT and overcomes the data problem by
creating fictitious data series on Nordic real estate total return. This allows a statistical
analysis and a framework for how to allocate capital between the segments included in a
Nordic real estate property portfolio.

1.2 Commissioner and assignment

The assignment has been commissioned by one of the largest Nordic real estate property
funds. The fund is an unlisted, semi open-ended investment vehicle with an infinite lifetime
and an uncapped target size investing in retail, offices and logistics in Sweden, Finland and
Denmark. The fund aims to become the best diversified unlisted property fund in the region
and to be the obvious choice for investors seeking exposure to real estate in the Nordic region.
The fund management team made a heavy research when laying the foundations for the
strategy of the fund. Since their research was on a more qualitative level mainly based on
market expertise, a more quantitative complement to reinforce their arguments was desirable.
The assignment has thus concerned the development of an allocation structure for a Nordic
real estate property fund.

1.3 Purpose

The aim of this thesis is to find an optimal way of how to allocate capital in a Nordic
commercial real estate property portfolio based on diversification by country and types of
assets. The included countries are Denmark, Finland and Sweden and the types of assets are
office-, retail- and industrial properties. Are there preferred ways of allocating capital among
the countries and the types of assets?

To answer the question we aim to create a methodology that enables us to suggest the optimal
mixture among the different property types and/or geographic regions.
1.4 Delimitations

The choice of real estate as a subject for this thesis was done as a result of our interest to this class of assets and by the challenge related to the small datasets available for real estate returns. Thus, the short and infrequent time series of real estate return is a big delimitation when it comes to the relevance of the numbers in the result. Consequently, the attention of this thesis should not be on numerical details but instead on the methodology used when choosing the targets for real estate investors.

The geographical scope includes countries Sweden, Finland and Denmark. Norway is excluded, as insurance companies from EU countries are one of the target investor groups for the commissioner’s property fund. This client group is not allowed to invest in non-EU countries, among which Norway is one. Apart from that, both the Norwegian economy and property market are less mature, small and very much domestic-oriented compared to their Scandinavian peers. The sector focus of the thesis covers the office, retail and industrial real estate market. The residential real estate market falls outside the scope of the commissioner’s property fund. The reason for this is that the residential investment markets in the Nordic region are generally not very attractive to (foreign) investors. The markets are small and highly regulated (i.e. limited potential for rental growth) and only Sweden has a rented housing market of reasonable size when compared to other European countries.

Lastly, since this thesis includes different countries, member as well as non-members, of the EMU, another delimitation is that currency effects are not taken into account.
2 Method

The research approach and the work procedure are presented in this chapter. These include descriptions of how the assignment is related to the problem statement and as well as an overview of how the work procedure was designed. The working process can be followed below.

2.1 Research

The problem statement described in this thesis can be approached in different ways. Yet, our mathematical background makes a quantitative procedure feel most natural. Interviews (see Chapter 8) together with a literature study, covering working papers as well as academic reports of the area, was done to get a deeper understanding for the real estate industry. Contributions from the interviews are not stated in detail but instead reflect how we framed our hypothesis and approached the problem. The research process also gave us some ideas of which mathematical theories are to be used (see Chapter 3) and how to attack the problem (see Chapter 5 and 6). We divided our problem approach into three sections described below.

2.2 Econometric analysis of determinants of total return

The total returns of commercial real estate in a region change over time in response to changes in several economic background variables, such as population, income, unemployment rate, interest rate etc. In this section a regression analysis, between real estate returns and some background variables will be described. The regression analysis will be applied to data for IPD commercial real estate indexes in the U.K. during the period 1971-2008. The set of background variables that present the strongest explanation of the variations of the different commercial real estate sectors will be identified.

2.3 Comparison of the different markets

This section employs a methodology to specifically test whether the U.K. commercial real estate market is converging with the Nordic countries. Are the solutions of the regression analysis in the U.K. applicable on Nordic data?

The length of Nordic macroeconomic data series is greater than the real IPD return data. To take advantage of this, Nordic background variables are used as in data in the functions
describing the U.K. real estate return. The result is Nordic fictitious return data, longer than the actual return data.

The real Nordic IPD return data is compared with the solutions from the regression analysis to test whether the return in the Nordic countries can be described by the same background variables as in the U.K.

2.4 Optimal portfolio allocation

In this section we use the fictitious data series created in the previous section to find an optimal portfolio allocation. By combining the different assets (country and sector specific) whose returns are not perfectly correlated, we seek to reduce the total variance of the portfolio. To draw some relevant conclusions from the results calculate the expected returns, their standard deviation and how the assets are correlated with each other.
3 Overview of investment mathematics and statistical concepts

In order to simplify for the reader, this part presents an overview of the mathematical phenomena encountered and involved in the modelling process. The aim of this part is only to provide a basic set of tools central to the issue of the methods mentioned further on. It is thus recommended to skip this chapter when reading the report for the first time, and instead use it as a book of reference when reading chapters 5-7.

To formulate a mathematical problem that leads to minimum-variance portfolios we need to review the following theoretical constructs:

- Returns
- Regression analysis
- Portfolio optimisation

3.1 Returns

Returns are the fundamental measure of investment performance. In essence, the return on an investment is what you get, minus what you started out with, expressed as a percentage of what you started out with. The historical return on any investment is defined as the appreciation (or depreciation) in the asset’s price plus any cash flow received as compared to the asset’s beginning price.

The total return is the most basic and complete measure of performance, over a given evaluation period.

The formula for the total return during period \( t \) is

\[
    r_t = \frac{C_F_t + V_t - V_{t-1}}{V_{t-1}}
\]

where

- \( r_t \) = total return in \% during period \( t \)
- \( C_F_t \) = net income during period \( t \)
- \( V_t \) = market value in the end of period \( t \)
- \( V_{t-1} \) = market value in the beginning of period \( t \)
Note that formula (1) can also be written as

$$r_t = \left[ \frac{CF_t + V_t}{V_{t-1}} \right] - 1$$

The total return can be broken down into two components, known as the income return and the appreciation return. These are relevant for the two major types of investment objectives, income and growth.

The **income return** during period $t$, $y_t$, also referred to as the current yield, equals the cash flow paid out to the asset owner during period $t$, as a fraction of the value of the asset at the beginning of the period, and is defined as

$$y_t = \frac{CF_t}{V_{t-1}}$$

The **appreciation return** during period $t$, $g_t$, also referred to as the capital return, is the change in the asset market value during period $t$, as a fraction of the asset market value at the beginning of period $t$, and is defined as

$$g_t = \frac{V_t - V_{t-1}}{V_{t-1}}$$

Note that the income return and the appreciation return components sum to the total return, during the period $t$:

$$r_t = y_t + g_t$$

Thus, the total return is the most important measure of the periodic return since it is more complete than either the growth or income component alone.
3.2 **Regression analysis**

A regression analysis is concerned with the study of, if, and how a variable (the *dependent variable*) is related to one or more other variables (the *explanatory variables*) in order to estimate and/or predict the mean of the former in terms of the known values of the latter.

### 3.2.1 Linear Regression

A regression model is characterized by the way the dependent variable is seen as a function of the explanatory variables. A linear regression means that the regression is linear in the coefficients (the coefficients are raised to the first power only). It may or may not be linear in the explanatory variables.

A linear relationship between \( k \) explanatory variables \( X_1, X_2, \ldots, X_k \) and the dependent variable \( Y \), may be written as

\[
Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k + \epsilon
\]

(5)

where \( \beta_1, \beta_2, \ldots, \beta_k \) are the coefficients that measure the effect that the associated explanatory variable has on \( Y \). Hence, the coefficients indicate the sensitivity of the dependent variable to the explanatory variable. Since a linear relationship isn’t generally perfectly linear, the error term, \( \epsilon \) is added. The reason is that all the explanatory variables that affect the dependent variable haven’t been taken to account in the equation and can therefore be represented by \( \epsilon \).

### 3.2.2 Ordinary least square (OLS)

To estimate the coefficients \( \beta_1, \beta_2, \ldots, \beta_k \) the Ordinary Least Square (OLS) method is used.

Observations on both \( Y \) and \( X_1, X_2, \ldots, X_k \) are required to be able to implement the OLS method. When having \( n \) observations, expression (5) can be written as

\[
Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + \epsilon_i \quad i = 1, 2, 3, \ldots, n
\]

(6)

where \( i \) represents observation number.

The task is to find the \( \beta \) values that fit all samples of \( Y_i \) and \( X_i \) in a linear model, in the “best” possible way. Given that the “best” coefficients \( \hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_k \) are found, it is possible to
predict outcomes of $\hat{Y}$ based on new $X$’s. The difference between the true $Y_i$ and the predicted $\hat{Y}_i$ is the error term, and the coefficients are obtained by minimizing the sum of the square error terms:

$$\sum \hat{e}_i^2 = \sum (Y_i - \hat{Y}_i)^2 = \sum Y_i - \hat{\beta}_o - \hat{\beta}_1 X_{1,i} - ... - \hat{\beta}_k X_{k,i}^2$$  

(7)

### 3.2.3 Goodness-of-fit

When performing a regression, it is interesting to know how much the set of chosen explanatory variables affects the dependent variable, i.e. how well the regression fits the observed data. One wants to know to what extent the regression explains variation in the observed $Y$’s and how much of the variation that must be absorbed by the error terms $\varepsilon$.

The coefficient determination, $R^2$, is the most commonly used measure of the goodness of fit of a regression line. Verbally, it measures the proportion or percentage of the total variation in $Y$ explained by the regression model. To define $R^2$, three basic concepts must be introduced:

- **Total Sum of Squares (TSS):** $\sum (Y_i - \bar{Y})^2 = \text{total variation of the actual } Y \text{ values about their sample mean.}$
- **Explained Sum of Squares (ESS):** $\sum (\hat{Y}_i - \bar{Y})^2 = \text{total variation of the estimated } Y \text{ values about their mean.}$
- **Residual Sum of Squares (RSS):** $\sum (\hat{e}_i)^2 = \text{residual or unexplained variation of the } Y \text{ values about the regression line.}$

Through calculations one can see the following:

$$\text{TSS} = \text{ESS} + \text{RSS}$$  

(8)

Verbally the above expression means that the total variation in the observed $Y$ values about their mean value can be partitioned into two parts, one attributable to the regression line and the other to random forces since not all actual $Y$ observations lie on the fitted line.

The following will be obtained by dividing with TSS on both sides of equation (8);
\[ 1 = \frac{ESS}{TSS} + \frac{RSS}{TSS} \]  

(9)

\[ R^2 \text{ is defined as} \]

\[ R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS} \]  

(10)

Two properties of \( R^2 \) may be noted. The first one is that it is a nonnegative quantity. The second one is that its limits are \( 0 \leq R^2 \leq 1 \). An \( R^2 \) of 1 means a perfect fit, that is, \( \hat{Y}_i = Y_i \) for each \( i \). An \( R^2 \) of zero means that there is no relationship between the explanatory variables and the dependent variable. When comparing two regression models with the same dependent variable but with different number of \( X \) variables, one should be very wary of choosing the model with the highest \( R^2 \). To compare two \( R^2 \) terms, one must take into account the number of \( X \) variables present in the model. This is done by using the alternative coefficient of determination, the adjusted \( R^2 \).

Unlike \( R^2 \), \( R_{adj}^2 \) increases only if the new explanatory variable improves the model more than would be expected by chance. \( R_{adj}^2 \) can be negative, and will always be less than or equal regular \( R^2 \).

It is defined as

\[ R_{adj}^2 = 1 - (1 - R^2) \frac{n-1}{n-k} \]  

(11)

\( R_{adj}^2 \) does not have the same interpretation as regular \( R^2 \), and is mainly useful for deciding if an explanatory variable improves the results to such an extent that it should be included in the regression.

### 3.2.4 Confidence intervals for regression coefficients

The estimations of \( \beta_1, \beta_2, \ldots, \beta_k \) are single point estimations. In statistics the reliability of a point estimator is measured by its standard error. Therefore, instead of relying on the point estimate alone, an interval around the point estimator is constructed.

An interval around \( \beta_k \) is constructed as follows:
\[
\Pr \left[ \hat{\beta}_k - t_{\alpha/2} \sigma_{\hat{\beta}_k} \leq \beta_k \leq \hat{\beta}_k + t_{\alpha/2} \sigma_{\hat{\beta}_k} \right] = 1 - \alpha
\]  \hfill (12)

where

- \( \hat{\beta}_k \) = estimator
- \( \sigma_{\hat{\beta}_k} \) = estimated standard deviation of the estimator
- \( \alpha \) = level of significance
- \( t_{\alpha/2} \) = value of the \( t \) variable obtained from the \( t \) distribution for a \( \alpha / 2 \) level of significance and \( n-k \) degrees of freedom

The width of the confidence interval is proportional to the standard error of the estimator. That is, the larger the standard error, the larger is the width of the confidence interval. Explained differently, the larger the standard error of the estimator (\( \sigma_{\hat{\beta}_k} \)), the greater is the uncertainty of estimating the true value of the unknown parameter. One can use the computed confidence interval to determine if a regression coefficient is significant. If the confidence interval holds the value zero, one can say that the related regression coefficient can with some probability be insignificant.

### 3.2.5 Multicollinearity

Multicollinearity is a statistical phenomenon in which two or more explanatory variables in a multiple regression model are highly correlated. We have perfect multicollinearity if the correlation between two explanatory variables is equal to 1 or -1. In practice, we rarely face perfect multicollinearity in a data set. More commonly, the issue of multicollinearity arises when there is a high degree of correlation (either positive or negative) between two or more explanatory variables. The clearest sign of multicollinearity is when \( R^2 \) is very high but none of the regression coefficients is statistically significant. Another rule of thumb is that if intercorrelations between explanatory variables are higher than the \( R^2 \) of the entire regression, multicollinearity will cause a problem. Yet, it has been said that if the sole purpose of the regression analysis is prediction or forecasting, then multicollinearity is not a serious problem because the higher the \( R^2 \), the better prediction. This is true as long as the values of the explanatory variables for which predictions are desired obey the same near-exact linear dependencies as the original design (data) matrix [2].
3.3 Portfolio Optimisation

3.3.1 Variance and covariance of returns

Understanding how risk is related to return is a basic part of real estate investment. The most widely used statistical measure to quantify risk is known as the standard deviation of the probability distribution of the future return possibilities. This measure of investment risk is also known as volatility. The greater is the standard deviation in the possible return (for a given future time horizon), the greater will usually the risk be. Essentially this statistically measures the dispersion of returns around an average value. To examine the dispersion of historical returns, the standard deviation of these returns can be written as:

$$\sigma = \sqrt{\frac{\sum_{t=1}^{T} (r_t - \bar{r})^2}{T-1}}$$  \hspace{1cm} (13)$$

where
- $T$ = total number of time periods under analysis
- $r_t$ = return in period $t$
- $\bar{r}$ = average return over $T$ time periods

Most important to understand in this definition of risk and return is that risk is represented by the range or deviation of the possible future return outcomes around the prior expectation of the return, which may be thought of roughly as a best guess.

Another measure of risk is covariance, which measures the way the total variance of two assets is altered when the assets’ returns are held in the same portfolio. Covariance, in turn, is dependent on the correlation between the returns of the two assets, as measured by the correlation coefficient. This coefficient can range from +1 to -1. The formula for computing the covariance and correlation coefficient are given, respectively, in equations (14) and (15):

$$\sigma_{i,j}^2 = \sigma_i \times \sigma_j \times \rho_{i,j}$$  \hspace{1cm} (14)$$

where
- $\sigma_{i,j}^2$ = covariance between the returns for asset $i$ and $j$
- $\sigma_i$ = standard deviation of asset $i$’s return distribution
\( \sigma_j \) = standard deviation of asset \( j \)'s return distribution

\( \rho_{i,j} \) = correlation coefficient between the returns of asset \( i \) and asset \( j \)

\[
\rho_{i,j} = \frac{\sum_{t=1}^{T} (r_{it} - \bar{r}_i) \times (r_{jt} - \bar{r}_j)}{\sqrt{\sum_{t=1}^{T} (r_{it} - \bar{r}_i)^2 \times \sum_{t=1}^{T} (r_{jt} - \bar{r}_j)^2}}
\]  

(15)

The notion of identifying the degree to which asset returns move together, their correlation, is central to what Markowitz described as mean/variance efficiency and later became the foundation for Modern Portfolio Theory. This notion is central to efficient diversification, which fundamentally rests on identifying assets with different return patterns while enjoying high absolute returns.

### 3.3.2 Modern Portfolio Theory

The objective in Modern Portfolio Theory is to minimize the portfolio volatility (or variance) subject to an expected return target. In fact, this objective is mathematically equivalent to its “dual” description: the maximization of portfolio expected return subject to a volatility constraint. Harry Max Markowitz (1952) demonstrated that the focus on maximizing the expected return is not a sufficient portfolio consideration; rather, the returns and variance of assets must be considered in combination. A diversified portfolio will create the optimal combination. The mathematical problem that minimizes the variance of the portfolio can be expressed as follows;

\[
\begin{align*}
\text{min} & \quad \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\
\text{s.t.} & \quad \sum_{i=1}^{n} w_i \bar{r}_i = r_{\text{avg et}} \\
& \quad \sum_{i=1}^{n} w_i = 1
\end{align*}
\]

(16)

where \( \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \) = variance of the return of a portfolio containing \( n \) different assets

\[
\sum_{i=1}^{n} w_i \bar{r}_i = \text{expected rate of return of a portfolio containing } n \text{ different assets.}
\]

The factor 1/2 in front of the variance is for convenience only. It makes the final form of the
equations neater. For further explanation of the problem statement (16), see appendix A.

### 3.3.3 Diversification

Portfolios with only a few assets can be subject to a high degree of risk, represented by relatively large variance. The process referred to as diversification can reduce that risk. By including additional assets in the portfolio the variance of the return can be reduced. This process reflects the philosophy, “Don’t put all your eggs in one basket” [3]. It is common sense that if all (or nearly all) of someone’s wealth is invested in only one type of asset, then this someone is overly exposed to loss in the case of a downside event that randomly affects that one type of asset. Portfolio theory quantifies the benefit of diversification in terms of portfolio risk and return by providing some rigorous guidance as how to, exactly, diversify, that is, how many eggs should this someone put in which baskets [4]. By holding a combination of assets that are not perfectly positively correlated, assets that do not move together, greater diversification benefits will be provided when they are combined in a portfolio. Hence, the volatility of such assets, combined in a portfolio, tends to cancel out.

### 3.3.4 Efficient frontier

Following the procedure outlined by Markowitz (1952), the optimal combination of assets weights can be determined based on the return, variance, and correlation coefficients for each asset in the portfolio. Every possible asset combination, portfolio, can be plotted in a risk-return space, as shown in Figure No.1, and the collection of all such possible portfolios defines a region in this space. These optimal portfolios are said to comprise what is known as the efficient frontier. Combinations along this line represent portfolios (explicitly excluding the risk-free alternative) with the lowest risk for a given level of return. Conversely, for a given amount of risk, the portfolio lying on the efficient frontier represents the combination offering the best possible return. Mathematically the efficient frontier is the intersection of the set of portfolios with minimum variance and the set of portfolios with maximum return. Formally, the efficient frontier is the set of portfolios for which one cannot improve both risk and return.
Allocations beyond the curve (the efficient frontier) reflect returns impossible to generate under current conditions. The area, below the curve, reflects portfolios that are inferior to the portfolios on the efficient frontier— they either offer the same returns but with more risk, or they offer less return for the same risk.
4 Description of Data

In this section we describe the data that will be used as input in our models. We divided the data into return data and macro economic background data.

4.1 Return Data

Historical periodic return data in the property market is difficult to obtain in most countries. The most transparent real estate market when it comes to IPD coverage is the U.K. which therefore is our choice of country for the regression analysis. Yearly total return series per sector (Retail, Office and Industrial) are covering the period 1971-2008, and can be seen below.

*Figure No. 2: Total U.K. Return 1971-2008*

As seen from *Figure No.2*, the returns from the different sectors follow the same pattern and can be expected to have high correlation between each other. From *Table No.1*, below, a rough estimation states that the industrial sector is the most attractive in between the three. It has the highest expected return and lowest standard deviation. The least attractive sector between the three is the office sector, which has the lowest expected return and the highest standard deviation.
Table No.1: Expected return and standard deviation for the different sectors.

<table>
<thead>
<tr>
<th></th>
<th>( r^{\text{retail}} )</th>
<th>( r^{\text{office}} )</th>
<th>( r^{\text{industrial}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp. Return (%)</td>
<td>11.11</td>
<td>9.61</td>
<td>11.73</td>
</tr>
<tr>
<td>Std. Dev. (%)</td>
<td>10.71</td>
<td>12.34</td>
<td>10.51</td>
</tr>
</tbody>
</table>

4.2 Macroeconomic Data

Various papers study determinants of real estate on a global scale. Case et al. (1999) attributed a substantial amount of the correlation across world property markets to the effects of changes in GNP, suggesting that real estate is a bet on fundamental economic variables, which are correlated across countries. A decomposition shows that a local production factor is more important in some countries than in others [5]. Ling and Naranjo (2000) perform a cross-country analysis on real estate returns. Their findings suggest that excess real estate returns are not constant across countries. Moreover, this is consistent with the findings by Case et al. (1999). They show that country-specific effects drive real estate returns, for example, the impact of local production factors varies between countries.

Later on Ronald Van Dijk (2003) investigated the determinants of direct real estate returns by analysing rents, capital appraisals, and total return. With a macro-view on the effects of the economic growth and supply and demand factors on nominal real estate returns he found that GDP, inflation, unemployment, vacancy rate, and the available stock all have an effect on real estate returns [9].

Our explanatory data set consists of macroeconomic demand indicators. Variables affecting the demand-side of properties are most likely to be economics-based. These variables are normally approximated by the real Gross Domestic Product and employment rates. GDP normally represents general economic conditions, as does the unemployment rate. In single-equation research, the real GDP figure remains a consistently significant influence on real estate returns. Empirical studies, [5] among others, noted that real GDP is the most appropriate and widely used demand-side measurement at an aggregate level, which gives a broad indicator of commercial real estate activity. Also interest rates are considered as predictors of economic conditions and provide an indication of the availability and cost of capital. Interest rates can also indicate directions of monetary policy and the dampening effect of high interest rates on economic activity is well established. Besides the above mentioned
demand side variables we also incorporate inflation in the model to pick up effects of shifts in value of money.

Additionally, we include a number of more sector specific demand variables in our analysis. In the retail sector regression analysis private consumption and retail sales will be included, whereas in the industrial sector regression analysis we additionally consider import, export and industrial output.

4.3 Weaknesses of data

Analysis about how returns and risk are related to each other when making real estate investments require great amount of observations of high quality and long time series. The data series used in this thesis are neither in great amounts nor of high quality. The fact that IPD data is only available on yearly basis makes the time series short. Also, the IPD index does not consider all transactions; it only includes properties which are directly owned by the organizations that contribute data to IPD. Especially in the Nordic countries the index quality is bad and the time series only contain a couple of years. IPD figures for Denmark and Finland are only available until respectively 2000 and 1998 and comparison is therefore somewhat distorted.
5 The Models

This section presents the models used to describe the total return. A general description of the models is followed by a specific presentation of each sector.

5.1 The models

The total returns observed for year \( t \) can be expressed as functions of the selected background variables for each sector. Our model should describe the absolute change in total return, \( r_t \), for a percentage change in the background variables. A model that can accomplish this purpose can be written as:

\[
 r_t = \beta_0 + \beta_1 \cdot \ln \left( \frac{X_t}{X_{t-1}} \right) + \epsilon_t
\]  

(17)

Let us interpret the slope coefficient \( \beta_1 \):\(^1\)

\[
 \beta_1 = \frac{\text{change in } r_t}{\text{change in } \ln \left( \frac{X_t}{X_{t-1}} \right)} = \frac{\text{change in } r_t}{\text{relative change in } \frac{X_t}{X_{t-1}}}
\]

The second step follows from the fact that a change in the log of a number is a relative change[2].

The values of the economic background variables recorded for year \( t \) will be denoted by subscript \( t \), i.e. the following notation will be used for the background variables considered: Gross Domestic Product (GDP), import (IMP), export (EXP), unemployment rate (UE), long term interest rate (LIR), short term interest rate (SIR), retail sales (RS), Consumer Price Index (CPI), private final consumption (PC) and industrial output (IO).

\(^1\)Let \( Z = X_t / X_{t-1} \), then using differential calculus, we have \( dr/dZ = \beta_1 (1/Z) \). Therefore \( \beta_1 = dr/dZ \). Symbolically, we have \( \beta_1 = \Delta r / \Delta Z \), where, as usual, \( \Delta \) denotes a small change. The previous equation can be written, equivalently, as \( \Delta r = \beta_1 \Delta Z / Z \).
5.1.1 Retail Sector

The total retail returns observed year $t$ can be expressed as functions of the set of the selected background variables as below;

$$r_{t, \text{retail}} = \beta_{0, \text{retail}} + \beta_{\text{GDP}, \text{retail}} \cdot \ln \left( \frac{\text{GDP}_{t}}{\text{GDP}_{t-1}} \right) + \beta_{\text{LIR}, \text{retail}} \cdot \ln \left( \frac{\text{LIR}_{t}}{\text{LIR}_{t-1}} \right) + \beta_{\text{SIR}, \text{retail}} \cdot \ln \left( \frac{\text{SIR}_{t}}{\text{SIR}_{t-1}} \right) + \beta_{\text{CPI}, \text{retail}} \cdot \ln \left( \frac{\text{CPI}_{t}}{\text{CPI}_{t-1}} \right) + ...$$

$$... + \beta_{\text{RS}, \text{retail}} \cdot \ln \left( \frac{\text{RS}_{t}}{\text{RS}_{t-1}} \right) + \beta_{\text{PC}, \text{retail}} \cdot \ln \left( \frac{\text{PC}_{t}}{\text{PC}_{t-1}} \right) + \beta_{\text{UE}, \text{retail}} \cdot \ln \left( \frac{\text{UE}_{t}}{\text{UE}_{t-1}} \right) + \epsilon_{t, \text{retail}}$$

(18)

where $\beta_{\text{GDP}, \text{retail}}, \beta_{\text{LIR}, \text{retail}}, \beta_{\text{SIR}, \text{retail}}, \beta_{\text{RS}, \text{retail}}, \beta_{\text{PC}, \text{retail}}, \beta_{\text{UE}, \text{retail}}, \beta_{\text{GDP}}$ and $\beta_{0, \text{retail}}$ are the constants to be found by regression analysis, i.e. by minimizing the quadratic error.

5.1.2 Office Sector

The total office returns observed year $t$ can be expressed as functions of the set of the selected background variables as below;

$$r_{t, \text{office}} = \beta_{0, \text{office}} + \beta_{\text{GDP}, \text{office}} \cdot \ln \left( \frac{\text{GDP}_{t}}{\text{GDP}_{t-1}} \right) + \beta_{\text{LIR}, \text{office}} \cdot \ln \left( \frac{\text{LIR}_{t}}{\text{LIR}_{t-1}} \right) + \beta_{\text{SIR}, \text{office}} \cdot \ln \left( \frac{\text{SIR}_{t}}{\text{SIR}_{t-1}} \right) + \beta_{\text{CPI}, \text{office}} \cdot \ln \left( \frac{\text{CPI}_{t}}{\text{CPI}_{t-1}} \right) + ...$$

$$... + \beta_{\text{UE}, \text{office}} \cdot \ln \left( \frac{\text{UE}_{t}}{\text{UE}_{t-1}} \right) + \epsilon_{t, \text{office}}$$

(19)

where $\beta_{\text{GDP}, \text{office}}, \beta_{\text{LIR}, \text{office}}, \beta_{\text{SIR}, \text{office}}, \beta_{\text{RS}, \text{office}}, \beta_{\text{PC}, \text{office}}, \beta_{\text{UE}, \text{office}}$ and $\beta_{0, \text{office}}$ are the constants to be found by regression analysis, i.e. by minimizing the quadratic error.

5.1.3 Industrial Sector

The total industrial returns observed year $t$ can be expressed as functions of the set of the selected background variables as below;

$$r_{t, \text{industrial}} = \beta_{0, \text{industrial}} + \beta_{\text{GDP}, \text{industrial}} \cdot \ln \left( \frac{\text{GDP}_{t}}{\text{GDP}_{t-1}} \right) + \beta_{\text{LIR}, \text{industrial}} \cdot \ln \left( \frac{\text{LIR}_{t}}{\text{LIR}_{t-1}} \right) + \beta_{\text{SIR}, \text{industrial}} \cdot \ln \left( \frac{\text{SIR}_{t}}{\text{SIR}_{t-1}} \right) + \beta_{\text{CPI}, \text{industrial}} \cdot \ln \left( \frac{\text{CPI}_{t}}{\text{CPI}_{t-1}} \right) + ...$$

$$... + \beta_{\text{IMP}, \text{industrial}} \cdot \ln \left( \frac{\text{IMP}_{t}}{\text{IMP}_{t-1}} \right) + \beta_{\text{EXP}, \text{industrial}} \cdot \ln \left( \frac{\text{EXP}_{t}}{\text{EXP}_{t-1}} \right) + \beta_{\text{IO}, \text{industrial}} \cdot \ln \left( \frac{\text{IO}_{t}}{\text{IO}_{t-1}} \right) + \beta_{\text{UE}, \text{industrial}} \cdot \ln \left( \frac{\text{UE}_{t}}{\text{UE}_{t-1}} \right) + \epsilon_{t, \text{industrial}}$$

(20)
where $\beta^{\text{industr}}_{\text{GDP}}, \beta^{\text{industr}}_{\text{LIR}}, \beta^{\text{industr}}_{\text{SIR}}, \beta^{\text{industr}}_{\text{CPI}}, \beta^{\text{industr}}_{\text{IMP}}, \beta^{\text{industr}}_{\text{EXP}}, \beta^{\text{industr}}_{\text{IO}}, \beta^{\text{industr}}_{\text{UE}}$ and $\beta^{\text{industr}}_0$ are the constants to be found by regression analysis, i.e. by minimizing the quadratic error.

6 Analysis

In this section we present all the results conducted in this thesis together with discussions. Usually when it comes to building a model, the testing part of the process is of considerable importance. Hence, here we test the results; if they are reasonable and how sensitive they are to changes in the parameters.

6.1 Econometric analysis of determinants of total return

6.1.1 Hypotheses

To get an idea of how the macroeconomic variables affect the total real estate return we below state our hypothesis.

General

- Demand variables are positively related to total returns. Consequently, the proxies for demand, GDP and UE are, respectively, positively and negatively related to total return.
- SIR is negatively correlated with total return. The impact is mainly through the cost of capital, cost of equity and the effect on rents. Yet, a change in short term interest rate will have small immediate effect on property return, but a large effect a few years later. [7].
- LIR, treasury bonds return, is negatively correlated with total return. One reason for this is the effect of the cost of capital.
- Change in CPI is a determinant of short-term adjustments in prices within the real estate market. This short-term price mechanism is revealed by the positive effect of change in inflation on change in rent. Property yields should be driven primarily by real interest rates since most lease agreements are inflation adjusted [7]. As a consequence, property investments are assumed to generate relatively strong returns in times of high inflation and vice versa, i.e. total return is positively correlated with changes in CPI.
Sector Specific

- The retail sector return is positively correlated with consumer demand described by \( PC \) and \( RS \).
- The office sector return has close links to the \( UE \) since it is believed that the majority of service sector activity takes place in an office environment [9]. When unemployment in the service sector rises, it decreases demand for office space and hence is negatively correlated with office sector returns.
- The demand on the industrial markets is, besides the overall economic conditions, driven by developments in \( EXP, IMP \) and \( IO \) growth. This is the case since, when trade increases the demand for logistic properties, the main part of the industrial market, will go up.

6.1.2 Retail Sector

6.1.2.1 Correlation Analysis

In Table No.2 the correlation factors between the retail sectors total return and the background variables are shown.

<table>
<thead>
<tr>
<th>( r^{retail} )</th>
<th>( GDP )</th>
<th>LIR</th>
<th>SIR</th>
<th>CPI</th>
<th>RS</th>
<th>PC</th>
<th>UE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r^{retail} )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GDP</td>
<td>0.3958</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LIR</td>
<td>-0.1729</td>
<td>-0.0102</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SIR</td>
<td>0.0071</td>
<td>0.2691</td>
<td>0.6439</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CPI</td>
<td>0.1230</td>
<td>-0.4929</td>
<td>0.2800</td>
<td>0.0710</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RS</td>
<td>0.2709</td>
<td>0.6359</td>
<td>-0.2238</td>
<td>-0.1004</td>
<td>-0.5914</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>PC</td>
<td>0.3185</td>
<td>0.6884</td>
<td>0.0506</td>
<td>0.3268</td>
<td>-0.3850</td>
<td>0.4849</td>
<td>1</td>
</tr>
<tr>
<td>UE</td>
<td>-0.3287</td>
<td>-0.8725</td>
<td>-0.1037</td>
<td>-0.3177</td>
<td>0.6046</td>
<td>-0.6089</td>
<td>-0.5892</td>
</tr>
</tbody>
</table>

It can be observed in this table that \( GDP, PC \) and \( UE \) have a reasonably high correlation with the retail returns \( r^{retail} \). Furthermore, the table reveals that \( SIR \) has virtually no correlation with the returns. Also, since pair wise correlations between the explanatory variables are high (note especially the correlation of -0.8725 between \( UE \) and \( GDP \)), we conclude that there is a high degree of multicollinearity in the data set.


6.1.2.2 Multivariate Regression Analysis

The solution of the regression analysis given by equation (18) is the following:

<table>
<thead>
<tr>
<th>Table No.3: Retail Sector; Multivariate Regression Analysis Result 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>( \beta_{\text{retail}} )</td>
</tr>
<tr>
<td>( \beta_{0.95,\text{retail}} )</td>
</tr>
<tr>
<td>( \beta_{0.95} )</td>
</tr>
</tbody>
</table>

The coefficients of determination corresponding to the above regression analysis are:

\[
R^2 = 0.4971 \\
R^2_{\text{adj}} = 0.3757
\]

From Table No. 3, one can see that \( LIR, CPI \) and \( UE \) are the only variables with significant coefficients at a 95% significance level. The other four (\( GDP, SIR, RS \) and \( PC \)) have a 95% confidence interval for their coefficients that contains the value zero.

In order to maximize \( R^2_{\text{adj}} \) we eliminate explanatory variables. The maximum \( R^2_{\text{adj}} \) is obtained when \( SIR \) is removed. This can be explained by the low regression coefficient for \( SIR \) as well as the correlation between the variable and the return. The following table shows the regression results with \( SIR \) removed.

<table>
<thead>
<tr>
<th>Table No.4: Retail Sector; Multivariate Regression Analysis Result 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>( \beta_{\text{retail}} )</td>
</tr>
<tr>
<td>( \beta_{0.95,\text{retail}} )</td>
</tr>
<tr>
<td>( \beta_{0.95} )</td>
</tr>
</tbody>
</table>

The coefficients of determination corresponding to the above regression analysis are:

\[
R^2 = 0.4964 \\
R^2_{\text{adj}} = 0.3957
\]

From Table No. 4, one can notice that the significant and insignificant explanatory variables are the same as in the previous regression. As a result of the fewer explanatory variables, the
value of $R^2$ is, as expected, lower. Yet, since the important metric is $R^2_{adj}$, which is somewhat higher than in the previous regression, we can safely rule out SIR as an explanatory variable. The values of the GDP coefficient in both tables can be rather misleading. One can see that GDP is a variable with insignificant negative coefficient. Logically the GDP should have a positive impact on the real estate return. Due to the high negative correlation between UE and GDP, seen in Table No. 2, it can be stated that multicollinearity is present in the data set. As a result our regression has “trouble” determining on which variable to place the coefficient weight that “rightfully” belongs entirely to one of the variables. Hence, some variables can be seen as insignificant and their coefficients do not have proper values. The following table shows the regression results with UE removed.

<table>
<thead>
<tr>
<th></th>
<th>Constant</th>
<th>GDP</th>
<th>LIR</th>
<th>CPI</th>
<th>RS</th>
<th>PC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{retail}$</td>
<td>0.00687</td>
<td>2.6185</td>
<td>-0.2929</td>
<td>1.2650</td>
<td>0.7656</td>
<td>0.6521</td>
</tr>
<tr>
<td>$\beta_{0.95+}$</td>
<td>0.0203</td>
<td>5.1620</td>
<td>-0.0007</td>
<td>2.0207</td>
<td>2.5630</td>
<td>2.2740</td>
</tr>
<tr>
<td>$\beta_{0.95-}$</td>
<td>-0.1576</td>
<td>0.0749</td>
<td>-0.5851</td>
<td>0.5094</td>
<td>-0.8319</td>
<td>-0.9699</td>
</tr>
</tbody>
</table>

The coefficients of determination corresponding to the above regression analysis are:

$$R^2 = 0.4125$$
$$R^2_{adj} = 0.3178$$

From Table No.5 it can be seen that the GDP now has a high positive coefficient, significant at the 95% significance level. The reason for that GDP obtains a misleading value when UE is present can together with multicollinearity be explained by that UE has a strong influence on the retail return. However, since the purpose of our regression analysis is prediction we are not concerned about multicollinearity. Instead we focus on the regression model that computes the highest $R^2_{adj}$, which is the following one:

$$r^2_{retail} = -0.0249 - 0.4884 \cdot \ln \left( \frac{GDP}{GDP_{t-1}} \right) - 0.4363 \cdot \ln \left( \frac{LIR}{LIR_{t-1}} \right) + 1.7100 \cdot \ln \left( \frac{CPI}{CPI_{t-1}} \right) + 0.6729 \cdot \ln \left( \frac{RS}{RS_{t-1}} \right) + ...$$

$$... + 0.8704 \cdot \ln \left( \frac{PC}{PC_{t-1}} \right) - 0.3687 \cdot \ln \left( \frac{UE}{UE_{t-1}} \right)$$
It can be noticed that the signs of the coefficients, besides the GDP, in front of each explanatory variable coincide with the hypothesis stated earlier. The negative sign in front of the GDP can, as mentioned earlier, be explained by the presence of multicollinearity.

6.1.3 Office Sector

6.1.3.1 Correlation Analysis

In Table No.6 the correlation factors between the office sector’s total return and the background variables is shown.

<table>
<thead>
<tr>
<th></th>
<th>r_{office}</th>
<th>GDP</th>
<th>LIR</th>
<th>SIR</th>
<th>CPI</th>
<th>UE</th>
</tr>
</thead>
<tbody>
<tr>
<td>r_{office}</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GDP</td>
<td>0.5699</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LIR</td>
<td>-0.0215</td>
<td>-0.0102</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SIR</td>
<td>0.1758</td>
<td>0.2691</td>
<td>0.6439</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CPI</td>
<td>0.0214</td>
<td>-0.4929</td>
<td>0.2800</td>
<td>0.0710</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>UE</td>
<td>-0.5136</td>
<td>-0.8725</td>
<td>-0.1037</td>
<td>-0.3177</td>
<td>0.6046</td>
<td>1</td>
</tr>
</tbody>
</table>

It can be observed in this table that GDP and UE have a reasonably high correlation with the office returns r_{office}. Furthermore, the table reveals that LIR and CPI have virtually no correlation with the returns.

6.1.3.2 Multivariate Regression Analysis

The solution of the regression analysis given by (19) is the following:

<table>
<thead>
<tr>
<th></th>
<th>Constant</th>
<th>GDP</th>
<th>LIR</th>
<th>SIR</th>
<th>CPI</th>
<th>UE</th>
</tr>
</thead>
<tbody>
<tr>
<td>β_{office}</td>
<td>-0.0366</td>
<td>2.0935</td>
<td>-0.2804</td>
<td>0.0138</td>
<td>1.3425</td>
<td>-0.3482</td>
</tr>
<tr>
<td>β_{0.05}</td>
<td>0.0626</td>
<td>5.7872</td>
<td>0.1398</td>
<td>0.2297</td>
<td>2.2313</td>
<td>0.0324</td>
</tr>
<tr>
<td>β_{0.05}</td>
<td>-0.1357</td>
<td>-1.6002</td>
<td>-0.7007</td>
<td>-0.2021</td>
<td>0.4537</td>
<td>-0.7288</td>
</tr>
</tbody>
</table>

The coefficients of determination corresponding to the above regression analysis are:

\[ R^2 = 0.4859 \]
\[ R_{adj}^2 = 0.4030 \]

From Table No.7, one can notice that CPI is the only variable with significant coefficient at a 95% significance level. The high number of insignificant coefficients is a result of the high
degree of multicollinearity existing in the data set. By eliminating the lowest regression coefficient, \( SIR \), we want to see if we will receive a higher \( R^2_{adj} \) value. The following table shows the regression results with \( SIR \) removed.

<table>
<thead>
<tr>
<th></th>
<th>Constant</th>
<th>GDP</th>
<th>LIR</th>
<th>CPI</th>
<th>UE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_{office} )</td>
<td>-0.0374</td>
<td>2.1266</td>
<td>-0.2644</td>
<td>1.3465</td>
<td>-0.3497</td>
</tr>
<tr>
<td>( \beta_{0.95+} )</td>
<td>0.0593</td>
<td>5.7226</td>
<td>0.0670</td>
<td>2.2183</td>
<td>0.0238</td>
</tr>
<tr>
<td>( \beta_{0.95-} )</td>
<td>-0.1341</td>
<td>-1.4694</td>
<td>-0.5957</td>
<td>0.4747</td>
<td>-0.7232</td>
</tr>
</tbody>
</table>

The coefficients of determination corresponding to the above regression analysis are:

\[
R^2 = 0.4856 \\
R^2_{adj} = 0.4213
\]

By eliminating \( SIR \), we receive, as expected, a lower \( R^2 \). Yet, we will proceed with this model since it has the highest possible \( R^2_{adj} \) value. We want to predict returns with a regression model that includes explanatory variables that improves the goodness of fit more than would be expected by chance and still have a high \( R^2 \). Hence, the best description of the office returns is given in Table No. 8 and can be seen in the following function:

\[
r_{office}^t = -0.0374 + 2.1266 \cdot \ln \left( \frac{GDP_t}{GDP_{t-1}} \right) - 0.2644 \cdot \ln \left( \frac{LIR_t}{LIR_{t-1}} \right) + 1.3465 \cdot \ln \left( \frac{CPI_t}{CPI_{t-1}} \right) - 0.3497 \cdot \ln \left( \frac{UE_t}{UE_{t-1}} \right)
\]

It can be noticed that the signs of the coefficients in front of each explanatory variable coincide with the hypothesis presented earlier.

6.1.4 Industrial Sector

6.1.4.1 Correlation Analysis

In Table No. 9 the correlation factors between the industrial sector’s total return and the background variables is shown.
Table No.9: Correlation factors between observed industrial returns and a set of background variables.

<table>
<thead>
<tr>
<th>$r_{\text{industrial}}$</th>
<th>GDP</th>
<th>LIR</th>
<th>SIR</th>
<th>CPI</th>
<th>IMP</th>
<th>EXP</th>
<th>IO</th>
<th>UE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{\text{industrial}}$</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GDP</td>
<td>0.3615</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LIR</td>
<td>-0.0635</td>
<td>-0.0102</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SIR</td>
<td>0.0622</td>
<td>0.2691</td>
<td>0.6439</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CPI</td>
<td>0.1383</td>
<td>-0.4929</td>
<td>0.2800</td>
<td>0.0710</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IMP</td>
<td>0.0753</td>
<td>0.1956</td>
<td>0.5177</td>
<td>0.3483</td>
<td>0.3587</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EXP</td>
<td>0.0620</td>
<td>-0.1625</td>
<td>0.4064</td>
<td>0.1700</td>
<td>0.6323</td>
<td>0.8091</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>IO</td>
<td>0.3715</td>
<td>0.8559</td>
<td>0.1086</td>
<td>0.2269</td>
<td>-0.2645</td>
<td>0.3722</td>
<td>0.0999</td>
<td>1</td>
</tr>
<tr>
<td>UE</td>
<td>-0.3900</td>
<td>-0.8725</td>
<td>-0.1037</td>
<td>-0.3177</td>
<td>0.6046</td>
<td>-0.1223</td>
<td>0.2606</td>
<td>-0.7833</td>
</tr>
</tbody>
</table>

It can be observed in this table that GDP, IO and UE have a reasonably high correlation with the industrial returns $r_{\text{industrial}}$. Furthermore, the table reveals that some explanatory variables have very high correlation between each other.

6.1.4.2 Multivariate Regression Analysis

The solution of the regression analysis given by (20) is the following:

Table No.10: Industrial Sector; Multivariate Regression Analysis Result 1

<table>
<thead>
<tr>
<th>$\beta_{\text{industrial}}$</th>
<th>Constant</th>
<th>GDP</th>
<th>LIR</th>
<th>SIR</th>
<th>CPI</th>
<th>IMP</th>
<th>EXP</th>
<th>IO</th>
<th>UE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{0.95%}$</td>
<td>0.0072</td>
<td>0.8733</td>
<td>-0.2504</td>
<td>-0.0606</td>
<td>1.8642</td>
<td>-0.4435</td>
<td>0.3829</td>
<td>-1.2028</td>
<td>-0.6447</td>
</tr>
<tr>
<td>$\beta_{0.95%}$</td>
<td>0.1117</td>
<td>4.8957</td>
<td>0.1390</td>
<td>0.1295</td>
<td>2.7660</td>
<td>0.2221</td>
<td>1.1255</td>
<td>0.9317</td>
<td>-0.2923</td>
</tr>
<tr>
<td>$\beta_{0.95%}$</td>
<td>-0.0972</td>
<td>-3.1492</td>
<td>-0.6398</td>
<td>-0.2507</td>
<td>0.9623</td>
<td>-1.1091</td>
<td>-0.3597</td>
<td>-3.3373</td>
<td>-0.9971</td>
</tr>
</tbody>
</table>

The coefficients of determination corresponding to the above regression analysis are:

$R^2 = 0.5515$

$R^2_{\text{adj}} = 0.4234$

By studying Table No.10 one can observe that CPI and UE are the only two variables with significant coefficients at the 95% significance level. Wanting to obtain the highest $R^2_{\text{adj}}$, we eliminate explanatory variables. The highest $R^2_{\text{adj}}$ is received by removing SIR. The below table shows the regression results with SIR removed.
Table No.11: Industrial Sector; Multivariate Regression Analysis Result 2.

<table>
<thead>
<tr>
<th>β_Industrial</th>
<th>Constant</th>
<th>GDP</th>
<th>LIR</th>
<th>CPI</th>
<th>IMP</th>
<th>EXP</th>
<th>IO</th>
<th>UE</th>
</tr>
</thead>
<tbody>
<tr>
<td>β_0.95</td>
<td>0.0157</td>
<td>0.5254</td>
<td>-0.3216</td>
<td>1.8135</td>
<td>-0.4404</td>
<td>0.3828</td>
<td>-1.0291</td>
<td>-0.6285</td>
</tr>
<tr>
<td>β_0.95</td>
<td>0.1155</td>
<td>4.3526</td>
<td>-0.0064</td>
<td>2.6911</td>
<td>0.2175</td>
<td>1.1168</td>
<td>1.011</td>
<td>-0.2839</td>
</tr>
<tr>
<td>β_0.95</td>
<td>-0.0842</td>
<td>-3.3017</td>
<td>-0.6368</td>
<td>0.9359</td>
<td>-1.0983</td>
<td>-0.3513</td>
<td>-3.0691</td>
<td>-0.9732</td>
</tr>
</tbody>
</table>

The coefficients of determination corresponding to the above regression analysis are:

\[ R^2 = 0.5447 \]
\[ R^2_{adj} = 0.4348 \]

By eliminating SIR, the LIR coefficient is now significant at the 95% significant level and the best description of the industrial returns is given by the following function:

\[ r_{industrial} = 0.0157 + 0.5254 \cdot \ln \left( \frac{GDP}{GDP_{t-1}} \right) - 0.3216 \cdot \ln \left( \frac{LIR}{LIR_{t-1}} \right) + 1.8135 \cdot \ln \left( \frac{CPI}{CPI_{t-1}} \right) - 0.4404 \cdot \ln \left( \frac{IMP}{IMP_{t-1}} \right) + ... \\
\ldots + 0.3828 \cdot \ln \left( \frac{EXP}{EXP_{t-1}} \right) - 1.0291 \cdot \ln \left( \frac{IO}{IO_{t-1}} \right) - 0.6285 \cdot \ln \left( \frac{UE}{UE_{t-1}} \right) \]

The result from this regression analysis does not entirely correspond with the hypothesis presented in the earlier section where IMP and IO where predicted positive coefficients. Yet, since the correlations between IMP and EXP, and IO and GDP are close to 1 (see Table No. 9), one can notify the existence of multicollinearity. This prevents the regression analysis to give suitable weights to each explanatory variable.

6.1.5 Prediction

To further analyse the chosen models we use them to predict returns on the basis of known values of the explanatory variables. While it may be considered false to perform prediction on historical data, a deep prediction analysis is beyond the scope of this paper as we are only interested in analysing market data for structure. The following figures show the predicted versus the actual returns (IPD data).
The results are good, as one can see that the predicted returns appear to, more or less, follow the same pattern as the corresponding actual returns. The predictions look as if they work
unexpectedly well on in-sample data, despite the correlations between returns and explanatory variables being low. Due to the regression results, the predicted returns seem to have smaller variance compared with the actual returns.

6.1.6 Discussion

The overall results of the explanatory degree from the regression analysis are around 50%. Considering that no supply indicators are taken into account this is a fairly good number. To better describe return data it would have been relevant to consider real estate supply variables such as vacancy rate and stock. Yet, the lack of country specific historical time series makes it impossible to take those variables under consideration in this analysis. What we have been able to conclude using multivariate regression is that GDP and UE both have a high degree of influence on the overall real estate returns. They also have a high correlation between each other, which will unable us to understand how much each variable influence the real estate returns. However, since the sole purpose of this regression analysis is prediction, the multicollinearity is nothing we have to worry about.

Before this study we hypothesized that both the LIR and the SIR influenced the real estate returns. What we didn’t know was which one of them influenced the most. Through the regression analysis, in terms of correlation, one can see that the SIR is found to be insignificant whereas the LIR has a great influence on the real estate return. One explanation for this can be that we have not considered the effect of time lag. As said in the hypothesis we predicted that SIR would not have an effect on the return until a few years later. The reason for neglecting the time lag is that the in-data frequency is on yearly basis. Time lag between the macroeconomic variables and the real estate returns are usually identified on monthly basis and it has therefore been difficult to draw any relevant conclusions from the time lag results. If a lag dependence between these background variables with the stronger explanatory power could be revealed, the models could be used in an early warning system. This would pave the way for the possibility to make predictions for these background variables in order to forecast the future development of the real estate return.

Last but not least, since real estate markets are persistent a positive relation exists between current return and return in the previous period. To use auto regression in our regression analysis would increase the explanatory level. Yet, since our aim is to create fictitious return series in the Nordic countries as a result of insufficient historical time series, this approach is not applicable in our case.
6.2 Comparison of the different markets

In this section we test the hypothesis that the U.K. and Nordic real estate markets are correlated through our chosen background variables. We use the U.K. regression models computed in the previous section, now with the Nordic macroeconomic variables, as in-data. This is done in order to create longer time series of Nordic return data for the portfolio optimization performed later on.

The figures (Figure No.6, 7, 8, 9, 10, 11, 12, 13 and 14) below show the returns predicted by our models together with the actual IPD data. One should keep in mind two sources of error when obtaining the final result. First, the U.K. models themselves are not perfectly descriptive, but has an explanatory degree of ~50%. Second, the correlations between the explanatory background variables in U.K. and the considered country are not equal to 1, and hence; the U.K. model is not perfectly applicable in the Nordic countries. The second source of error can be further analyzed in the below tables (Table No. 12, 13, 14, 15, 16, 17, 18, 19 and 20).

6.2.1 Sweden

Swedish IPD data covers the period 1984-2008. When comparing the predicted return with this data we have to keep in mind that the supply variables have been left outside the regression analysis. The overall correlation between the predicted returns and the IPD data lies between 0.5 and 0.8. CPI can be identified as the most important explanatory variable for commercial real estate returns in Sweden.
The figure above indicates a fairly good result. The pattern from the predicted returns seems to follow the actual return. A deeper understanding about how adequate the U.K. retail model is on the Swedish retail sector can be obtained from the table below.

<table>
<thead>
<tr>
<th>Retail</th>
<th>Explanatory Weight U.K.</th>
<th>Correlation with U.K.</th>
<th>Explanatory Weight Sweden</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPI</td>
<td>39%</td>
<td>0.82</td>
<td>32%</td>
</tr>
<tr>
<td>PC</td>
<td>16%</td>
<td>0.38</td>
<td>6%</td>
</tr>
<tr>
<td>RS</td>
<td>15%</td>
<td>0.19</td>
<td>3%</td>
</tr>
<tr>
<td>GDP</td>
<td>11%</td>
<td>0.45</td>
<td>5%</td>
</tr>
<tr>
<td>LIR</td>
<td>10%</td>
<td>0.62</td>
<td>6%</td>
</tr>
<tr>
<td>UE</td>
<td>9%</td>
<td>0.28</td>
<td>2%</td>
</tr>
<tr>
<td></td>
<td><strong>100%</strong></td>
<td></td>
<td><strong>54%</strong></td>
</tr>
</tbody>
</table>

The high correlation between the CPI in U.K. and Sweden is the greatest contribution to the good result. Nevertheless, keeping in mind that the U.K. model has an explanatory degree of ~50% and isn’t perfect in itself, the final explanatory degree for the total return in the retail sector in Sweden would be ~27%.

A similar analysis as for the retail sector can be done for the Swedish office sector.
A deeper understanding about how adequate the U.K. retail model is on the Swedish office sector can be obtained from the table below.

<table>
<thead>
<tr>
<th>Office</th>
<th>Explanatory Weight U.K.</th>
<th>Correlation with U.K.</th>
<th>Explanatory Weight Sweden</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDP</td>
<td>52%</td>
<td>0.45</td>
<td>23%</td>
</tr>
<tr>
<td>CPI</td>
<td>33%</td>
<td>0.82</td>
<td>27%</td>
</tr>
<tr>
<td>UE</td>
<td>9%</td>
<td>0.28</td>
<td>2%</td>
</tr>
<tr>
<td>LIR</td>
<td>6%</td>
<td>0.62</td>
<td>4%</td>
</tr>
</tbody>
</table>

100% 56%

Here *GDP* and *CPI*, both with a high explanatory weight and considerably high correlation with the U.K., give the greatest contribution to the return prediction for Swedish office properties. Given the explanatory degree of ~50% for the U.K. model, the final explanatory degree for the total return in the Swedish office sector would be ~28%.

Besides the *GDP* and *CPI*; *IO* and *UE* are important variables when describing industrial return.
The fairly low correlation between UE in U.K. and Sweden makes the contribution to the final explanatory degree low. Instead, the high correlation of IMP and EXP between the countries gives those variables a more important role when describing Swedish industrial return. A deeper understanding about how adequate the U.K. industrial model is on the Swedish industrial sector can be obtained from the table below.

<table>
<thead>
<tr>
<th>Industrial</th>
<th>Explanatory Weight U.K.</th>
<th>Correlation with U.K.</th>
<th>Explanatory Weight Sweden</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPI</td>
<td>35%</td>
<td>0.82</td>
<td>29%</td>
</tr>
<tr>
<td>IO</td>
<td>20%</td>
<td>0.36</td>
<td>7%</td>
</tr>
<tr>
<td>UE</td>
<td>12%</td>
<td>0.28</td>
<td>3%</td>
</tr>
<tr>
<td>GDP</td>
<td>11%</td>
<td>0.45</td>
<td>5%</td>
</tr>
<tr>
<td>IMP</td>
<td>9%</td>
<td>0.74</td>
<td>6%</td>
</tr>
<tr>
<td>EXP</td>
<td>7%</td>
<td>0.59</td>
<td>4%</td>
</tr>
<tr>
<td>LIR</td>
<td>6%</td>
<td>0.62</td>
<td>4%</td>
</tr>
<tr>
<td></td>
<td><strong>100%</strong></td>
<td><strong>58%</strong></td>
<td></td>
</tr>
</tbody>
</table>

Given the explanatory degree of ~50% for the U.K. model, the final explanatory degree for the return in the Swedish industrial sector would be ~29%. 

Figure No.8: Swedish Industrial Sector; Actual vs. Predicted Return
6.2.2 Denmark

Danish IPD data covers the period 2000-2008.

Figure No.9: Danish Retail Sector; Actual vs. Predicted Return

One can not draw any reasonable conclusion from the figure above due to the lack of IPD data. Some shifting between the time series can be noticed. The correlation between the actual returns and the predicted returns is 0.0395. By adding a one-year time lag, the correlation increases to 0.8212. To obtain a wider and deeper understanding about how sufficient the U.K. retail model is on the Danish retail sector, one has to also consider the result presented in the table below.

Table No.15: U.K. model on Danish Retail Sector.

<table>
<thead>
<tr>
<th>Retail</th>
<th>Explanatory Weight U.K.</th>
<th>Correlation with U.K.</th>
<th>Explanatory Weight Denmark</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPI</td>
<td>39%</td>
<td>0.85</td>
<td>33%</td>
</tr>
<tr>
<td>PC</td>
<td>16%</td>
<td>-0.06</td>
<td>-1%</td>
</tr>
<tr>
<td>RS</td>
<td>15%</td>
<td>-0.03</td>
<td>0%</td>
</tr>
<tr>
<td>GDP</td>
<td>11%</td>
<td>0.56</td>
<td>6%</td>
</tr>
<tr>
<td>LIR</td>
<td>10%</td>
<td>0.69</td>
<td>7%</td>
</tr>
<tr>
<td>UE</td>
<td>9%</td>
<td>0.37</td>
<td>3%</td>
</tr>
<tr>
<td></td>
<td><strong>100%</strong></td>
<td></td>
<td><strong>48%</strong></td>
</tr>
</tbody>
</table>

Since the CPI is the variable that has most influence on the U.K. retail return and the correlation between CPI in U.K. and Denmark is high one can fairly state that CPI is the most important explanatory variable when applying the U.K. retail model on the Danish retail
sector. The final explanatory degree for the total return on the Danish retail sector, considering the explanatory degree of the U.K. model, is \( \sim 24\% \).

The shifting between the actual and predicted returns observed in the retail sector can also be noticed in the office sector.

*Figure No.10: Danish Office Sector; Actual vs. Predicted Return*

The correlation between the actual and the predicted returns is 0.5943. By adding a one-year time lag, the correlation increases to 0.8091. To obtain a deeper understanding about how sufficient the U.K. office model is on the Danish office sector, one has to also consider the result presented in the table below.

*Table No.16: U.K. model on Danish Office Sector.*

<table>
<thead>
<tr>
<th>Office</th>
<th>Explanatory Weight U.K.</th>
<th>Correlation with U.K.</th>
<th>Explanatory Weight Denmark</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDP</td>
<td>52%</td>
<td>0.56</td>
<td>29%</td>
</tr>
<tr>
<td>CPI</td>
<td>33%</td>
<td>0.85</td>
<td>28%</td>
</tr>
<tr>
<td>UE</td>
<td>9%</td>
<td>0.37</td>
<td>3%</td>
</tr>
<tr>
<td>LIR</td>
<td>6%</td>
<td>0.69</td>
<td>4%</td>
</tr>
<tr>
<td></td>
<td><strong>100%</strong></td>
<td></td>
<td><strong>64%</strong></td>
</tr>
</tbody>
</table>

*\( CPI \) and *GDP* are the explanatory variables that make the U.K. office model relatively suitable on the Danish office sector. They have high explanatory weight in the U.K. office model and their correlation between U.K. and Denmark, are relatively high. Given the*
explanatory degree of ~50% for the U.K. office model, the final explanatory degree for the return in Denmark is ~32%.

The industrial sector shows the same patterns as the retail and office sectors.

*Figure No.11: Danish Industrial Sector; Actual vs. Predicted Return*

The correlation between the actual and the predicted returns is -0.2730. By adding a one-year time lag, the correlation increases to 0.9164. To better understand the sufficiency of the U.K. industrial model on the Danish industrial sector, one has to consider the table below.

*Table No.17: U.K. model on Danish Industrial Sector.*

<table>
<thead>
<tr>
<th>Industrial</th>
<th>Explanatory Weight U.K.</th>
<th>Correlation with U.K.</th>
<th>Explanatory Weight Denmark</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPI</td>
<td>35%</td>
<td>0.85</td>
<td>30%</td>
</tr>
<tr>
<td>IO</td>
<td>20%</td>
<td>0.35</td>
<td>7%</td>
</tr>
<tr>
<td>UE</td>
<td>12%</td>
<td>0.37</td>
<td>4%</td>
</tr>
<tr>
<td>GDP</td>
<td>10%</td>
<td>0.56</td>
<td>6%</td>
</tr>
<tr>
<td>IMP</td>
<td>9%</td>
<td>0.69</td>
<td>6%</td>
</tr>
<tr>
<td>EXP</td>
<td>8%</td>
<td>0.63</td>
<td>5%</td>
</tr>
<tr>
<td>LIR</td>
<td>6%</td>
<td>0.69</td>
<td>4%</td>
</tr>
</tbody>
</table>

Since the CPI is the variable that has most influence on the U.K. industrial return and the correlation between CPI in U.K. and Denmark is high one can fairly state that CPI is the most important explanatory variable when applying the U.K. model on the Danish industrial sector. Keeping in mind that the U.K. model has an explanatory degree of ~50% and isn’t perfect in
it self, the final explanatory degree for the total return in the industrial sector in Denmark is ~31%.

6.2.3 Finland

Finnish IPD data covers the period 1998-2008. As in Denmark, the short IPD data series makes it hard to interpret the result in the figure below. The correlation between the predicted returns and the actual ones is -0.0896 for the retail sector, 0.7336 for the office sector and 0.0692 for the industrial sector. No big difference occurs on the correlation front when considering time-lag.

Figure No.12: Finnish Retail Sector; Actual vs. Predicted Return

No relevant pattern can be seen when comparing the actual IPD data with our predicted returns. Yet, as seen in the table below, a high explanatory degree is obtained when the U.K. retail model is used for Finland.

Table No.18: U.K. model on Finnish Retail Sector.

<table>
<thead>
<tr>
<th>Retail</th>
<th>Explanatory Weight U.K.</th>
<th>Correlation with U.K.</th>
<th>Explanatory Weight Finnish</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPI</td>
<td>39%</td>
<td>0.93</td>
<td>36%</td>
</tr>
<tr>
<td>PC</td>
<td>16%</td>
<td>0.40</td>
<td>6%</td>
</tr>
<tr>
<td>RS</td>
<td>15%</td>
<td>0.24</td>
<td>5%</td>
</tr>
<tr>
<td>GDP</td>
<td>11%</td>
<td>0.54</td>
<td>6%</td>
</tr>
<tr>
<td>LIR</td>
<td>10%</td>
<td>0.56</td>
<td>6%</td>
</tr>
<tr>
<td>UE</td>
<td>9%</td>
<td>0.33</td>
<td>3%</td>
</tr>
</tbody>
</table>

|           | 100%                    | 62%                   |
Besides the *CPI*, the *PC*, *GDP* and *LIR* have high explanatory weights when describing the Finnish retail sector return. Keeping in mind the explanatory degree of ~50% for the U.K. retail model a final explanatory degree for the model describing Finnish retail total return is ~31%.

The office sector model seems to describe the actual return better than the retail model, yet marginally. Also, the short time series makes a conclusion irrelevant.

*Figure No.13: Finnish Office Sector; Actual vs. Predicted Return*

To better understand how good the U.K. office model describes Finish office return, one can study the table below.

*Table No.19: U.K. model on Finnish Office Sector.*

<table>
<thead>
<tr>
<th>Office</th>
<th>Explanatory Weight U.K.</th>
<th>Correlation with U.K.</th>
<th>Explanatory Weight Sweden</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDP</td>
<td>52%</td>
<td>0.54</td>
<td>28%</td>
</tr>
<tr>
<td>CPI</td>
<td>33%</td>
<td>0.93</td>
<td>31%</td>
</tr>
<tr>
<td>UE</td>
<td>9%</td>
<td>0.33</td>
<td>3%</td>
</tr>
<tr>
<td>LIR</td>
<td>6%</td>
<td>0.56</td>
<td>4%</td>
</tr>
</tbody>
</table>

The office sector is as seen best described by *GDP* and *CPI* which are, especially *CPI*, highly correlated to the same in U.K., the variables also have a great importance in the U.K. model. The final explanatory degree for the total return on the Finnish office sector, considering the explanatory degree of the U.K. model, is ~33%.
The last figure, *Figure No. 14*, describes the actual versus the predicted return in the Finnish industrial sector.

*Figure No.14: Finnish Industrial Sector; Actual vs. Predicted Return*

To further analyze the sufficiency of the U.K. industrial model in describing Finnish office sector return, the table below can be studied.

*Table No.20: U.K. model on Finnish Industrial Sector.*

<table>
<thead>
<tr>
<th>Industrial</th>
<th>Explanatory Weight U.K.</th>
<th>Correlation with U.K.</th>
<th>Explanatory Weight Sweden</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPI</td>
<td>35%</td>
<td>0.93</td>
<td>33%</td>
</tr>
<tr>
<td>IO</td>
<td>20%</td>
<td>0.39</td>
<td>8%</td>
</tr>
<tr>
<td>UE</td>
<td>12%</td>
<td>0.33</td>
<td>4%</td>
</tr>
<tr>
<td>GDP</td>
<td>11%</td>
<td>0.54</td>
<td>7%</td>
</tr>
<tr>
<td>IMP</td>
<td>9%</td>
<td>0.33</td>
<td>3%</td>
</tr>
<tr>
<td>EXP</td>
<td>7%</td>
<td>0.49</td>
<td>4%</td>
</tr>
<tr>
<td>LIR</td>
<td>6%</td>
<td>0.56</td>
<td>3%</td>
</tr>
<tr>
<td><strong>100%</strong></td>
<td></td>
<td><strong>62%</strong></td>
<td></td>
</tr>
</tbody>
</table>

IO, as the second most explanatory variable for Finnish industrial return, is given a high weight in the U.K. model. Yet, the high correlations between GDP in the countries make it almost as strong as the IO in describing Finnish industrial return. Given the explanatory degree of ~50% for the U.K. industrial model, the final explanatory degree for the total return in Finland is ~31%.
6.3 Optimal portfolio allocation

Stoesser and Hess (2000) have concluded that skillful portfolio managers can add value (on a risk-adjusted basis) in the private real estate market through asset selection and investment timing strategies. That many market participants believe real estate portfolio managers can add value is evidenced by the dramatic growth of the commercial investment management industry since the 1970s and by the recent proliferation of investment funds and management strategies offered by the industry. Typically, the excess returns expected to be produced by these strategies are thought to arise from the manager’s ability to successfully “target” geographical markets and/or property types [8].

To make sure that the results we obtain from the portfolio optimization represent real life conditions, we made some adjustments on the optimization problem, (16), described in Chapter 4. In our case the optimization problem has a different return condition. The reason for this is that in real life, an investor is not interested in knowing what the risk is for a certain specific return. The return target should instead be the lower limit, for which the expected return for a certain risk cannot fall below.

\[
\begin{align*}
\min & \quad \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\
\text{s.t.} & \quad \sum_{i=1}^{n} w_i r_i \geq r_{\text{target}} \\
& \quad \sum_{i=1}^{n} w_i = 1 \\
& \quad w_i \geq 0 \quad i = 1, \ldots, n
\end{align*}
\]

The fictitious data series of the three commercial real estate sectors in Sweden, Denmark and Finland have been used to calculate correlations between the different real estate markets and to estimate returns and standard deviations.

<table>
<thead>
<tr>
<th>Table No.21: Expected Return (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Retail</td>
</tr>
<tr>
<td>Office</td>
</tr>
<tr>
<td>Industrial</td>
</tr>
<tr>
<td>Average</td>
</tr>
</tbody>
</table>
As seen the office sector in Finland has the highest expected return whereas the same in Denmark has the lowest. Overall across the countries Sweden has the highest expected return and across the sectors industrial properties are the winners.

<table>
<thead>
<tr>
<th></th>
<th>Sweden</th>
<th>Denmark</th>
<th>Finland</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Retail</td>
<td>8.25</td>
<td>12.70</td>
<td>9.94</td>
<td>10.30</td>
</tr>
<tr>
<td>Office</td>
<td>9.36</td>
<td>12.10</td>
<td>12.81</td>
<td>11.42</td>
</tr>
<tr>
<td>Industrial</td>
<td>12.42</td>
<td>16.90</td>
<td>12.49</td>
<td>13.94</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>10.01</td>
<td>13.90</td>
<td>11.75</td>
<td></td>
</tr>
</tbody>
</table>

The industrial sector in Denmark has the highest risk whereas the retail sector in Sweden has the lowest. Overall across the countries the Danish market is the most volatile and across the sectors industrial properties are most risky. The above tables can be summarized in the figure below.

If an investor were to strive to reach the highest possible return he would invest all his money on the Finnish office sector. Yet, this action would expose him to a great amount of risk since the corresponding standard deviation is very high. The best single asset investment strategy for a risk averse investor, is to bet all the money on the Swedish retail sector, since it has the lowest standard deviation among the ones represented in Table No. 22.
A single asset investment is of great importance in the cases when the correlations between different assets are close to perfect positively correlated. In order to see how the investment objects coexist the table below shows the correlation matrix for all the asset types.

<table>
<thead>
<tr>
<th></th>
<th>Sweden</th>
<th></th>
<th>Denmark</th>
<th></th>
<th>Finland</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Retail</td>
<td>Office</td>
<td>Industrial</td>
<td>Retail</td>
<td>Office</td>
<td>Industrial</td>
</tr>
<tr>
<td>Sweden</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Retail</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Office</td>
<td>0.8972</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Industrial</td>
<td>0.9125</td>
<td>0.8636</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Denmark</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Retail</td>
<td>0.0707</td>
<td>0.0680</td>
<td>0.0263</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Office</td>
<td>0.0566</td>
<td>0.1080</td>
<td>0.0163</td>
<td>0.8969</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Industrial</td>
<td>-0.0216</td>
<td>0.0001</td>
<td>0.0017</td>
<td>0.9240</td>
<td>0.8906</td>
<td>1</td>
</tr>
<tr>
<td>Finland</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Retail</td>
<td>0.6829</td>
<td>0.6795</td>
<td>0.5399</td>
<td>-0.022</td>
<td>-0.1182</td>
<td>-0.1113</td>
</tr>
<tr>
<td>Office</td>
<td>0.6452</td>
<td>0.7595</td>
<td>0.5502</td>
<td>0.0300</td>
<td>0.0138</td>
<td>-0.0165</td>
</tr>
<tr>
<td>Industrial</td>
<td>0.7213</td>
<td>0.7367</td>
<td>0.6507</td>
<td>-0.1468</td>
<td>-0.2333</td>
<td>-0.2158</td>
</tr>
</tbody>
</table>

Since there exist correlations that are close to zero or even negative, the result presented in Table No. 23 give us an indication that a single asset investment is not always of current interest when seeking to obtain the lowest possible standard deviation. As seen in the table the highest correlations exist between the different sectors within each country, i.e. it would not be the wisest choice to diversify among these specific assets if striving for a risk reduction. By instead investing in different countries, having low or negative correlation, diversification would better reduce the portfolio risk. Within the choice of country combination one can see that between Sweden and Finland the correlation is much higher than between Denmark and the two. Hence, a greater risk reduction would be obtained if combining Denmark with Sweden or Finland instead of combining only Sweden and Finland.

### 6.3.1 Nordic Allocation

Below one can see the solutions to the optimization problem, the frontier. The efficient frontier, the curve above the minimum-variance point (a standard deviation of 6.71%), presents the optimal portfolios. As seen, the efficient frontier stretches from a return of 6.72% (corresponding to the minimum-variance point) to 8.54% (everything invested in the Finnish office sector-the highest possible return).
The minimum standard deviations corresponding to some returns can be seen in the table below.

<table>
<thead>
<tr>
<th>Target Return (%)</th>
<th>6.72</th>
<th>7.00</th>
<th>7.50</th>
<th>8.00</th>
<th>8.50</th>
<th>8.54</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Deviation (%)</td>
<td>6.71</td>
<td>7.06</td>
<td>8.15</td>
<td>9.47</td>
<td>12.32</td>
<td>12.76</td>
</tr>
</tbody>
</table>

It can be seen on the efficient frontier that the minimum standard deviation remains constant for a small range of target return requirements. The reason for this is that the optimal solution, which is 32% in Danish office sector, 42% in Swedish retail sector and 26% in Finnish retail sector, will always generate the lowest yearly standard deviation for any given target return up to 6.72%; the minimum-variance point. This in turn can be explained by that our return condition is an inequality, which will leave out the bullet shaped frontier. The highest expected return, corresponding to the lowest risk (6.71%), is 6.72%. All target returns below this value will always generate the same standard deviation.

The optimal allocations for a set of target returns are given below.
According to our results, when looking to invest in the Nordic market, a risk averse investor would choose to invest in the Danish office sector together with the retail sector in Sweden and Finland. This is the case since that allocation represents the portfolio with the lowest standard deviation. When looking towards a higher portfolio return, up to around 8%, a more diversified allocation is required and also industrial properties in Denmark and Sweden as well as offices in Finland are included in the portfolio. The most risk preferring investor, seeking to obtain returns over 8.5%, would “put all his eggs” in Finnish office properties.

### 6.3.2 Swedish Allocation

When allowing investments only in Sweden the return limits stretches from 6.56% (corresponding to the minimum-variance point) to 7.42% (everything invested in the industrial sector).
The minimum standard deviations corresponding to some returns can be seen in the table below.

<table>
<thead>
<tr>
<th>Target Return (%)</th>
<th>6.56</th>
<th>6.80</th>
<th>7.00</th>
<th>7.20</th>
<th>7.40</th>
<th>7.42</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Deviation (%)</td>
<td>8.25</td>
<td>9.17</td>
<td>10.10</td>
<td>11.13</td>
<td>12.28</td>
<td>12.42</td>
</tr>
</tbody>
</table>

As seen the lowest reachable risk is a standard deviation of 8.25% corresponding to a maximum return of 6.56%. Yet a risk preferring investor can reach a maximum yearly return of 7.42% if accepting a standard deviation of 12.42%.

The optimal allocations for a set of target returns are given below.
According to our results, when looking to invest only in the Swedish market, a risk averse investor would choose to invest all his money in the retail sector. This is the case since it represents the highest return (6.56%) for the lowest possible risk (8.25%). When looking towards a higher portfolio return a more diversified allocation is required and also industrial and office properties are included in the portfolio. The most risk preferring investor looking for the highest possible return portfolio would “put all his eggs” in the industrial sector.

### 6.3.3 Danish Allocation

When allowing investments only in Denmark the return limits stretches from 6.48% (corresponding to the minimum-variance point) to 7.41% (everything invested in the industrial sector).
The minimum standard deviation corresponding to some returns can be seen in the table below.

<table>
<thead>
<tr>
<th>Target Return (%)</th>
<th>6.48</th>
<th>6.80</th>
<th>7.00</th>
<th>7.20</th>
<th>7.40</th>
<th>7.41</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Deviation (%)</td>
<td>12.00</td>
<td>13.41</td>
<td>14.47</td>
<td>15.61</td>
<td>16.83</td>
<td>16.90</td>
</tr>
</tbody>
</table>

As seen, the lowest reachable risk is a standard deviation of 12.00% corresponding to a return of 6.48%. Yet a risk preferring investor can reach a yearly return of 7.41%, but then has to accept a standard deviation of 16.90%.

The optimal allocations for a set of target returns are given below.
According to our results, when looking to invest only in the Danish market, a risk averse investor would choose to split his investment on offices and retail properties. This since it represents the highest return (6.48%) for the lowest possible risk (12.00%). When looking towards a higher portfolio return a more diversified allocation is required and also industrial properties are included in the portfolio. The most risk preferring investor looking for the highest possible return portfolio would “put all his eggs” in the industrial sector.

### 6.3.4 Finnish Allocation

When allowing investments only in Finland the return limits stretches from 7.33% (corresponding to the minimum-variance point) to 8.54% (everything invested in the office sector).
The minimum standard deviation corresponding to some returns can be seen in the table below.

Table No.27: Finland; Target Return vs. Standard Deviation

<table>
<thead>
<tr>
<th>Target Return (%)</th>
<th>7.33</th>
<th>7.40</th>
<th>7.80</th>
<th>8.20</th>
<th>8.40</th>
<th>8.54</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Deviation (%)</td>
<td>9.94</td>
<td>10.50</td>
<td>10.79</td>
<td>11.79</td>
<td>12.37</td>
<td>12.81</td>
</tr>
</tbody>
</table>

As seen, the lowest reachable risk is a standard deviation of 9.94% corresponding to a return of 7.33%. Yet a risk preferring investor can reach a yearly return of 8.54% if accepting a standard deviation of 12.81%.

The optimal allocations for a set of target returns are given below.
According to our results, when looking to invest only in the Finnish market, a risk averse investor would choose to invest all his money in the retail sector. This since it represents the highest return (7.33%) for the lowest possible risk (9.94%). When looking towards a higher portfolio return a more diversified allocation is required and also offices are included in the portfolio. The most risk preferring investor looking for the highest possible return portfolio would “put all his eggs” in the office sector. It can be noticed that industrial properties are not included in the portfolio. Logically as a result of a low return combined with a high standard deviation, see Table No. 21 and Table No. 22.

6.3.5 Discussion

When analysing the fictitious data series through expected return, their standard deviation and their correlation it can be seen that when having the whole Nordic market as an investment playground, one can, for the same return requirement, reach a lower risk if investing in different countries and sectors, than by putting all money in the least volatile asset alone. For example one could think that the lowest risk possible would be to put all money in the Swedish retail sector since it has the lowest standard deviation, 8.25%. Yet, by adding office properties in Denmark and retail properties in Finland (low and negatively correlated with the Swedish retail market) a standard deviation of 6.71% can be obtained. Hence, by analysing the correlation matrix one can draw the conclusion that a risk reduction of the portfolio is best reached through international diversification.
When only allowing investments in one country the lowest reachable risk in Sweden and Finland equals “putting all eggs in one basket”, the retail sector. The high sector correlations within the countries explain this. Yet, when requiring higher returns, diversification is beneficial up to a certain limit. The reason for that the optimal allocation becomes less diversified along a higher required target return stream is that most expected asset returns are too low. The assets that have low risk have expected returns that are too low to meet the investors return requirement for the higher risk he is exposed to. As long as the investor is not able to short-sell, the highest possible return he can obtain is equal to the highest expected return for an individual asset, i.e. he will be forced to invest all his money in the same asset and not be able to diversify. This in turn will expose him to a higher standard deviation, but this is also the beauty of MPT.
7 Conclusion and Future Work

What kind of insights has this modelling process provided us? To what level is our model valid and how may it be criticized? These questions are answered in this last part of the report together with suggestions of further research in the area.

7.1 Conclusion

Analysis about how returns and risk are related to each other when making real estate investments require great amounts of observations of high quality and long time series. Since this hasn’t been the case in this thesis our conclusion is only based on the results from our models and should not be interpreted as any reality. As said our work should be used as a methodology giving broad indications on how to think when creating an optimal commercial real estate portfolio based on MPT instead of as exact numbers on how to allocate capital.

The final optimal allocations are results of a statistical analysis on fictitious Nordic return data created from a function describing U.K real estate returns. According to the results from the regression analysis the U.K. real estate return, independent on which sector, is highly correlated with the GDP and UE (more than with any other macroeconomic variables). The functions that best describes the three sector returns has an explanatory degree of around 50% which is fairly good since no supply variables are taken into account. The retail sector return is regarding to the regression best described by the growth in RS and PC whereas the office sector’s main affecting variables are the growth in GDP and CPI. The industrial sector return has growth in inflation and industrial output as it’s most affecting background variables.

The correlations between the actual IPD data and the predicted returns in the Nordic real estate markets differ between the countries. In order to see how good our models describe the reality in the Nordic we use the U.K. regression models computed with the Nordic background variables, as input.

In Sweden, the country with the longest IPD data, the result is reasonably good; the predicted returns follow the same pattern as the IPD data. Consequently the U.K. and the Swedish real estate markets are rather correlated through our chosen background variables. The IPD data in Denmark and Finland are too short to draw any relevant conclusion from the comparison of the actual data with the longer fictitious time-series. Yet, it can be noticed that the predicted
returns in Denmark are shifted backwards comparing to the actual returns and the correlation between the predicted and the actual IPD returns with one-year time lag are around 0.85. In Finland the only conclusion one can draw, with such short IPD data available, is that the predicted returns do not follow the same pattern as the actual returns.

When analysing the fictitious data it can be stated that having the whole Nordic market available for investments is a great advantage against a country specific investment. The figure below, Figure No. 24, clearly addresses the benefit of risk reduction through international diversification. As known every point, below an efficient frontier are suboptimal, and hence, a rational investor will hold a portfolio only on the frontier. Since the curves corresponding to the different countries are below the Nordic efficient frontier one can state that the most favourable allocation can never be obtained by only investing in one of the countries.

The figure tells us, spontaneously, to not choose the Danish market for investments. Yet, as a market with returns very low or negatively correlated with Swedish and Finish returns, it is a good ingredient in a portfolio but not as a single market investment.

*Figure No.24: Efficient Frontiers*

![Efficient Frontiers](image)

To come back to reality from our fictitious world one should be reminded that cross-country correlation is increasing, due perhaps to the growing interdependence among the international markets. Hence, benefits of international portfolio diversification may be overstated.
Recently, the advantage of country diversification relative to sector diversification has been questioned especially against the background of the European monetary and financial integration. Correct estimates of the correlation matrix are central for the evaluation of the relative diversification gains. These estimates should take into account the time-varying and asymmetric behaviour of the correlation process particularly in the context of major changes in volatility and market trends.

Lastly, it should not be forgotten that history is no mirror to the future and it is worth noting that the results in this thesis are based on historical returns. Therefore, the optimal portfolio weights derived here may not be applicable to future asset allocations if the underlying economic conditions have changed. However, this study has at least demonstrated the benefits of diversification in a commercial real estate property portfolio. Furthermore, the approaches developed are certainly useful for portfolio managers in solving their asset allocation problems.

### 7.2 Future Work

Real estate is becoming more important as an exotic asset type in a portfolio. The future will come up with much more research concerning different areas around how econometric analysis and MPT can be used as tools to analyse the real estate market.

In our broad modelling process with a certain time limitation, we had to leave out many import aspects. Some of them are presented below.

An improvement of the model presented in this thesis would be to further understand how the macroeconomic variables, specifically, affect the total return. This can be done by computing the regression analysis on both the change in income return and in yield, instead of doing it only on the total return. In this way one can, in more detail, understand which part of the total return each background variable influence and how.

In order to accomplish more correct results regarding the background variables affecting the commercial real estate return greater focus should be on the regression analysis itself. Here it would be appropriate to eliminate the multicollinearity from the background variables so that the regression could determine on which variable to place the coefficient weight that “rightfully” belongs entirely to that variable. In this approach auto regression could also be
included as a background variable. By using the previous years return when describing the current year return, one would probably increase the model’s explanatory degree.

Previous studies show that the convergence of the European commercial real estate markets very much depends on the time period covered, i.e. convergence is time-varying. Hence, a more correct model would be achieved if dividing the analysis, in time, and by looking at the correlation in certain periods; crises times, normal times etc. Also a time-lag analysis would be of great interest. Since our time-series were too short, we were not able to compute any deeper analysis regarding time-lag existence. Yet, having that possibility one may reveal a stronger explanatory power and the model could be used in an early warning system.
8 References

This part contains a list of references as well as a list of the interviewees.

8.1 Written Sources


[7] Leimdörfer, (June 2009), “Approaching zero- how do record low interest rates affect the property market?”


### 8.2 Interviews


9 Appendix

Mean return of a portfolio with n assets
Suppose that the n assets have random rates of return \( r_1, r_2, \ldots, r_n \). These have expected values \( E(r_1) = \bar{r}_1, E(r_2) = \bar{r}_2, \ldots, E(r_n) = \bar{r}_n \).

Suppose that the n assets with their weights \( w_1, w_2, \ldots, w_n \) shape the portfolio. The rate of return of the portfolio in terms of the return of the individual returns is

\[
    r = w_1 r_1 + w_2 r_2 + \ldots + w_n r_n.
\]

Owing to the linearity of the expected value, the expected rate of return of the portfolio can be expressed as follows;

\[
    E(r) = w_1 E(r_1) + w_2 E(r_2) + \ldots + w_n E(r_n) = w_1 \bar{r}_1 + w_2 \bar{r}_2 + \ldots + w_n \bar{r}_n = \sum_{i=1}^{n} w_i \bar{r}_i.
\]

The expected rate of return of the portfolio is found by taking the weighted sum of the individual expected rates of return.

Variance of a portfolio with n assets
Suppose that the n assets have the variances \( \sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2 \). The variance of the return of the portfolio is denoted by \( \sigma^2 \) and the covariance of the return of asset \( i \) with \( j \) is symbolized by \( \sigma_{ij} \).

\[
    \sigma^2 = E \left[ \left( r - \bar{r} \right)^2 \right] = E \left[ \left( \sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \bar{r}_i \right)^2 \right] =
\]

\[
    = E \left[ \left( \sum_{i=1}^{n} w_i (r_i - \bar{r}_i) \right) \left( \sum_{j=1}^{n} w_j (r_j - \bar{r}_j) \right) \right] =
\]

\[
    = E \left[ \sum_{i,j=1}^{n} w_i w_j (r_i - \bar{r}_i)(r_j - \bar{r}_j) \right] = \sum_{i,j=1}^{n} w_i w_j E \left[ (r_i - \bar{r}_i)(r_j - \bar{r}_j) \right] =
\]

\[
    = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}.
\]