Lecture 2: Splitting of separatrices

Master Class

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The case of one and a half degrees of freedom: the Melnikov method

Let us consider a Hamiltonian with $1 + \frac{1}{2}$ degrees of freedom with $2\pi-$periodic time dependence:

$$H(p, q, t; \mu) = H_0(p, q) + \mu H_1(p, q, t; \mu),$$

where $H_0(p, q) = P(p, q)$ is a pendulum: $P(p, q) = \frac{1}{2}p^2 + V(q)$.
Associated differential equations:

$$\dot{x} = f(x, t; \mu) = J\nabla H(x, t; \mu) = f_0(x) + \mu f_1(x, t; \mu), \quad x = (q, p), t \in \mathbb{T}$$

with:

$$f_0(x) = J\nabla H_0(x), \quad f_1(x, t; \mu) = J\nabla H_1(x, t; \mu), \quad x = (q, p), t \in \mathbb{T}$$

and denote by $\Phi(t; \theta_0, x_0; \mu)$ the general solution such that $\Phi(\theta_0; \theta_0, x_0; \mu) = x_0$.

The unperturbed system

Observe that, for $\mu = 0$, we have: $\Phi(t; \theta_0, x_0; 0) = \varphi(t - \theta_0, x_0)$, where $\varphi(t, x)$ is the flow of

$$\dot{x} = f_0(x) = J\nabla H_0(x), \quad \text{such that} \quad \varphi(0, x) = x.$$

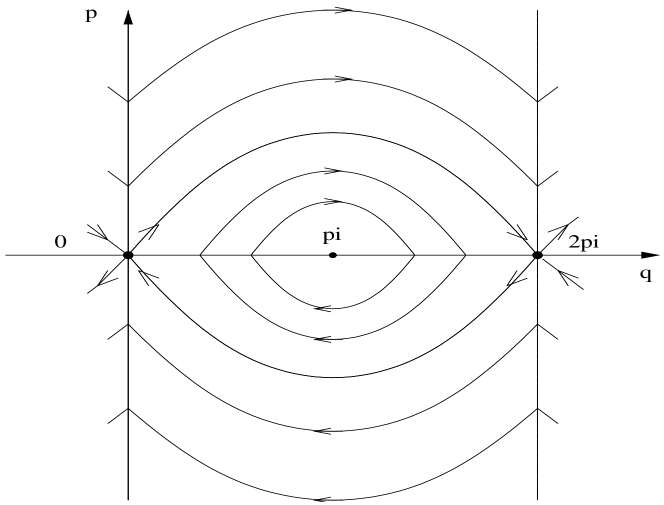
Assumptions:

- $H_0(p, q) = P(p, q)$ is a pendulum: $P(p, q) = \frac{1}{2}p^2 + V(q)$, with $V(q)$ $2\pi-$periodic with a unique non-degenerate maximum, say at $q = 0$ and take, for instance, $V(0) = 0$.
  Therefore, $x^* = (0, 0)$ is and equilibrium of saddle type of $\dot{x} = f_0(x)$, that is, the eigenvalues of $Df_0(x^*)$ are $\lambda_1 = \lambda < 0$ and $\lambda_2 = -\lambda > 0$.
- One branch of the stable and unstable manifolds of x^* coincide along a separatrix Γ included in $P^{-1}(0) = \{(q, p), \ \frac{p^2}{2} + V(q) = 0\}$.
- $f_1(x, t + 2\pi) = f_1(x, t)$

We want to study what happens with the critical point x^* and its stable and unstable manifolds for $\mu > 0$ small.

Unperturbed system

The dynamics of $\dot{x} = f_0(x),\ x \in \mathbb{R}^2$



- We have a critical point at x^* with a homoclinic orbit Γ.
- $x_h(t)$ is a parameterizaton of the homoclinic orbit Γ such that: $\dot{x}_h(t) = f_0(x_h(t))$ and $x_h(t) \to x^*$ as $t \to \pm\infty$.
- This gives us a parameterizaton of the homoclinic manifold (curve)

$$\Gamma = \{x = x_h(v), v \in \mathbb{R}\} \subset W^u(x^*) \cap W^s(x^*)$$

which satisfies: $\varphi(t, x_h(v)) = x_h(v + t)$ (because for $\varepsilon = 0$ the system is autonomous).

The Poincaré (stroboscopic) map

Recall: a way to study a non-autonomous periodic differential equation is to consider the global section

$$\Sigma_\theta = \{(x,\theta),\ x \in \mathbb{R}^2\}$$

and the Poincaré map (identifying $\theta \simeq \theta + 2\pi$):
$\mathcal{P}_{\theta,\mu} : \Sigma_\theta \to \Sigma_\theta$ given by

$$\mathcal{P}_{\theta,\mu}(x) = \Phi(\theta + 2\pi; \theta, x; \mu)$$

$\Phi(t,\theta,x;\mu)$ is the solution of the system such that $\Phi(\theta,\theta,x;\mu) = x$

Unperturbed system: the Poincaré map $\mu = 0$

Let's denote $\mathcal{P}_{\theta,0} := \mathcal{P}_\theta$, we have

- $\mathcal{P}_\theta(x) = \Phi(\theta + 2\pi, \theta, x; 0) = \varphi(2\pi, x)$
- x^* is a fixed point of the Poincaré map $\mathcal{P}_\theta$ for any θ, because it is a critical point of the vector field:
$$\dot{x}(t) = f_0(x(t)), \tag{1}$$
Therefore $\varphi(t, x^*) = x^*$, $\forall t$, and $\mathcal{P}_\theta(x^*) = \varphi(2\pi, x^*) = x^*$.
- Moreover:
$$D\mathcal{P}_\theta(x^*) = D_x\varphi(2\pi, x^*)$$
As $\varphi(t, x)$ is the solution of the equation (1) satifying $\varphi(0, x) = x$, $D_x\varphi(t, x^*)$ is a fundamental solution of the variational equations:
$$z' = Df_0(x^*)z, \ z(0) = \mathrm{Id}$$
Therefore $D_x\varphi(t, x^*) = e^{Df_0(x^*)t}$ and consequently:
$$D\mathcal{P}_\theta(x^*) = e^{Df_0(x^*)2\pi}$$

Unperturbed system: the Poincaré map

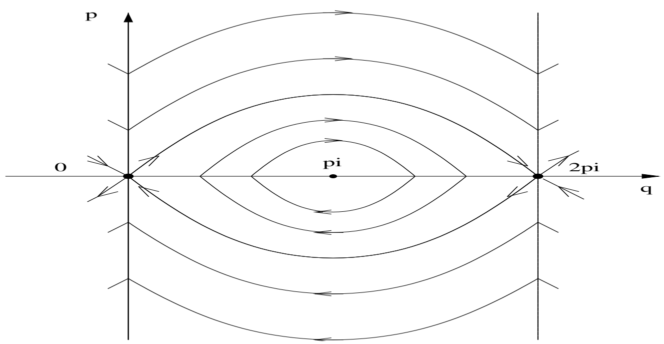
In conclusion we have seen that, for $\mu = 0$:

1. $\mathcal{P}_\theta(x^*) = x^*$
2. $D\mathcal{P}_\theta(x^*) = e^{Df_0(x^*)2\pi}$
3. As the eigenvalues of $Df_0(x^*)$ are $\lambda < 0 < -\lambda$, the eigenvalues of $D\mathcal{P}_\theta(x^*)$ are $e^{2\pi\lambda} < 1 < e^{-2\pi\lambda}$

Therefore, for $\mu = 0$, x^* is a hyperbolic fixed point of saddle type of the Poincaré map $\mathcal{P}_\theta$ for any θ and has one dimensional stable and unstable manifolds.

Unperturbed system: the Poincaré map

For $\mu = 0$, the dynamics of the Poincaré map $\mathcal{P}_\theta$ is "the same" as the flow ($\dot{x} = f_0(x)$ is autonomous) for any θ: (observe that $\mathcal{P}_\theta(x) = \varphi(2\pi, x)$)



- We have a fixed point at x^* with a homoclinic orbit Γ.
- $x_h(v)$ is a parameterizaton of the homoclinic manifold (curve) $\Gamma = \{x = x_h(v), v \in \mathbb{R}\} \subset W^u(x^*) \cap W^s(x^*)$
- For any $x_h(v) \in \Gamma$, $\mathcal{P}_\theta(x_h(v)) = x_h(v + 2\pi) \in \Gamma$
- $\mathcal{P}_\theta^n(x_h(v)) = x_h(v + 2\pi n) \to x^*$ as $n \to \pm\infty$.
- $\|\mathcal{P}_\theta^n(x_h(v)) - \mathcal{P}^n(x^*)\| \leq Ce^{2\pi\lambda|n|}$, for some constant $C > 0$.
- This inequality is a consequence of the hyperbolicity of the fixed point x^*.

$\mu \neq 0$: Existence of the periodic orbit Λ_μ

From now on we consider the full system:

$$\dot{x} = f_0(x) + \mu f_1(x, t; \mu), \ x \in \mathbb{R}^2, \ t \in \mathbb{T},$$

We have the following

Lemma

- *There exists $\mu_0 > 0$ such that for $0 \leq |\mu| \leq \mu_0$, it has a 2π-periodic solution $\Lambda(t; \mu)$.*
- *Moreover, there exists a constant $K > 0$ such that $|\Lambda(t; \mu) - x^*| \leq K\mu$ for any $t \in \mathbb{R}$.*
- *The periodic orbit $\Lambda_\mu = \{x = \Lambda(t; \mu), \ t \in \mathbb{T}\}$ is also hyperbolic of saddle type, and its characteristic multipliers are μ-close to $e^{2\pi\lambda}$, $e^{-2\pi\lambda}$.*

$\mu \neq 0$: Existence of the periodic orbit Λ_μ

Proof

Consider the Poincaré map $\mathcal{P}_{\theta,\mu}$ and look for a point $x = \Lambda(\theta;\mu)$ such that

$$M(x,\mu) = \mathcal{P}_{\theta,\mu}(x) - x = 0$$

Observe that

- $M(x^*, 0) = \mathcal{P}_\theta(x^*) - x^* = 0$
- $\det(\frac{\partial M}{\partial x})(x^*, 0) = \det D\mathcal{P}_\theta(x^*) - \mathrm{Id} = \det(e^{2\pi Df_0(x^*)} - \mathrm{Id}) \neq 0$

The second condition is satisfied because $e^{2\pi Df_0(x^*)}$ has eigenvalues $e^{2\pi\lambda} < 1 < e^{-2\pi\lambda}$ different from 1, therefore $(e^{2\pi Df_0(x^*)} - \mathrm{Id})$ has eigenvalues different from 0.

The implicit function theorem gives the existence of a fixed point $x = \Lambda(\theta, \mu)$ for $\mathcal{P}_{\theta,\mu}$, which is μ-close to x^*.

Moreover the eigenvalues of $D\mathcal{P}_{\theta,\mu}(\Lambda(\theta,\mu))$ are μ close to the ones of $D\mathcal{P}_\theta(x^*)$, which are $e^{2\pi\lambda}$, $e^{-2\pi\lambda}$.

$\mu \neq 0$: Existence of the periodic orbit Λ_μ

- The solution $\Phi(t, \theta, \Lambda(\theta, \mu); \mu)$ is 2π-periodic.

  Proof:

  As the differential equation is 2π-periodic in time and we have that
  $x_1(t) = \Phi(t, \theta, \Lambda(\theta, \mu); \mu)$ is a solution and
  $x_2(t) = \Phi(t + 2\pi, \theta, \Lambda(\theta, \mu); \mu)$ is also a solution.
  Moreover
  $x_1(\theta) = \Phi(\theta, \theta, \Lambda(\theta, \mu); \mu) = \Lambda(\theta, \mu) = \mathcal{P}_{\theta,\mu}(\Lambda(\theta, \mu)) =$
  $\Phi(\theta + 2\pi, \theta, \Lambda(\theta, \mu); \mu) = x_2(\theta)$
  therefore, by the existence and uniqueness theorem we have that
  $x_1(t) = x_2(t)$ for any $t \in \mathbb{R}$
  which gives:
  $\Phi(t, \theta, \Lambda(\theta, \mu); \mu) = \Phi(t + 2\pi, \theta, \Lambda(\theta, \mu))$,
  therefore the solution is a 2π-periodic solution.

- Moreover $\Phi(t, \theta, \Lambda(\theta, \mu); \mu) = \Lambda(t, \mu) \in \Sigma_t$ because is the fixed point of the Poincaré map $\mathcal{P}_{t,\mu}$.

Invariant manifolds of Λ_μ

- Now that we know about the existence of the hyperbolic periodic orbit $\Lambda_\mu = \cup_{\theta\in[0,2\pi]}\{\Lambda(\theta,\mu)\}$ which is of saddle type, we will find its stable and unstable manifolds.

- Fix $\theta \in [0, 2\pi]$ and work with the Poincaré map $\mathcal{P}_\theta$. By the stable manifold theorem, we know that the $W^{u,s}(\Lambda(\theta,\mu))$ are μ-close to $W^{u,s}(x^*) = \Gamma$ in a neighborhood U of the origin (independent of μ).

- Now we take any point $x_1^u \in W^u(\Lambda(\theta,\mu)) \cap U$ and $x_1^h \in \Gamma \cap U$ such that $x_1^u - x_1^h = \mathcal{O}(\mu)$.

- **exercice**
  For any $t^* > \theta$, there exists $\mu_0 > 0$, $K > 0$ such that the solution $\Phi(t,\theta,x_1^u;\mu)$ and the solution $\Phi(t,\theta,x_1^h;0)$ satisfy, for $0 \le \mu \le \mu_0$:

$$|\Phi(t,\theta,x_1^u;\mu) - \Phi(t,\theta,x_1^h;0)| \le K\mu \text{ for } \theta \le t \le t^*$$

Invariant manifolds of Λ_μ

- To prove the exercice use the Gronwall lemma:
  If $u(t)$ is a continuous non-negative function in $[a, b]$, such that, there exist $c > 0$, $L > 0$ such that:

$$0 \le u(t) \le c + L \int_a^t u(s)ds, \;\; t \in [a, b]$$

  Then:

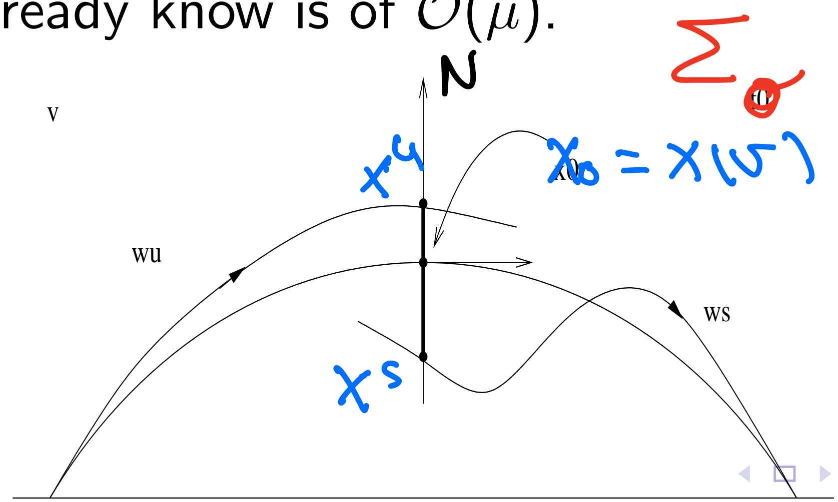
$$u(t) \le ce^{L(t-a)}$$

  .

- There is an analogous result for the stable manifold.
- This result tells us that the stable and unstable manifolds of the periodic orbit Λ_μ remain μ-close to the ones of x^* when we extend it for finite times.

Invariant manifolds of Λ_μ

- Fix a Poincaré section Σ_θ, we have the fixed point $\Lambda(\theta, \mu)$ with stable and unstable curves $W^{u,s}(\Lambda(\theta, \mu))$.
- Take a point in the unperturbed homoclinic manifold: $x_0 = x_h(v) \in \Gamma$
- Take straight line N transversal to Γ, for instance the ortogonal one: $N = N(x_0) = q_0 + \langle (f(x_0))^T \rangle = x_0 + \langle \nabla P(x_0) \rangle = x_h(v) + \langle \nabla P(x_h(v)) \rangle$
- As a consequence of the exercise, $W^{s,u}(\Lambda(\theta, \mu)) = W^{s,u}(x^*) + \mathcal{O}(\mu)$ and, as Γ intersects N transversally, both manifolds intersect N in unique points x^u, x^s, which are μ-close to $x_0 = x_h(v)$.
- Our goal is to obtain information about the distance between the points x^u and x^s, that we already know is of $\mathcal{O}(\mu)$.



Distance between the invariant manifolds of Λ_μ

We want to compute the distance between x^u and x^s:

$$d(v,\theta;\mu) = \|x^u - x^s\|$$

Theorem (Theorem 1, Melnikov-Poincaré)

$$d(v,\theta;\mu) = \frac{\mu}{|\nabla P(x_h(v))|} M(v,\theta) + \mathcal{O}(\mu^2)$$

where

$$M(v,\theta) = \int_{-\infty}^{+\infty} \{H_0, H_1\}(x_h(v+s), \theta + s; 0)\, ds$$

where:

$$\{P,Q\} = \frac{\partial P}{\partial x_1}\frac{\partial Q}{\partial x_2} - \frac{\partial P}{\partial x_2}\frac{\partial Q}{\partial x_1}$$

is the Poisson bracket of P and Q.

The Melnikov function

The function:

$$M(v,\theta) = \int_{-\infty}^{+\infty} \{H_0, H_1\}(x_h(v+s), \theta + s; 0)\, ds \tag{2}$$

is called the Melnikov function.

Exercice:

It is 2π-periodic respect to θ and satisfies:

$$M(v,\theta) = M(0, \theta - v) = \mathcal{M}(\theta - v)$$

and

$$\mathcal{M}(\alpha) = \int_{-\infty}^{+\infty} \Omega\left(f_0(x_h(t)), f_1(x_h(t), \alpha + t; 0)\right) dt$$

is 2π-periodic.

The Melnikov potential

Exercice:

- Prove that $M(v,\theta) = \frac{\partial L}{\partial v}(v,\theta)$, where

$$L(v,\theta) = \int_{-\infty}^{+\infty} (H_1(x_h(v+s), \theta+s; 0) - H_1(x^*, \theta+s; 0))ds$$

  is called the Melnikov potential or Poincaré function ans also satisfies: $L(v,\theta) = L(0,\theta-v) = \mathcal{L}(\theta-v)$, where:

$$\mathcal{L}(\alpha) = \int_{-\infty}^{+\infty} (H_1(x_h(t), \alpha+t; 0) - H_1(x^*, \alpha+t; 0))dt$$

- Analogously: $\mathcal{M}(\alpha) = \mathcal{L}'(\alpha)$

Now we have the following:

Theorem (Theorem 2)

- *If* $\forall\theta$ $M(v,\theta)$ *has a simple zero* $v = v^*(\theta)$, *there exists* $\mu_0 > 0$ *such that for any* $0 < \mu \leq \mu_0$*:*
  - *The stable and unstable manifolds of the fixed point* $\Lambda(\theta,\mu)$ *of the Poincaré map* $\mathcal{P}_{\theta,\mu}$ *intersect transversally in a point* $x^*(v(\theta;\mu))$ *where* $v(\theta;\mu) = v^*(\theta) + \mathcal{O}(\mu)$, $x^*(v) = x_h(v) + \mathcal{O}(\mu)$.
  - *The stable and unstable manifolds of the periodic orbit* Λ_μ *intersect transversally along a curve*

$$\Gamma_\mu = \{x = x^*(v(\theta;\mu), \theta \in \mathbb{T}\},$$

- *If* $M(v,\theta) > 0$, *for all* v,θ, *then there exists* $\mu_0 > 0$ *such that for any* $0 < \mu \leq \mu_0$ *the stable and unstable manifolds of the periodic orbit* Λ_μ *do not intersect.*

Observe that as

$$M(v,\theta) = \mathcal{M}(\theta - v)$$

If there exists α^* such that $\mathcal{M}(\alpha^*) = 0$ and $\mathcal{M}'(\alpha^*) \neq 0$,then, for any $\theta \in [0, 2\pi]$, taking $v^*(\theta) = \theta - \alpha^*$ is a simple zero of $M(v,\theta)$.

Exercice: Prove the theorem 2.

T-periodic case

If we consider a T-periodic Hamiltonian system:

$$H(q,p,\omega t;\mu) = H_0(q,p) + \mu H_1(q,p,\omega t;\mu), \ t \in \mathbb{R}, \ \omega = \frac{1}{T} \qquad (3)$$

the Melnikov function becomes:

$$M(v,\theta) = \int_{-\infty}^{+\infty} \{H_0, H_1\} (x_h(v+s), \theta + \omega s; 0)) \, ds \qquad (4)$$

Exercice:

Prove the equivalent properties of the Melnikov function:

$$M(v,\theta) = M(0,\theta - \omega v) = \mathcal{M}(\theta - \omega v) = \mathcal{L}'(\theta - \omega v).$$

Example

Consider the second order equation:

$$\ddot{x} = x - x^3 + \mu \cos(\omega t)$$

If we call $y = \dot{x}$ we have a Hamiltonian system in the plane:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^3 + \mu \cos(\omega t) \end{aligned}$$

Exercice 1

1. When $\mu = 0$ we have the Duffing equation, which is a Hamiltonian system of $H_0(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$, and has a saddle point at the origin $(0,0)$.
2. $(0,0)$ has an homoclinic orbit given by:

$$x_h(v) = \frac{\sqrt{2}}{\cosh v}, \quad y_h(v) = -\frac{\sqrt{2}\sinh v}{\cosh^2 v},$$

example

Exercice 2

1. Prove that the system has a periodic orbit Λ_μ if μ is small enough and that the corresponding Melnikov function satisfies: $M(v,\theta) = \mathcal{M}(\theta - \omega v)$ where

$$\mathcal{M}(\alpha) = \int_{-\infty}^{+\infty} y_h(t) \cos(\alpha + \omega t)\, dt$$

2. Prove that:

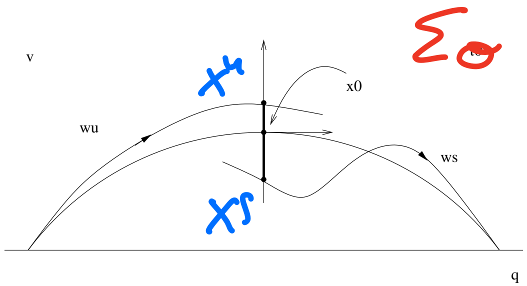
$$\mathcal{M}(\alpha) = -\sqrt{2}\pi\omega \frac{\sin \omega\alpha}{\cosh \frac{\pi\omega}{2}}$$

3. Prove that the manifolds $W^u(\Lambda_\mu)$, $W^s(\Lambda_\mu)$ intersect for μ small enough.

Proof of Theorem 1: Distance between the invariant manifolds of Λ_μ

Remember, we have to compute: $d(v,\theta;\mu) = \|x^u - x^s\|$

- The points $x^{u,s} \in W^{u,s}(\Lambda(\theta,\mu)) \cap N$, with $N = x_0 + \nabla P(x_0)$, $x_0 = x_h(v)$.



- There exist $\alpha^{u,s} \in \mathbb{R}$ such that $\alpha^{u,s} = \mathcal{O}(\mu)$ and:

$$x^u = x_0 + \alpha^u \nabla P(x_0),\ x^s = x_0 + \alpha^s \nabla P(x_0) \ \rightarrow \|x^u - x^s\| = |\alpha^u - \alpha^s| \|\nabla P(x_0)\|$$

Let's compute $P(x^*)$, $* = u, s$, expanding by Taylor:

$$P(x^*) = P(x_0) + DP(x_0)\alpha^* \nabla P(x_0) + \mathcal{O}(\mu^2) = P(x_0) + \alpha^* \|\nabla P(x_0)\|^2 + \mathcal{O}(\mu^2)$$

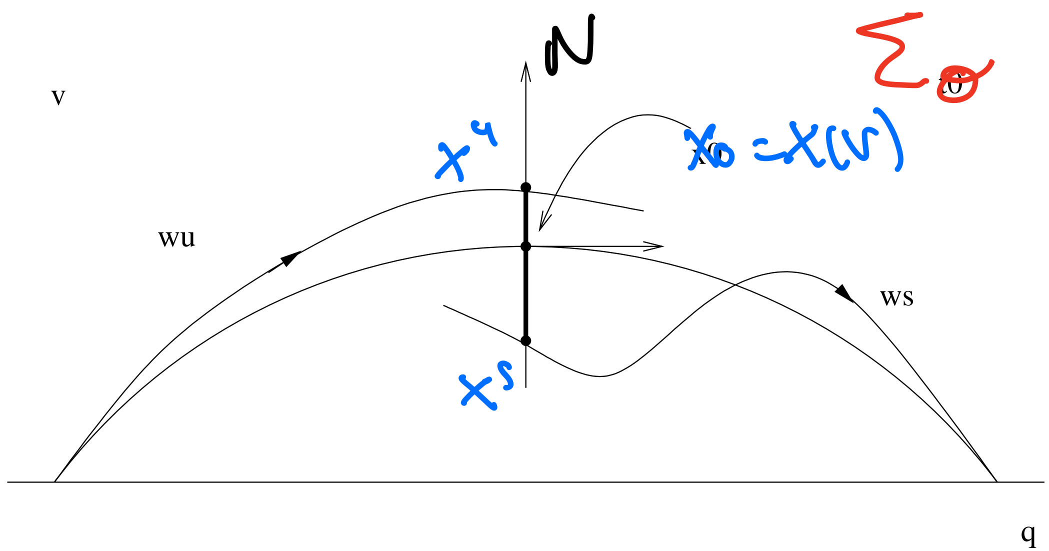
up to order μ, the quantities $|P(x^u) - P(x^s)|$ and $\|x^u - x^s\|$ are the same:

$$|P(x^u) - P(x^s)| = |\alpha^u - \alpha^s| \|\nabla P(x_0)\|^2 + \mathcal{O}(\mu^2) = \|x^u - x^s\| \|\nabla P(x_0)\| + \mathcal{O}(\mu^2)$$

Proof of Theorem 1: Distance between the invariant manifolds of Λ_μ

We have to compute: $d(v,\theta;\mu) = \|x^u - x^s\|$. We will compute $P(x^u) - P(x^s)$ instead.

- Denote $\tilde{x}(t) = x_h(v - \theta + t)$ the solution of the unperturbed system such that $\tilde{x}(\theta) = x_h(v) = x_0$.
- Consider the solutions of the system $x^*(t) = \Phi(t,\theta,x^*;\mu)$ such that $x^*(\theta) = x^*$, $* = s, u$.
- Then $P(x^u) - P(x^s) = P(x^u(\theta)) - P(x^s(\theta))$.



proof of Theorem 1: Distance between the invariant manifolds of Λ_μ

In general, given a point x, the solution $\Phi(t;\theta,x;\mu)$ is solution of:

$$\dot{x} = J\nabla P(x) + \mu J\nabla H_1(x,t;\mu) = J\nabla H(x,t;\mu)$$

If we consider $m(t) = P(\Phi(t;\theta,x;\mu))$, using the fundamental theorem of calculus:

$$m(t) = m(\theta) + \int_\theta^t \frac{d}{dt} m(\sigma)\, d\sigma$$

$$P(\Phi(t;\theta,x)) = P(x) + \mu\int_\theta^t \{P,H_1\}(\Phi(\sigma,\theta,x;\mu),\sigma;\mu)d\sigma,$$

where $\{P,h\} = \frac{\partial P}{\partial q}\frac{\partial h}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial h}{\partial q}$ is the Poisson bracket of P and h ($P = P(p,q)$).

Proof of Theorem 1: Distance between the invariant manifolds of Λ_μ

To compute $P(x^u) = P(x^u(\theta))$ and $P(x^s) = P(x^s(\theta))$ we use the previous computation and recall $x^{u,s}(t) = \Phi(t; \theta, x^{u,s}; \mu)$.

$$P(x^{u,s}(t)) = P(x^{u,s}(\theta)) + \mu \int_\theta^t \{P, H_1\}(x^{u,s}(\sigma), \sigma; \mu) d\sigma,$$

The same is true for the periodic solution $\Lambda(t, \mu) = \Phi(t; \theta, \Lambda(\theta; \mu))$:

$$P(\Lambda(t, \mu)) = P(\Lambda(\theta, \mu)) + \mu \int_\theta^t \{P, H_1\}(\Lambda(\sigma, \mu), \sigma; \mu) d\sigma,$$

then:

$$P(x^{u,s}(t)) - P(\Lambda(t, \mu)) = P(x^{u,s}(\theta)) - P(\Lambda(\theta, \mu)) + \mu \int_\theta^t \left(\{P, H_1\}(x^{u,s}(\sigma), \sigma; \mu) - \{P, H_1\}(\Lambda(\sigma, \mu), \sigma; \mu)\right) d\sigma,$$

Proof of Theorem 1: Distance between the invariant manifolds of Λ_μ

Recall that the points $x^{u,s}(\theta) \in W^{u,s}(\Lambda(\theta,\mu))$, therefore:

$$|\mathcal{P}_\theta^n(x^{u,s}(\theta)) - \mathcal{P}_\theta^n(\Lambda(\theta,\mu))| \leq Ce^{\gamma|n|}, \;\; \gamma = 2\pi\lambda + \mathcal{O}(\mu) < 0$$

and the solutions

$$|x^{u,s}(t) - \Lambda(t,\theta)| = |\Phi(t,\theta,x^{u,s};\mu) - \Phi(t,\theta,\Lambda(\theta,\mu);\mu)| \leq \tilde{C}e^{\gamma|t|}, \text{ for } \mp t \in [0,\infty)$$

Use the previous formula for the stable manifold at $t = T > 0$ and the unstable at $t = -T < 0$:

$$\begin{aligned} P(x^s(T)) - P(\Lambda(T,\mu)) &= P(x^s(\theta)) - P(\Lambda(\theta,\mu)) \\ &+ \mu \int_\theta^T (\{P,H_1\}(x^s(\sigma),\sigma;\mu) - \{P,H_1\}(\Lambda(\sigma,\mu),\sigma;\mu)) \\ P(x^u(-T)) - P(\Lambda(-T,\mu)) &= P(x^u(\theta)) - P(\Lambda(\theta,\mu)) \\ &+ \mu \int_\theta^{-T} (\{P,H_1\}(x^u(\sigma),\sigma;\mu) - \{P,H_1\}(\Lambda(\sigma,\mu),\sigma;\mu) \end{aligned}$$

proof of Theorem 1: Distance between the invariant manifolds of Λ_μ

We can write, taking $T = +\infty$ in the previous expressions (the integrals are convergent!):

$$P(x^s(\theta)) - P(\Lambda(\theta,\mu)) = -\mu \int_\theta^{\infty} (\{P, H_1\}(x^s(\sigma), \sigma; \mu) - \{P, H_1\}(\Lambda(\sigma,\mu), \sigma; \mu))\, d$$

$$P(x^u(\theta)) - P(\Lambda(\theta,\mu)) = -\mu \int_\theta^{-\infty} (\{P, H_1\}(x^s(\sigma), \sigma; \mu) - \{P, H_1\}(\Lambda(\sigma,\mu), \sigma; \mu))\, d$$

Up to here these computations are exact. Now we use:

- $H_1(x, \sigma; \mu) = H_1(x, \sigma; 0) + \mathcal{O}(\mu)$
- $\Lambda(\theta, \mu) = x^* + \mathcal{O}(\mu)$, then $P(\Lambda(\theta,\mu)) = P(x^*) + DP(x^*)\mathcal{O}(\mu) = \mathcal{O}(\mu^2)$, because $DP(x^*) = 0$. Analogously $\{P, H_1\}(\Lambda(\sigma,\mu)) = \{P, H_1\}(x^*) + \mathcal{O}(\mu) = \mathcal{O}(\mu)$.
- $\{P, H_1\}(x^s(\sigma,\mu), \sigma; 0) = \{P, H_1\}(x_h(\sigma - \theta + v), \sigma; 0) + \mathcal{O}(\mu)$ for $\sigma \geq \theta$, where x_h is the unperturbed homoclinic!
- $\{P, H_1\}(x^u(\sigma,\mu), \sigma; 0) = \{P, H_1\}(x_h(\sigma - \theta + v), \sigma; 0)) + \mathcal{O}(\mu)$ for $\sigma \leq \theta$.

Proof of Theorem 1: Distance between the invariant manifolds of Λ_μ

Using the previous approximations in the expressions:

$$P(x^s(\theta)) - P(\Lambda(\theta,\mu)) = -\mu \int_\theta^{\infty} \left(\{P,H_1\}(x^s(\sigma),\sigma;0) - \{P,H_1\}(\Lambda(\sigma,\mu),\sigma;0)\right) d$$

$$P(x^u(\theta)) - P(\Lambda(\theta,\mu)) = -\mu \int_\theta^{-\infty} \left(\{P,H_1\}(x^s(\sigma),\sigma;0) - \{P,H_1\}(\Lambda(\sigma,\mu),\sigma;0)\right) d$$

We obtain:

$$P(x^s(\theta)) = -\mu \int_\theta^{\infty} \{P,H_1\}(x_h(\sigma-\theta+v),\sigma;0) + \mathcal{O}(\mu^2),$$

$$P(x^u(\theta)) = \mu \int_{-\infty}^{\theta} \{P,H_1\}(x_h(\sigma-\theta+v),\sigma;0) + \mathcal{O}(\mu^2),$$

$$P(x^u(\theta)) - P(x^s(\theta)) = \mu \int_{-\infty}^{+\infty} \{P,H_1\}(x_h(\sigma-\theta+v),\sigma;0) + \mathcal{O}(\mu^2)$$

$$= \mu \int_{-\infty}^{+\infty} \{P,H_1\}(x_h(s+v),s+\theta;0) + \mathcal{O}(\mu^2) = \mu M(v,\theta) + \mathcal{O}(\mu^2)$$

Proof of Theorem 1: Distance between the invariant manifolds of Λ_μ

Model $\dot{x} = J\nabla P(x,y) + \mu H_1(x,t;\mu)$. H_1 can be T-periodic in time.

- Recall that the points $x^u(\theta), x^s(\theta) \in W^u(\Lambda_\mu) \cap N$
- We wanted to compute $\|x^u(\theta) - x^s(\theta)\|$
- Take the pendulum $P(q,p) = \frac{p^2}{2} + V(q)$
- We saw that $|P(x^u) - P(x^s)| = \|x^u - x^s\| \|\nabla P(q_0)\| + \mathcal{O}(\mu^2)$.
- We have seen that:

$$|P(x^u) - P(x^s)| = \mu M(v,\theta) + \mathcal{O}(\mu^2)$$

- Consequently:

$$\|x^u - x^s\| = \frac{\mu}{\|\nabla P(q_0)\|} M(v,\theta) + \mathcal{O}(\mu^2)$$

where $M(v,\theta) = \mathcal{M}(\theta - \omega v) = \frac{\partial}{\partial v} L(v,\theta) = \mathcal{L}'(\theta - \omega v)$ is the Melnikov function and L the melnikov potential.

An example of fast forcing: The perturbed pendulum

In our original model $\omega = \frac{1}{\sqrt{\varepsilon}}$. Let's do an examle.

$$H\left(p, q, \frac{t}{\varepsilon}\right) = \frac{p^2}{2} + (\cos q - 1) + \mu(\cos q - 1)\sin\frac{t}{\sqrt{\varepsilon}}$$

- $\Lambda_\mu = \{(0,0)\}$ is a hyperbolic periodic orbit for this system. $\Lambda(\theta;\mu) = (0,0)$ is the fixed point of the Poincaré map $\mathcal{P}_\theta$.
- The Melnikov potential is:

$$L(v,\theta) = 4\pi e^{-\frac{\pi}{2\sqrt{\varepsilon}}}\left(\sin(v - \frac{\theta}{\sqrt{\varepsilon}})\right).$$

$$P(z^s) - P(z^u) = 4\pi e^{-\frac{\pi}{2\sqrt{\varepsilon}}}\left(\sin(v - \frac{\theta}{\sqrt{\varepsilon}})\right) + \mathcal{O}(\mu^2)$$

We need $\mu = \mathcal{O}(e^{-\frac{\pi}{2\sqrt{\varepsilon}}})$ to make the error term smaller!