Abstract. In this work the so-called $\mu$-$k$ method for robust flutter analysis is generalized to the Laplace domain. Although the generalization as such is a straightforward matter, the resulting $\mu$-$p$ method is far more versatile than the previous formulation. With the new method, a single structured singular value (or $\mu$) evaluation can be used to determine if a certain eigenvalue $p$ is a solution to the uncertain flutter equation or not. This result makes it possible to compute regions of feasible eigenvalues in the complex plane, as well as extreme eigenvalues that can be used to predict damping bounds and perform robust flutter analysis. The capability to predict damping bounds at any flight condition is a very attractive feature of the $\mu$-$p$ method, as flight testing is rarely taken to the flutter point. This feature also makes it possible to estimate the magnitude of the uncertainty based on the measured frequency and/or damping of a particular mode, which can reduce the potential conservativeness of the robust analysis. Finally, the capabilities of the new method is demonstrated by applying it to a low-speed wind tunnel model.

1 INTRODUCTION

Flight flutter testing is always associated with high risk, since it can potentially lead to structural damage or failure. A reliable flutter analysis is therefore a vital component of any test campaign, since it enables more confident decisions through a successive comparison of predictions and test results. However, the development of a reliable flutter analysis still poses a great challenge to aeroelasticians, because the complex character of aeroelastic phenomena give rise to many different sources of error and uncertainty.

A fairly recent approach to deal with deterministic uncertainty and variation of aeroelastic models was initiated by Lind and Brenner [1, 2], who first demonstrated the use of so-called $\mu$-analysis to perform robust flutter analysis. Later on, Borglund developed the $\mu$-$k$ method [3, 4, 5], which is based on traditional frequency-domain flutter analysis. As a result, the robust analysis can be performed using existing numerical models, and inherently produces match-point flutter solutions. The $\mu$-$k$ formulation also opened up new possibilities to model aerodynamic uncertainty, which is of primary importance in aeroelastic applications. Some additional developments in this direction can be found in Refs. [6, 7, 8].

Although significant progress has been made, the current methods for robust flutter analysis suffer from an apparent shortcoming. So far, the main objective of the robust analysis has been to compute a worst-case flutter speed subject to a given uncertainty description. While a worst-case flutter speed is certainly an important result, its usefulness when it comes to flight flutter testing is limited. The reason for this is that the flight testing is rarely taken to the flutter point. It is rather desirable to clear the flight envelope without reaching a flutter condition at all. This means that the validity of the robust analysis may remain unknown, which is clearly an unsatisfactory situation.
The practical usefulness of such a method would increase substantially if it could be used to perform robust analysis of properties that can be estimated at sub-critical flight conditions, such as the damping of a particular mode. In this paper, the $\mu$-$k$ method will be generalized to the Laplace domain for this purpose. The new formulation makes it possible to answer questions like “How much can the damping of a particular mode vary as a result of the system uncertainties?”, or “How much uncertainty is required to make the flutter equation have a certain eigenvalue $p$?”. This also provides new means for model validation, where frequency and damping data from flight testing can be used to estimate the level of uncertainty present in the model.

2 UNCERTAIN FLUTTER EQUATION

To introduce the concept of robust flutter analysis, we consider the Laplace-domain equations of motion with uncertainty and/or variation,

$$P(p, \delta) \eta = \left[ M(\delta) p^2 + (L^2/V^2) K(\delta) - (\rho L^2/2) Q(p, \delta) \right] \eta = 0,$$

(1)

where $M(\delta)$ is the mass matrix, $K(\delta)$ is the stiffness matrix, $Q(p, \delta)$ is the aerodynamic transfer matrix, and $\eta$ is the vector of modal coordinates. Further, $V$ is the airspeed, $L$ is the aerodynamic reference length and $\rho$ is the air density. The nondimensional Laplace variable is denoted $p = g + ik$, where $g$ is the damping and $k$ is the reduced frequency. Note that the dependence on the Mach number and structural damping have been omitted for conciseness.

The uncertainty of the mass, stiffness and aerodynamic properties are represented by unknown but bounded parameters in the vector $\delta$. We will here focus on parametric uncertainty, since it can be used to represent many different uncertainty mechanisms in aeroelastic systems. It is then convenient to assume that the uncertainty has been normalized such that $|\delta_j| \leq 1$ holds for each parameter, and that $\delta = 0$ corresponds to the nominal model without uncertainty. This means that $\delta \in D$ where the set $D = \{ \delta : ||\delta||_\infty \leq 1 \}$. Note that both real and complex parameters are possible, typically representing mass/stiffness and aerodynamic uncertainty, respectively.

The uncertain flutter equation (1) is a nonlinear eigenvalue problem that for each $\delta \in D$ defines a set of eigenvalues $p(\delta)$ and eigenvectors $\eta(\delta)$. In particular, the eigenvalues are the values of $p$ making the flutter matrix $P(p, \delta)$ singular, such that $\det[P(p, \delta)] = 0$ holds. Depending on the formulation used for the aerodynamic forces, the eigenvalue problem can (for a given $\delta$) be solved using for example the $p$-$k$ method [9] or the $g$-method [10]. While the system is nominally stable if all nominal eigenvalues $p_0 = p(0)$ have negative real parts, robust stability requires that all eigenvalues $p(\delta)$ have negative real parts for all $\delta \in D$. In the same manner, the system is robustly unstable if some $p(\delta)$ has a positive real part for all $\delta \in D$.

As the eigenvalues are continuous functions of the matrix elements in the flutter equation (11), each nominal eigenvalue will expand to a ball of feasible eigenvalues in the complex plane for $\delta \in D$. This is illustrated for one distinct eigenvalue in Figure 1. Apparently, the corresponding mode is robustly stable if the ball is restricted to the left half plane. The robust flutter boundary is reached when the ball crosses the imaginary axis at $\text{Re}(p) = 0$, since some $\delta \in D$ can then destabilize the mode. Of particular interest are the eigenvalues $p_-$ and $p_+$ having the minimum and maximum real part, respectively. These eigenvalues provide damping bounds for the mode, and if $p_+$ has a negative real part the mode is robustly stable.

It would be very desirable to be able to compute the boundary of the eigenvalue ball in the complex plane, as well as the extreme eigenvalues $p_-$ and $p_+$. In simple cases it may be possible to characterize the eigenvalue ball by performing a systematic sweep of the parameter space [5], but this method becomes computationally infeasible even for a modest number of parameters. It
is also possible to pose an explicit optimization problem to minimize or maximize the real part of an eigenvalue, using the uncertainty parameters as variables [12, 13]. Here the main obstacle is that the optimization problem is in general non-convex, so robustness cannot be guaranteed. These approaches also lack a structured framework, and tend to become very dependent on the specific problem. The next Section will introduce the $\mu$-$p$ method, which utilizes $\mu$-analysis to accomplish the desired objectives.

3 THE $\mu$-$p$ METHOD

Using the $\mu$-$p$ method, the first step is to pose the flutter equation (1) in the uncertainty feedback form

$$[I - F(p)\Delta]z = 0,$$

which governs the dynamics of the equivalent feedback loop in Figure 2. This can be accomplished in a systematic way using Linear Fractional Transformation (LFT) matrix operations as described in Refs. [3, 8, 14]. The uncertainty parameters $\delta \in D$ are now isolated to a block-structured uncertainty matrix $\Delta$ that belongs to a corresponding set $B$ defined as

$$B = \{\Delta : \Delta \text{ structured and } \bar{\sigma}(\Delta) \leq 1\},$$

where $\bar{\sigma}(\cdot)$ denotes the maximum singular value (the induced 2-norm). The system matrix $F(p)$ is composed of the nominal elements of the flutter equation, and scaling matrices originating from the uncertainty description.

It is now possible to apply structured singular value analysis to investigate the possible solutions to the uncertain flutter equation. The structured singular value $\mu$ of the system matrix $F(p)$ is
defined as the reciprocal of the minimum norm of any structured \( \Delta \) making \([I - F(p)\Delta]\) singular,

\[
\mu[F(p)] = \frac{1}{\min_{\Delta} \{ \sigma(\Delta) : \det(I - F(p)\Delta) = 0 \text{ for structured } \Delta \}}. 
\] (4)

If no such structured \( \Delta \) exists, then \( \mu[F(p)] = 0 \). By definition, this means that if

\[
\mu(p) = \mu[F(p)] \geq 1, 
\] (5)

then \( p \) is a possible eigenvalue of (1) for some \( \Delta \in B \). Correspondingly, \( p \) is not a possible eigenvalue if \( \mu(p) < 1 \). This is a very powerful result, meaning that a single \( \mu \) evaluation can be used to determine if a certain \( p \) is a possible solution to the uncertain flutter equation or not. The \( \mu-k \) method [3, 4, 5] only considered \( p \)-values along the imaginary axis, where \( \mu(ik) \) was evaluated to investigate if a critical flutter eigenvalue \( p = ik \) was possible for some \( \Delta \in B \). The \( \mu-k \) method is therefore a special case of the more versatile \( \mu-p \) method.

The most basic application of the \( \mu-p \) method is to visualize a region (ball) of feasible eigenvalues by evaluating \( \mu(p) \) for a grid of \( p \)-values in the neighborhood of a nominal eigenvalue \( p_0 \) (this is essentially what is illustrated in Figure 1). Using a dense grid of sufficient size it is in principle possible to track the ball in the complex plane to compute the robust flutter speed, and to estimate damping bounds for the mode. However, as outlined in the next Section, this can be accomplished more efficiently by computing the extreme eigenvalues \( p_- \) and \( p_+ \).

In practice, the criterion (5) is evaluated using computable upper bounds for \( \mu \) [15], which is in fact what guarantees robustness. In the present context, this means that the regions in the complex plane where \( \mu(p) \geq 1 \) will be slightly expanded, depending on the quality of the upper bound. Basically, utilizing \( \mu \) for robust flutter analysis is a sophisticated search in the parameter space for the most critical perturbation. Unfortunately, the most critical perturbation as such is not a result of the upper bound computation, but it can be estimated by other means [8, 16]. In this work, the \( \mu \)-solver in Matlab [17] was used for the \( \mu \)-analysis, using the default settings.

### 3.1 Basic algorithm

The \( \mu \) criterion (5) can also be used to compute the boundary of a (distinct) eigenvalue ball for a particular mode, as well as the extreme eigenvalues \( p_- \) and \( p_+ \) shown in Figure 1. The first step is to compute the nominal eigenvalue \( p_0 \). From the definition (4) it is clear that \( \mu(p) \to \infty \) when \( p \to p_0 \), because the nominal eigenvalue \( p_0 \) is a solution to the uncertain flutter equation for \( \Delta = 0 \). Figure 1 can thus be seen as a contour plot of the positive real-valued function \( \mu(p) \) at \( \mu(p) = 1 \), and there is a distinct peak (out of the plane) at the location of the nominal eigenvalue \( p_0 \). To find one point at the boundary (or contour) where \( \mu(p) = 1 \), one possibility is to begin at \( p_0 \) and step in a certain direction until a value \( \mu(p) < 1 \) is detected. From there a bisection iteration [18] can be applied to find an accurate solution \( p \) such that \( \mu(p) = 1 \). Repeating this procedure for an arbitrary number of directions will allow for a visualization of the boundary, which will be illustrated in the subsequent sample test case.

Exploiting that the parameter search is embedded in the function \( \mu(p) \), the eigenvalue \( p_+ \) with the maximum real part can be defined as the solution to an optimization problem with only two variables (the real and imaginary parts of \( p \)),

\[
\begin{align*}
\text{maximize} & \quad \text{Re}(p) \\
\text{subject to} & \quad \mu(p) \geq 1,
\end{align*}
\] (6)
It is here assumed that the optimization is initiated at a feasible point with respect to the constraint $\mu(p) \geq 1$, such that the corresponding extreme eigenvalue is found. Hence the optimization problem (6) has to be solved for each mode of interest.

In this work a simple coordinate search was used to find the optimal solution to (6). First a golden section search [18] in frequency was implemented to define the function

$$
\mu_k(g) = \max_k \mu(g + ik).
$$

This function computes the (closest) maximum value of $\mu(p)$ along a line of constant $g$ in the complex plane, as well as the maximizing frequency $k^*$. In the basic algorithm, the nominal frequency $k_0$ (the imaginary part of $p_0$) was used as initial value for each golden section search. Using that the real part $g^*$ of the extreme point $p_+$ satisfies $\mu_k(g^*) = 1$, upper and lower bounds for $g^*$ were obtained by starting at the nominal value $g_0$ and stepping in the positive $g$ direction until a value $\mu_k(g) < 1$ was detected. Then a bisection iteration [18] was applied to find an accurate value of $g^*$. Finally, the converged values of $g^*$ and $k^*$ were used to define the eigenvalue $p_+ = g^* + ik^*$ sought for.

The described algorithm is easy to implement and requires no derivatives of $\mu$. As described in the next Section, the function $\mu_k(g)$ defined in (7) is also useful for model validation purposes. The basic algorithm described above was modified slightly to make it more efficient. For example was the golden section search terminated if a value $\mu(p) \geq 1$ was detected at some point, because the corresponding value of $g$ is then known to be a lower bound of the real part of $p_+$. The eigenvalue $p_-$ with the minimum real part was computed in the same manner, by performing a search in the negative $g$ direction.

It may occur that the eigenvalue balls of two different modes overlap in the complex plane, and that the algorithm converges to the same solution for both modes. This does not pose a significant problem for the robust flutter boundary because the most extreme solution can always be found, but it may obstruct the possibility to find the true damping bounds for some mode in some part of the flight envelope. If this situation would be critical for some application, it is always possible to evaluate $\mu(p)$ for a grid of $p$-values in the complex plane to investigate the complete picture in more detail.

In summary, the $\mu$-formulation makes it possible to compute robust eigenvalues $p_-$ and $p_+$ that can be tracked in a traditional root-locus graph, and the system is robustly stable as long as all eigenvalues $p_+$ have negative real parts. The real parts of the robust eigenvalues also provide damping bounds that can be compared with results from flight testing. These features will be exemplified in the final part of this paper.

### 4 MODEL VALIDATION IN THE $\mu$-FRAMEWORK

So far it has been assumed that the structure as well as the magnitude of the uncertainty are known. While the structure of the uncertainty will be a result of the selection of uncertain parameters in the model, the magnitude can be more difficult to estimate. This is particularly true for aerodynamic uncertainty [3]. A too low magnitude will result in a robust analysis that may not capture the worst-case perturbation, and a too high magnitude will lead to an unnecessary conservative prediction of the robust flutter boundary.

The basic principle of model validation is to adjust the magnitude of the uncertainty to an appropriate (minimum) level by matching robust predictions against experimental data. So far, the $\mu$ framework has relied on model validation based on single-input single-output fre-
frequency responses of the system [19]. While this has indeed been proven very useful in aeroelastic applications [3, 20], it is also associated with some difficulties. One problem is that the frequency-response data can be influenced by uncertainty in the excitation (such as uncertain control surface aerodynamics), which may not be possible to isolate from the primary source of uncertainty. Another problem is simply the fact that the model is validated against frequency-response data rather than the aeroelastic damping, which is the critical parameter in flight testing. In the following it will be shown how the $\mu$p framework can be used for this purpose.

4.1 $p$-validation

Assume that an eigenvalue $p_{\exp} = g_{\exp} + ik_{\exp}$ has been obtained in flight testing at some given flight condition. Then the minimum magnitude of the uncertainty (in terms of the norm of the uncertainty matrix) required to make the uncertain flutter equation (1) have an eigenvalue $p_{\exp}$ is given by

$$\bar{\sigma}_p = \frac{1}{\mu(p_{\exp})}$$

where $\mu(p_{\exp})$ is evaluated at the same flight condition. This result follows directly from the definition of the function $\mu(p)$ in (4) and is illustrated in Figure 3, where $\bar{\sigma}_p$ is the minimum magnitude of the uncertainty required to expand the eigenvalue ball of the particular mode to include the eigenvalue $p_{\exp}$. Assume for example that the initial uncertainty description has been scaled such that $\bar{\sigma}(\Delta) \leq 1$ corresponds to a 100% maximum variation of the uncertain parameters in the model. Then a value $\bar{\sigma}_p = 0.1$ would mean that a 10% variation is sufficient to make the uncertain flutter equation have an eigenvalue $p_{\exp}$. This is the most basic application of the result (8), where the uncertainty bounds of all parameters are scaled uniformly in order to validate the model (other applications are possible).

4.2 $g$-validation

In order to avoid an excessively large uncertainty bound, it can in some cases be beneficial to validate the model against the aeroelastic damping only. This is particularly true when a discrepancy in frequency is known to be caused by a mechanism that is not captured by the

![Figure 3: Model validation in the complex plane.](image-url)
uncertainty description (and has a small influence on the damping). This type of validation is here referred to as $g$-validation.

The minimum magnitude of the uncertainty required to make an eigenvalue of the uncertain flutter equation have the real part $g_{\text{exp}}$ can be computed as

$$\tilde{\sigma}_g = \frac{1}{\mu_k(g_{\text{exp}})}$$  \hspace{1cm} (9)

where the function $\mu_k(g)$ is defined in (7). This is also illustrated in Figure 3, where $\tilde{\sigma}_g$ is the uncertainty bound required to expand the eigenvalue ball until it touches the line $g = g_{\text{exp}}$ in the complex plane. Using this type of validation we thus accept that the uncertain flutter equation does not have an eigenvalue $p = p_{\text{exp}}$, but an eigenvalue such that $\text{Re}(p) = g_{\text{exp}}$. This eigenvalue is $p = g_{\text{exp}} + ik^*$, where $k^*$ is the frequency that maximizes $\mu(g_{\text{exp}} + ik)$.

### 4.3 $k$-validation

By maximizing $\mu(p)$ along a line of constant frequency $k = k_{\text{exp}}$ in the complex plane, it would also be possible to perform a corresponding $k$-validation to ensure that the flutter equation has an eigenvalue such that $\text{Im}(p) = k_{\text{exp}}$. Although validation based on frequency can be useful, $p$- and $g$-type validation are considered more central because of the important role played by the aeroelastic damping.

Note that the new means for model validation introduced above are performed on a mode-by-mode basis, meaning that an individual uncertainty bound can be computed for each mode of interest. The critical flutter mechanism can therefore be isolated in the model validation, which can reduce the potential conservatism of the robust analysis. It also makes it possible to capture the frequency dependence of aerodynamic uncertainty in a well-defined and structured manner, by successively updating the modal uncertainty bounds in flight testing.

### 5 APPLICATION TO A WIND TUNNEL MODEL

In this Section the $\mu$-$p$ method is applied to the same test case as in Ref. [3], allowing for comparison with some results obtained using the $\mu$-$k$ method. A 1.2 meter semi-span wing model with a controllable trailing edge flap was tested in the low-speed wind tunnel L2000 at the Royal Institute of Technology, as illustrated in Figure 4. The details of the wind-tunnel model is not accounted for here, but can be found in previous work by the author [3, 21]. The model was found to suffer a 6.40 Hz flutter instability at the critical airspeed 16.0 m/s.

#### 5.1 Nominal flutter analysis

The nominal model used in Ref. [3] was obtained by combining a beam finite-element model for the structural dynamics and a doublet-lattice model [22] for the unsteady aerodynamics (a standard modal formulation was used). Here, the $g$-approximation by Chen [10] will be used for the aerodynamic forces in order to perform analysis in the $p$-domain. Using this model, the frequency-domain aerodynamic matrix $Q(ik)$ and its derivative $Q'(ik)$ with respect to the reduced frequency are utilized to obtain an aerodynamic transfer matrix $Q(p)$ that is correct to the first order in $g$,

$$Q(p) = Q(ik) + gQ'(ik).$$  \hspace{1cm} (10)

This formulation is very practical since $Q(ik)$ can be computed using well-known linear aerodynamic methods [22, 23]. In this work, B-splines [24] was used to obtain smooth representations
of $Q(ik)$ and $Q'(ik)$ from discrete matrix data of $Q(ik)$. Further, the nominal eigenvalues were computed using an extended version of the $p$-$k$ solver in Ref. [9].

Although a $p$-domain aerodynamic model is always desirable, the $\mu$-$p$ method can be applied regardless of the formulation used. If $\mu(p) \geq 1$, then $p$ is a possible eigenvalue of the corresponding flutter equation. Consequently, robust analysis of a weakly damped mode can be performed using $Q(ik)$, if this approximation is considered to be sufficiently accurate. As the $\mu$-$p$ method extends standard linear flutter analysis to take uncertainty into account, the same considerations regarding modeling have to be made.

The result of the nominal flutter analysis is presented in Figure 5 for the two most critical modes. The wing is predicted to flutter in the second mode at 14.7 m/s with a frequency of 6.43 Hz, which corresponds fairly well with the experimental flutter point. Still, the damping of the flutter mode is not fully captured by the numerical model.
5.2 Uncertainty description

In this case the aerodynamic model is known to be incomplete, since the wing tip plates shown in Figure 4 are not included. Using the same technique as described in Ref. [3], an aerodynamic uncertainty description is introduced to capture their influence on the aeroelastic behavior of the wing. The aerodynamic panels in the wing tip region are divided into three spanwise patches as shown in Figure 6. Within each patch, the pressure coefficients are allowed to vary in a uniform manner according to \( c_p = (1 + w_j \delta_j)c_{p0} \), where \( c_{p0} \) are the nominal pressure coefficients and \( w_j > 0 \) a real-valued bound such that the complex-valued uncertainty parameter \( \delta_j \) satisfies \( |\delta_j| \leq 1 \). Thus, a value \( w_j = 0.1 \) means that the (complex-valued) pressure coefficients within the corresponding patch can vary up to 10%.

To pose the uncertainty description in a form suitable for \( \mu \)-analysis, the aerodynamic transfer matrix \( Q(p) \) is partitioned according to \( Q(p) = R \cdot S(p) \), where \( S(p) \) computes the wing pressure coefficients from the modal deformations and \( R \) computes the modal forces from the pressure coefficients. Using the \( g \)-approximation for the aerodynamic forces, the partition \( S(p) \) is obtained from (10) as

\[
S(p) = S(ik) + gS'(ik),
\]

where \( S(ik) \) is the corresponding partition of the computable matrix \( Q(ik) \). As described in more detail in Ref. [5], the partitioned form of the aerodynamic matrix can be used to write the uncertain aerodynamic transfer matrix in the form

\[
Q(p, \delta) = Q(p) + Q_L(p) \Delta Q_R(p),
\]

where \( Q(p) \) is the nominal matrix, \( \Delta = \text{diag}(\delta_j I_j) \) is a diagonal aerodynamic uncertainty matrix, and \( Q_L(p) \) and \( Q_R(p) \) are scaling matrices computed from \( R, S(p) \) and the bounds \( w_j \).

The uncertain flutter equation is obtained by inserting (12) in (1). In terms of the nominal flutter matrix

\[
P(p) = Mp^2 + (L^2/V^2)K - (\rho L^2/2)Q(p),
\]

the feedback form (2) is defined by the system matrix

\[
F(p) = (\rho L^2/2)Q_R(p)P(p)^{-1}Q_L(p),
\]

along with the aerodynamic uncertainty matrix \( \Delta \) and the variables \( z = Q_R(p)\eta \) [5]. Having derived \( F(p) \) and the structure of \( \Delta \), it is now possible to perform \( \mu \)-\( p \) model validation and robust analysis.

5.3 Model validation

In the present case the only experimental data available is the flutter point at 16.0 m/s, at which the critical eigenvalue is located on the imaginary axis in the complex plane. Still, this is sufficient to demonstrate \( \mu \)-\( p \) model validation in practice.
The uncertainty bounds for the pressure coefficients in each patch was initially set to $w_j = 1$. Using $p$-validation as described in Section 4.1, a norm $\bar{\sigma}_p = 0.17$ was required to validate the model at the experimental flutter speed (corresponding to a 17% maximum variation). The bounds were therefore updated to $w_j = 0.17$ and the boundary of the eigenvalue ball was computed using the approach in Section 3.1. The resulting boundary is shown in Figure 7 and was found to be almost circular in this case. As expected, the computed uncertainty bound expands the ball to include the experimental eigenvalue. Using $g$-validation, outlined in Section 4.2, the uncertainty bound was reduced slightly to $\bar{\sigma}_g = 0.16$. The ball is now expanded to include an eigenvalue having the same damping (real part) as the experimental eigenvalue. As can be seen in the Figure, the modest reduction of the uncertainty bound using $g$-validation is a result of the nominal and experimental eigenvalues being close in frequency.

The overall result of the model validation is that a 17% uncertainty in the spanwise load distribution is required to match the experimental flutter point. This agrees well with previous results, where a 20% uncertainty bound was obtained using model validation based on frequency response data [3]. However, the previous result is considered to be less reliable, since it was based on an assumption about the impact of aerodynamic uncertainty in the control surface excitation.

### 5.4 Robust flutter analysis

Next, the updated model was used to perform a $\mu$-$p$ robust flutter analysis as described in Section 3.1. Nominal as well as robust eigenvalues were computed for airspeeds in the range 10 to 28 m/s. The computational effort using the $\mu$-$p$ method strongly depends on the dimension of the uncertainty description, but was here comparable to that of the nominal analysis. A robust eigenvalue was thus computed in a couple of seconds. The different eigenvalues are displayed in a root-locus graph in Figure 8. The worst- and best-case flutter speeds for each mode are the airspeeds where the corresponding eigenvalues $p_+$ and $p_-$ crosses the imaginary axis, respectively. To give an example of how the different eigenvalues are related to each other, a ball of feasible eigenvalues is shown for the first mode at 21 m/s.

The robust flutter speeds are more apparent in the damping graph of Figure 8. It is important to note that it is the damping $2g/k$ of the extreme eigenvalues (with max/min $g$) that is displayed.
not the max/min values of $2g/k$. The second mode is predicted to flutter in the range [13.6, 16.1] m/s, while [23.6, 25.4] m/s is the corresponding range for the first mode. Note, however, that the uncertainty bound obtained for the second mode was used for the first mode as well (since no experimental data was available for this mode). Finally, it is noted that the experimental flutter speed is within the predicted range of flutter speeds for the critical mode, as should be the case.

6 CONCLUSIONS

In this paper it has been shown how $\mu$-analysis can be used for robust aeroelastic analysis in the Laplace domain. In essence, the $\mu$-$p$ method extends standard linear flutter analysis to take deterministic uncertainty and variation into account. The new method does not only allow for robust analysis of the flutter speed, but also of sub-critical properties such as the frequency and damping of a particular mode. Adding the capability to perform model validation based on modal flight data, the $\mu$-$p$ framework has a potential to make the current procedures for flutter clearance of aircraft more reliable and efficient.

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8 REFERENCES


