Micromechanical model of cross-over fibre bridging –
Prediction of mixed mode bridging laws

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Abstract

The fracture resistance of fibre composites can be greatly enhanced by crack bridging. In situ observations of mixed mode crack growth in a unidirectional carbon-fibre/epoxy composite reveal crack bridging by single fibres and by beam-like ligaments consisting of several fibres. Based on the observed bridging mechanism, a micromechanical model is developed for the prediction of macroscopic mixed mode bridging laws (stress-opening laws). The model predicts a high normal stress for very small openings, decreasing rapidly with increasing normal and tangential crack opening displacements. In contrast, the shear stress increases rapidly, approaching a constant value with increasing normal and tangential openings. The solutions for the bridging laws and the resulting toughening due to the bridging stresses are obtained in closed analytical form.

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1. Introduction

Some fibre composites experience crack bridging during cracking. Crack bridging is defined here as ligaments that connect the crack faces in the wake behind the tip of an advancing crack. The crack bridging zone can be regarded as a part of the fracture process zone. Crack bridging can increase the fracture resistance of the composite considerably (Foote et al., 1986; Hashemi et al., 1990; Spearing and Evans, 1992; Albertsen et al., 1995; Sørensen and Jacobsen, 1998). A complication in the analysis of crack bridging in composites is that the bridging zone size can be comparable to or larger than the smallest relevant specimen dimensions. This is called large-scale-bridging (LSB). Under LSB, data analysis using linear elastic fracture mechanics is inadequate; classic R-curves (i.e., the fracture resistance as a function of crack extension) are specimen geometry and size dependent and cannot be considered being material properties (Bao and Suo, 1992; Suo et al., 1992; Spearing and Evans, 1992; Sørensen...
Bridging laws describe the response (under monotonic opening) of crack bridging within the fracture process zone in terms of stress–displacement laws, viz.,

\[
\sigma_n = \sigma_n(\delta_n, \delta_t) \quad \text{and} \quad \sigma_t = \sigma_t(\delta_n, \delta_t),
\]

(1)

where \(\sigma_n\) is the normal stress and \(\sigma_t\) is the shear stress, \(\delta_n\) and \(\delta_t\) denote the normal and tangential crack opening displacements, respectively. In the analysis of bridged cracks, crack bridging ligaments are replaced by the bridging stresses as sketched in Fig. 1. Note from Eq. (1), that the normal and tangential stresses are assumed to be functions of both \(\delta_n\) and \(\delta_t\) but otherwise independent of position within the bridging zone. It is often assumed that the bridging stresses vanish when the normal and tangential crack opening displacements have exceeded critical values, denoted \(\delta_n^0\) and \(\delta_t^0\), respectively. The bridging laws are taken as material properties. Their importance on the fracture resistance can be seen by evaluating the path-independent \(J\) integral (Rice, 1968) locally around the fracture process zone. The result is (Sørensen and Kirkegaard, 2006)

\[
J_{\text{loc}} = \int_0^{\delta_t^*} \sigma_t(\delta_n, \delta_t) \, d\delta_t + \int_0^{\delta_n^*} \sigma_n(\delta_n, \delta_t) \, d\delta_n + J_{\text{tip}},
\]

(2)

where \(J_{\text{tip}}\) is the \(J\) integral value evaluated around the crack tip (\(\Gamma_{\text{tip}}\) in Fig. 1b), while \(\delta_t^*\) and \(\delta_n^*\) are the end-opening and end-sliding of the bridging zone. Eq. (2) shows that the bridging laws are central to the energy dissipation within the bridging zone. Eq. (2) can be given a physical interpretation as the work (per unit cross-section area) of the bridging stresses. Bridging laws can be determined from experiments or from micromechanical models.

Fibre bridging in unidirectional fibre composites is traditionally divided into two basic cracking modes: crack bridging by fibres oriented perpendicular to the crack plane (Marshall et al., 1985; McCartney, 1987; Hutchinson and Jensen, 1990; Zok and Hom, 1990; Thouless and Evans, 1988; Deve and Maloney, 1991; Kardomateas and Carlson, 1996) and crack bridging by fibres oriented parallel to the cracking plane, denoted fibre cross-over bridging (Bao and Suo, 1992; Spearing and Evans, 1992; Kaute et al., 1993; Shercliff et al., 1994; Sørensen and Jacobsen, 1998; Jacobsen and Sørensen, 2001). The present paper focuses at the latter.

A number of micromechanical models have been developed for Mode I cracking with fibre cross-over bridging (Spearing and Evans, 1992; Kaute et al., 1993; Ivens et al., 1995). Spearing and Evans (1992) modelled cross-over bridging by a beam peeling off along an interface. Neglecting shear deformation, their model predicts a relationship between the normal stress and normal opening as \(\sigma_n \propto \delta_n^{-1/2}\). Since this result comes from a model based on classic beam theory, it is not expected to
be accurate for very short ligaments and the singular stress predicted for $\delta_n \to 0$ should not be taken literally. A bridging law having such a shape was found experimentally for a unidirectional carbon fibre-epoxy composite (Sørensen and Jacobsen, 1998) using a $J$ integral approach developed for cementitious materials by Li and Ward (1989). The agreement between the measured and predicted bridging law shapes suggests that a micromechanical model based on simple beam theory may be adequate to represent cross-over bridging. The model by Kaute et al. (1993) is based on fibre pull out from the matrix. This mechanism was observed in situ during testing of ceramic matrix composites (Kaute et al., 1993; Shercliff et al., 1994). The model predicts that a constant bridging stress is attained if the bridging fibres remain intact. However, the bridging stress decreases to zero because of fibre failure, which is predicted by the Weibull model (Weibull, 1939).

The present study concerns mixed mode crack bridging by the cross-over bridging mechanism. First, a mixed mode test fixture is developed for use in an environmental scanning electron microscope (ESEM). Testing in the ESEM allows us to make in situ observations to characterise the bridging mechanism. These observations form the basis for the development of a simple model for the prediction of mixed mode bridging laws for fibre-cross-over bridging.

2. Experimental

In order to obtain in situ observations of the cross-over bridging mechanism under mixed mode crack opening, some fracture mechanics tests were conducted in the vacuum chamber of an ESEM, using a special mixed mode test fixture. The authors are not aware of a similar mixed mode test fixture for fracture mechanics testing in ESEM. Therefore, we start off with a brief description of the mixed mode test method that we have developed.

The advantage of performing fracture experiments in an ESEM (or a low vacuum scanning electron microscope) is that it is possible to observe newly cracked surfaces, which have not been opened and closed (by earlier loading and unloading of the specimen) before the microscopic observations. In a conventional SEM, the surface to be investigated must be electron conductive. Specimens made of non-conducting materials must be coated with a thin layer of a conductive material (e.g., carbon or gold). However, fracture surfaces formed after the application of the coating would charge up. Furthermore, for ductile materials, such as thermoplastics, the coating can crack at a lower strain than the underlying material (Bradley, 1989). Such problems are eliminated by performing the experiments in an ESEM, where it is not required that the surface is conductive.

2.1. Test specimen geometry and loading

The chosen test specimen is depicted in Fig. 2. It is a geometrically symmetric double cantilever beam (DCB) specimen loaded with a bending moment, $M$, and an axial force, $P$, applied at the neutral axis of the cracked beam ends ($x_1 = -a$, $x_2 = H/2$, where $a$ is the crack length and $H$ is the beam height, and $x_1$ and $x_2$ are coordinate axis, see Fig. 2). This configuration has been analysed as a general bimaterial specimen by Suo and Hutchinson (1990) and as an orthotropic specimen by Suo (1990). The analysis shows that $P$ and $M$ can represent any load case involving axial forces and moments at the edges.

Evaluating the $J$ integral along the external boundaries of a specimen having isotropic elastic properties gives for plane strain (Suo and Hutchinson, 1990)

$$J_{\text{ext}} = \frac{1 - \nu^2}{E} \left[ \frac{P^2}{B^2H} + 12 \frac{M^2}{B^2H^2} + 12 \frac{PM}{B^2H^2} \right],$$

where $E$ and $\nu$ are the Young’s modulus and the Poisson’s ratio, respectively, and $B$ is the beam thickness. An important feature of the specimen configuration is that the $J$ integral equation is independent of the crack length, $a$, and valid for LSB problems, since it does not depend on details of the bridging law, as elaborated by Suo et al. (1992) (obviously, $J_{\text{loc}}$ equals $J_{\text{ext}}$ due to the path independence of the $J$ integral, so that the value of

![Fig. 2. A DCB specimen loaded with axial force, $P$, and a bending moment, $M$, at the beam-ends. The moment $M^*$ is given by moment equilibrium.](image-url)
depends on the bridging law, as indicated by (2)). In the study of LSB problems, this is advantageous, since this allows the bridging law to be determined directly from experiments (Bao and Suo, 1992; Suo et al., 1992; Sørensen and Jacobsen, 1998, 2003). This is, however, beyond the scope of the present study.

In practice, the combination of an axial force and a bending moment is created by applying a force that is placed a distance \( D \) from the crack plane, see Fig. 3. The moment is then given as

\[
M = P \left( D - \frac{H}{2} \right).
\]

Inserting (4) into (3) gives

\[
J_{\text{ext}} = \frac{4(1 - \nu^2)P^2}{B^2HE} \left( 1 + \frac{3D^2}{H^2} \right). \tag{5}
\]

In case of a small scale fracture process zone it is relevant to characterise the stress state by \( J \) and the mode mixity, defined as the phase angle of the stress intensity factors,

\[
\psi = \tan^{-1} \left( \frac{K_{\text{II}}}{K_{\text{I}}} \right), \tag{6}
\]

where \( K_{\text{I}} \) and \( K_{\text{II}} \) denote the Mode I and the Mode II stress intensity factors, respectively. For the present test configuration the mode mixity can be obtained from the analysis of Suo and Hutchinson (1990). The result is

\[
\psi = \tan^{-1} \left( \frac{H}{\sqrt{3D}} \right). \tag{7}
\]

Fig. 4 shows \( J \) and \( \psi \) as a function of the length \( D \). Note that \( \psi \) is rather sensitive to \( D \) for \( D/H < 1/2 \) (\( \psi \) varies 45° for \( 0 < D/H < 1/2 \)). This, coupled with the fact that the DCB specimen deflects under loading makes it difficult to control the value of \( \psi \) accurately during an experiment. In the present study, however, this is of minor concern, since the purpose of our experiments is to observe the micromechanics of mixed mode crack bridging, not to measure actual fracture resistance data. Another issue is that the loading fixture may not be able to deliver a sufficiently high load for high values of \( \psi \) (low \( D \)), since \( J \) decreases rapidly with decreasing \( D \).

2.2. Specimen manufacturing

DCB specimens were cut from a 1 mm thick plate made of unidirectional carbon-fibre/epoxy composites. The plate was made by stacking plies of prepregs (Sigrafil CE 1007), followed by curing of the laminate in a vacuum bag at 120 °C for 2 h in an autoclave under a pressure of 0.5 MPa. The nominal fibre volume fraction was 60%. Following consolidation, the plate was cut to specimens with nominal dimensions 50 mm × 20 mm × 1 mm. The fibre direction was parallel to the \( x_1 \)-direction. A
notch was cut to act as a crack starter. The crack plane \((x_2 = 0)\) was in the direction of the fibres, orthogonal to the plane of the plies. The cracking was thus intralaminar. The specimen surface to be examined in the ESEM was polished to ease observations.

2.3. Testing procedure

The experiments were conducted in an ESEM (ElectroScan, model E3) using water as the ionising gas. Typically, the water vapour pressure was 3–5 Torr (\(\sim 500\) Pa) and the temperature was about 20–30 °C. Details of the practical design of the test fixture are given in Appendix A. The loading fixture, including the DCB specimen, was mounted at the \(x\)--\(y\)--\(z\) table of the microscope. After the application of the low pressure, the notch and crack tip was brought into the viewing area of the microscope. Then, the load was increased gradually until crack growth was detected visually at the ESEM monitor. Images (secondary electrons) were acquired of microstructural events, such as the formation of bridging ligaments. Experiments were made using different values of \(D\), imposing different normal and tangential crack opening displacements.

2.4. Experimental results

Crack bridging, both in the form of single fibre and ligament bridging, was observed during the mixed mode cracking experiments. Some representative examples are shown in Figs. 5 and 6. In these examples, the loading was mixed mode \((D/H \approx 0.5)\). In both cases, the loading was such that one of the beams \((x_2 > 0)\) was displaced in the positive

![Fig. 5](image1.png)

**Fig. 5.** Micrographs of two ligaments, (A) and (B), that bridge the main crack (a). With increasing opening (b), the ligament length increases and a fibre fails at the point of ligament detachment (C).

![Fig. 6](image2.png)

**Fig. 6.** Micrographs of a large ligament, (A), connecting the crack faces (a). The tip of the interface crack is indicated (B). With increasing opening (b), the crack extends (C), a small ligament loaded in compression forms (D). Single fibre bridging is also seen (E).
\( x_1 \)-direction and in the positive \( x_2 \)-direction (mixed mode opening) relative to the other beam of the specimen \( (x_2 < 0) \), in accordance with Fig. 3.

Fig. 5 shows two beam-like ligaments that have formed at the same \( x_1 \)-position. Both ligaments consist of a few fibres embedded in the matrix. The two ligaments, however, appear to behave differently. The one in front (labelled A) takes an S-shape as it transmits a compressive force (in the \( x_1 \)-direction). With increasing opening, the ligament length increases and a fibre fails (C) at the tip of the crack, possibly due to tensile stresses induced by bending. The ligament lying behind (labelled B) also displays fibration failure and appears to split up in single fibre bridges. Splitting of the ligaments was, however, only rarely seen. It is noted that the ligaments can remain intact over crack openings that are several times the thickness of the ligaments.

The development of a large beam-like ligament (A) is shown in Fig. 6. With increasing opening, the ligament increases in length as the tip of the interface crack moves from (B) to (C). A small compressive ligament (D) has formed as the crack shifted plane. The small compressive ligament is bend into an S-shape. Single fibre bridging (E) as well as fibre failure is also seen.

The observations made here are in many respects similar to the observations made under pure Mode I experiments (Spearing and Evans, 1992; Sørensen and Jacobsen, 2000). One difference is that under mixed mode crack bridging, ligaments or fibres can be subjected to tensile or compressive force in the fibre direction, dependent on the ligament orientation, as shown in Fig. 5. Mechanically, they deform differently. The ligament subjected to a tensile force is likely to straighten as a string, whereas a ligament subjected to compression is likely to buckle.

3. Micromechanical model

3.1. Model specification

As mentioned above, the mechanism of mixed mode cross-over bridging in the investigated composite is similar to the cross-over bridging mechanism observed under pure Mode I. Therefore, we extend the existing Mode I model of Spearing and Evans (1992) to mixed mode by also including a tangential crack opening displacement. One complication is that ligaments in compression and tensile behave differently. Here, we disregard the ligaments loaded in compression, since they buckle easily and thereby their load-carrying capability vanishes. In contrast, ligaments loaded in tension act as strings and can thus carry a significant higher tensile load. This has been verified in another study of mixed mode ligament bridging where modelling was made by the finite element method (Østergaard et al., in preparation). Therefore, our model only includes ligaments that transmit tensile stresses.

Furthermore, we disregard plasticity and visco-elasticity. This allows us to make an analytical model. Although plastic deformation may occur at crack tips, neglecting plasticity is justified by the fact that fibres are usually very closely spaced (typically, a few microns); it is then anticipated that the effect of plasticity diminishes (Tvergaard and Hutchinson, 1994). We assume that the ligament peels off along a fibre/matrix interface that possesses a constant, mode mixity independent fracture energy, \( G_c \). This assumption will be justified later in the paper (Section 4.4). The mixed mode cross-over bridging mechanism is modelled using classic beam theory (small displacement, small rotations, small strains and shearing deformation is disregarded) and smeared-together isotropic elastic properties (extension to orthotropic elastic properties is straightforward). The only failure event that is modelled is the detachment of a ligament along a cracking plane (involving fibre/matrix interface and matrix cracking); fibre failure is not included in the bridging law model. Splitting of the ligaments is not modelled, since it was rarely observed and thus considered insignificant.

The model is shown in Fig. 7a shows a main crack with a few bridging ligaments. Fig. 7b shows details of a half bridging ligament. The model is planar. Each ligament is taken to have a rectangular cross-section. The height and width of the ligament are denoted \( h \) and \( b \), respectively. Depending on the value of \( h \), the model can represent cross-over bridging by single fibres or by beam-like ligaments consisting of several fibres. In the following, we will therefore make no distinction between the two cases; we will call both ligaments bridging. It is assumed that the number of bridging ligaments are uniformly distributed along the crack area and the number of ligaments per unit cracked areas is denoted \( \eta \). As a first approximation, \( \eta \) is taken to be constant, i.e., independent of the actual opening path.

The parameters that specify the problem are: \( \delta_n \), \( \delta_t \), \( h \), \( b \), \( E \), \( G_c \) and \( \eta \). Thus, we can express the nor-
malised bridging stresses as function of the non-
dimensional parameters for a plane problem:
\[
\frac{\sigma_n}{\eta bhE} = f\left(\frac{\delta_n}{h}, \frac{\delta_t}{h}, \frac{\gamma_c}{Eh}\right) \quad \text{and} \\
\frac{\sigma_t}{\eta bhE} = g\left(\frac{\delta_n}{h}, \frac{\delta_t}{h}, \frac{\gamma_c}{Eh}\right),
\]

where \(f\) and \(g\) denote functions that are presently unknown, but are to be determined in the remainder of the paper.

3.2. Prediction of bridging laws

In order to simplify the derivation, we find it convenient to use the half openings of the crack, \(\delta_x\) and \(\delta_y\), defined as
\[
\delta_x = \frac{\delta_t}{2} \quad \delta_y = \frac{\delta_n}{2}.
\]

Furthermore, we define the ligament length as \(2\ell\), see Fig. 7 (the ligament length is not an independent parameter, but must be determined as a part of the solution). In the midpoint of the bridging ligament, the moment vanishes since the plane problem contains 180° rotational symmetry around the midpoint. Then, the only forces at the midpoint are a normal force, \(p_x\) and a transverse force, \(p_y\). The relationship between displacements of the beam end (Fig. 7) and the forces can, by the use of classic beam theory (e.g., Beer and Russell Johnston, 1992), be written as
\[
\delta_x = \frac{p_x \ell}{bhE} \quad \delta_y = 4 \frac{p_y \ell^3}{bh^3 E}.
\]

We treat \(p_x, p_y\) and \(\ell\) as unknowns, with \(\delta_x\) and \(\delta_y\) prescribed. Eq. (10) provide two equations for three unknowns. The third equation for the determination of the third unknown, \(\ell\), is determined by the requirement that the energy release rate of the bridging mechanism equals the fracture energy of the interface, \(\gamma_c\). Following the Mode I micromechanical model of Spearing and Evans (1992), we use the compliance method for this purpose. To do so, we consider the resulting force, \(p\), and the displacement along the resulting force, \(\delta_p\). We can thus write the forces as
\[
p_x = p \sin \alpha \quad p_y = p \cos \alpha,
\]
where \(\alpha\) is the angle of attack of the force (see Fig. 7b). By the use of (10) and (11) we find the displacement in the direction of the force as
\[
\delta_p = 4 \frac{p \ell^3}{bh^3 E} \cos^3 \alpha + \frac{p \ell}{bhE} \sin^2 \alpha.
\]

According to the compliance method, the energy release rate is given by (Broek, 1986)
\[
\gamma = \frac{p^2}{2b} \frac{\partial C}{\partial \ell},
\]
where \(C\) is the compliance defined as the displacement per unit force
\[
C = \frac{\delta_p}{p}
\]
and \(\ell\) is the crack length for our mechanism. Inserting (12) into (14) and performing the differentiation gives
\[
\frac{\partial C}{\partial \ell} = 12 \frac{\ell^2}{bh^3 E} \cos^2 \alpha + \frac{\sin^2 \alpha}{bhE}.
\]

Inserting (15) into (13), while using (11), we obtain...
\[ \mathcal{G} = 6 \frac{p_x \ell^2}{b^2 h^2 E} + \frac{p_y^2}{2 b^2 h^2 E}. \]  
\[ \ell > 0. \]  
\[ \mathcal{G} = \frac{3 G h \delta_0^2}{8 \ell^4} + \frac{E h \delta^2}{2}. \]  
As the ligament peels off, the energy release rate must be identical to the fracture energy of the interface, \( \mathcal{G} \). Setting \( \mathcal{G} \) (from (17)) equal to \( \mathcal{G}_c \) leads to the following equation for the determination of \( \ell \):
\[ 8 \frac{\mathcal{G}_c}{E h} \left( \frac{\ell}{h} \right)^4 - 3 \left( \frac{\delta_x}{h} \right)^2 - 4 \left( \frac{\delta_y}{h} \right)^2 \left( \frac{\ell}{h} \right)^2 = 0. \]  
Eq. (18) can be considered being a second order equation in \((\ell/h)^2\). The only physically admissible solution \((\ell/h)^2 > 0\) is
\[ \left( \frac{\ell}{h} \right)^2 = \left\{ \frac{4 \mathcal{G}_c}{E h} \right\}^{-1} \left\{ \left( \frac{\delta_x}{h} \right)^2 + \sqrt{\left( \frac{\delta_x}{h} \right)^4 + \frac{6 \mathcal{G}_c}{E h} \left( \frac{\delta_y}{h} \right)^2} \right\}. \]  
Eq. (19) is the third equation that \(-\) together with the two equations given by (10) \(-\) are required for the determination of the three unknowns, \( \sigma_n \), \( \sigma_t \), and \( \ell \). Given \( \delta_x \) and \( \delta_y \), we can determine \( \ell/h \) from (19), so that we can calculate the bridging forces from (10).

Using the number of bridging ligaments per unit cracked area, \( \eta \), Eqs. (9), (10) and (19) we obtain the bridging stresses as
\[ \eta \mathcal{G}_c = \sigma_n \left[ \frac{\mathcal{G}_c}{h} \right]^{3/2} \left[ \left( \frac{\delta_x}{2h} \right)^2 + \sqrt{\left( \frac{\delta_x}{2h} \right)^4 + \frac{6 \mathcal{G}_c}{E h} \left( \frac{\delta_y}{2h} \right)^2} \right]^{3/2} \]  
and
\[ \sigma_t = \frac{\eta \mathcal{G}_c}{\eta h E} \left[ \left( \frac{\delta_t}{2h} \right)^2 + \sqrt{\left( \frac{\delta_t}{2h} \right)^4 + \frac{6 \mathcal{G}_c}{E h} \left( \frac{\delta_y}{2h} \right)^2} \right]^{1/2}. \]

Note that both bridging stresses, \( \sigma_n \) and \( \sigma_t \), depend on both \( \delta_x \) and \( \delta_y \). Thus, according to our model, the mechanism of cross-over bridging does not result in decoupled bridging laws; it is a coupled mechanism.

For pure Mode I opening \((\delta_t = 0)\), Eq. (20a) reduces to
\[ \sigma_n = \frac{1}{4} \left\{ \frac{8 \mathcal{G}_c}{E h} \right\}^{3/4} \frac{2h}{\delta_n} bh E \wedge \sigma_t = 0, \]  
i.e., \( \sigma_n \) is inversely proportional to the square root of the normal opening. This result is consistent with the earlier analysis of the pure Mode I fibre cross-over bridging problem when shear deformation is neglected (Spearing and Evans, 1992; Sørensen and Jacobsen, 1998).

Another special case is pure Mode II \((\delta_n = 0)\). Then (20b) becomes
\[ \sigma_t = \left[ \frac{2 \mathcal{G}_c}{E h} \right]^{1/2} \eta h E \wedge \sigma_n = 0. \]
Note, that (22) predicts that the shear stress under pure Mode II is constant, independent of the magnitude of the tangential displacement.

### 3.3. The existence of a potential function

In modelling, bridging laws and cohesive laws are frequently taken to be derived from a potential function (Tvergaard and Hutchinson, 1994). Then, as will be elaborated later, the fracture resistance due to bridging stresses does not depend on the opening path history. It is therefore of interest to investigate whether the bridging stresses of the mixed mode cross-over bridging mechanism can be derived from a potential function, viz.,
\[ \sigma_n = \frac{\partial \Phi}{\partial \delta_n} \wedge \sigma_t = \frac{\partial \Phi}{\partial \delta_t}, \]  
where \( \Phi(\delta_n, \delta_t) \) is the potential function and \( \Phi(0,0) = 0 \). A potential function exists if (Creighton and Buck, 1978), for a simply connected domain in the \( \delta_n-\delta_t \) plane
\[ \frac{\partial \sigma_n}{\partial \delta_t} = \frac{\partial \sigma_t}{\partial \delta_n}. \]  
From (8), we have \( \partial \sigma_n/\partial \delta_t = \partial \sigma_n/2 \partial \delta_x \) and \( \partial \sigma_t/\partial \delta_n = \partial \sigma_t/2 \partial \delta_y \). Then from (20a), we obtain
\[ \frac{\partial \sigma_n}{\partial \delta_t} = -3 \eta h \mathcal{G}_c^{3/2} \left( \frac{\mathcal{G}_c}{E h} \right)^{1/2} \left[ \left( \frac{\delta_x}{2h} \right)^2 + \sqrt{\left( \frac{\delta_x}{2h} \right)^4 + \frac{6 \mathcal{G}_c}{E h} \left( \frac{\delta_y}{2h} \right)^2} \right]^{-1/2} \times \left\{ \left( \frac{\delta_x}{2h} \right)^2 + \sqrt{\left( \frac{\delta_x}{2h} \right)^4 + \frac{6 \mathcal{G}_c}{E h} \left( \frac{\delta_y}{2h} \right)^2} \right\}^{5/2} \]  
while (20b) gives
\[ \sigma_t = -3 \eta b \frac{\phi^{3/2}}{E^{1/2} h^{3/2}} \left\{ \left( \frac{\phi}{2h} \right)^2 + \sqrt{\left( \frac{\phi}{2h} \right)^4 + \frac{\phi_c}{Eh} \left( \frac{\phi}{2h} \right)^2} \right\}^{1/2} \]

It can be shown that (25) is identical to (26). Then, (24) is fulfilled. We have thus proven that for a simply connected domain a potential function exists for the bridging stresses in Eq. (20). Now, we proceed to find the potential function.

We utilise the fact that the potential function is independent of the integration path. We can therefore select the integration path that we find most convenient. We choose to determine the potential function by integrating along the solid lines shown in Fig. 8,

\[ \Phi(\delta_n, \delta_t) = \int_0^{\delta_n} \sigma_n(\delta_n = 0, \delta_t) \, d\delta_t + \int_0^{\delta_t} \sigma_n(\delta_n, \delta_t) \, d\delta_n \]

where \( \delta_n \) and \( \delta_t \) are integration variables.

Inserting \( \sigma_n \) and \( \sigma_t \) from (20) into (27) and performing the integrations leads to the following equation for the potential function

\[ \Phi(\delta_n, \delta_t) = \frac{4}{\eta b h^2 E} \sqrt{\phi_c} \left\{ \left( \frac{\delta_n}{2h} \right)^2 + \sqrt{\left( \frac{\delta_n}{2h} \right)^4 + \frac{\phi_c}{Eh} \left( \frac{\delta_n}{2h} \right)^2} \right\} \]

It can be verified that partial differentiation of the potential function (28) according to (23) indeed gives \( \sigma_n \) and \( \sigma_t \) (Eq. (20)).

3.4. Model results

Fig. 9 shows the predicted bridging stresses, \( \sigma_n \) and \( \sigma_t \) (both normalised) as a function of the normalised openings, \( \delta_n/2h \) and \( \delta_t/2h \) with \( \phi_c/Eh = 10^{-4} \) (other values of \( \phi_c/Eh \) give qualitatively different, but qualitatively similar results). It is seen in Fig. 9 that \( \sigma_n \) decreases rapidly (from a singular value at \( \delta_n = \delta_t = 0 \)), towards zero with increasing \( \delta_n \) and \( \delta_t \). In contrast, \( \sigma_t \) increases rapidly, approaching the bridging stress under pure Mode II, given by (22).
Except for values of \( \delta_i/2h \) smaller than approximately 0.1, \( \sigma_i \) is significantly smaller than \( \sigma_n \).

Using \( \delta_n^0 \) and \( \delta_t^0 \) as (arbitrary) upper bounds for \( \delta_n \) and \( \delta_t \), respectively, the domain defined by \( \{(0 \leq \delta_n < \delta_n^0 \land 0 \leq \delta_t < \delta_t^0) \setminus (0;0)\} \) is simply connected so the potential function \( \Phi \) is valid within the domain. Fig. 10 shows a plot of \( \Phi \) as a function of \( \delta_n \) and \( \delta_t \) for a part of this domain. For pure Mode I opening (\( \delta_t = 0 \)), \( \Phi \) is proportional to \( \sqrt{\delta_n} \). For pure Mode II opening (\( \delta_n = 0 \)), \( \Phi \) increases linearly with \( \delta_t \). This follows from integration of (21) and (22). The mixed mode solution is a smooth transition between the two pure modes. Except for very small openings, the potential function attains a higher value under dominating Mode II (\( \delta_n < \delta_t \)) than under dominating Mode I (\( \delta_n > \delta_t \)).

The potential function depends on the two fracture parameters, \( \eta \) and \( \varphi_c \). From (28) it is clear that \( \Phi \) depends linearly on \( \eta \). However, the shape of the \( \Phi \)-surface does not depend on \( \eta \); it depends only on \( \varphi_c \).

When the bridging stresses are derived from the potential function according to (23), it follows from (2) that the fracture resistance (i.e., the value of the \( J \) integral when cracking takes place) can be obtained as

\[
J_R = \Phi(\delta_n^*, \delta_t^*) + J_0,
\]  

where \( J_0 \) is the fracture energy of the main crack tip. Eq. (29) reads that the fracture resistance comprises the energy dissipation of the bridging zone and the fracture energy of the main crack tip. The energy dissipation of the bridging zone is simply the potential function evaluated at the end-opening, \( \delta_n^* \), and the end-sliding, \( \delta_t^* \), of the bridging zone, see Fig. 1a.

Combining (28) and (29) we can simulate the fracture resistance for the cross-over bridging mechanism. First, define the magnitude of the end-opening, \( \delta^* \) as (shown as a dashed line in Fig. 8)

\[
\delta^* = \sqrt{\delta_n^* + \delta_t^*^2}
\]

and the phase angle of the end-openings as

\[
\varphi = \tan^{-1} \left( \frac{\delta_t^*}{\delta_n^*} \right).
\]

Then, \( \varphi = 0^\circ \) corresponds to pure normal opening (Mode I) and \( \varphi = 90^\circ \) corresponds to pure tangential opening (Mode II).

Predicted fracture resistance, calculated from the potential function according to (29), is shown in Fig. 11. Here, \( J_R = 0 \) and \( \varphi \) is taken to be constant for each curve. As mentioned, for \( \varphi = 0^\circ \), the fracture resistance increases as \( \sqrt{\delta^*} \) in agreement with earlier findings (Sørensen and Jacobsen, 1998). For higher values of \( \varphi \), \( \Phi \) increases almost linearly with \( \delta^* \). For a fixed value of \( \delta^* \), a higher value of \( \varphi \) gives a higher normalised \( \Phi \). The pure Mode II normalised \( \Phi \) is several times that of pure Mode I. Thus, under the assumption of constant \( \eta \) and constant \( h \), the cross-over bridging mechanism is a much more effective toughening mechanism under Mode II than under Mode I.

4. Discussion

4.1. Consequences of the existence of a potential function for the bridging stresses

The fact that the bridging stresses are derived from a potential function has two important consequences. Eq. (29) shows that the energy dissipation of the bridging zone is determined directly from the end-opening and end-sliding. More precisely, the energy dissipation within the bridging zone depends only on the actual end-opening and end-sliding and not on the opening path history. In contrast, if the bridging stresses were not derived from a potential function, the energy dissipation within the bridging zone can only be calculated by recording the stress-opening history for each point within the bridging zone and perform the integration, corresponding to Eq. (2), numerically (van den Bosch et al., 2006).

Furthermore, the fact that the energy dissipation within the bridging zone depends only on \( \delta_n^* \) and \( \delta_t^* \)
has implications for models made by the finite element method. In the finite element method the displacements are determined as the primary variables. Therefore, displacements are computed with a better accuracy than, e.g. stresses and strains that are obtained from displacement gradients. It is therefore anticipated that the energy dissipation within the bridging zone can be computed rather accurately by the use of fairly coarse meshes.

4.2. Criterion for fibre failure

Failure of bridging ligaments and bridging fibres is often observed experimentally. Obviously, breakage of the bridging ligaments precludes the full utilisation of the cross-over bridging mechanism. It is therefore relevant to investigate the conditions under which ligaments fail.

Fibres often fail just at the point where they peel off (Kauta et al., 1993; Shercliff et al., 1994), see also Fig. 5b. Assume that all the fibres possess the same tensile strength, \( \sigma_{fu} \) (a more advanced model could incorporate fibre strength variation and scale effects, e.g. by the use of the Weibull function (Weibull, 1939)). A criterion for the occurrence of fibre failure at the point where the ligament peels off can be made by the use of beam theory. The criterion can be stated as follows: failure is assumed to happen when the maximum tensile stress exceeds the fibre strength, 

\[
\frac{3}{2} \left( \frac{\delta_n}{2h} \right) \left( \frac{h}{\ell} \right)^2 + \left( \frac{\delta_t}{2h} \right) \left( \frac{h}{\ell} \right) \geq \frac{\sigma_{fu}}{E},
\]  

(33)

where \( \ell/h \) is given by (19) while \( \delta_n \) and \( \delta_t \) are known. It follows from (19), with all other parameters fixed, a higher value of \( G_c \) leads to a lower value of \( \ell \), which increases the left hand side of (33). For a sufficiently high \( G_c \), the left hand side will be larger than the right hand side of (33). Then, fibre failures are predicted, and fibre bridging is reduced or hindered.

Further insight can be gained by considering the pure Mode I and pure Mode II crack opening histories, since (33) simplify significantly under these conditions. Under pure Mode I (\( \delta_t = 0 \)), (19) reduces to

\[
\left( \frac{\ell}{h} \right)^2 = \sqrt{\frac{3Eh}{8G_c}} \left( \frac{\delta_n}{2h} \right).
\]

(34)

Inserting (34) into (33) leads to the following criterion for fibre failure

\[
\frac{G_cE}{\sigma_{fu}h} = \frac{1}{6}.
\]

(35)

The left hand side can be regarded as being a non-dimensional fracture parameter. The role of the interface fracture energy, \( G_c \), can be understood as follows. Starting from a sufficiently small value of \( G_c \), the cracking parameter will be lower than the right hand side of (35), so that the fibre failure criterion is not fulfilled. An increase in \( G_c \) increases the cracking parameter. For a sufficiently high \( G_c \), the cracking parameter exceed the right hand side of
Then, fibre fracture is predicted. Note also the effect of the ligament thickness, \( h \). With all other parameter fixed, a smaller value of \( h \) results in a higher value of the cracking parameter. Thus, thinner ligaments (e.g. single fibre ligaments) are anticipated to fail at a lower interface fracture energy value than thicker ligaments. For pure Mode II \((\delta_n = 0)\), (19) simply becomes

\[
\ell = \sqrt{\frac{E h}{2G_c}} \left( \frac{\delta}{2h} \right).
\]

Inserting (36) into (33) gives an equation identical to (35), except that the right hand side, the critical value of the non-dimensional parameter, is three times higher for Mode II than for Mode I. Consequently, with all other parameters fixed, the criterion for fibre breakage will be fulfilled at a lower value of \( G_c \) for Mode I crack opening than for pure Mode II.

### 4.3. Considerations for maximizing fracture resistance of composites

Consider the interplay between the toughening from cross-over bridging and the energy dissipation at the tip of the main crack. As mentioned, if the interface bonding is very strong, the stress in the fibres about to bridge can exceed the fibre strength. Then, fibres fail instead of forming crack bridging. When no fibre bridging develops, the only contribution to toughness comes from the crack tip. On the other hand, if the interface bonding is very weak, a lot of fibre bridging may develop. But the bridging stresses will be very low, since the fibres peel off very easily. Then, the resulting toughening will be low. If fibre bridging is completely absent, the only energy dissipation within the fracture process zone will be the fracture energy of the crack tip, \( J_0 \).

The tip of the main crack is likely to propagate along a fibre/matrix interface just as the crack tips of the bridging ligaments. When mode mixity effects are neglected, \( J_0 \) must be equal to \( G_c \), the fracture energy of the interface. This reasoning suggests that an optimum interface fracture energy exists, which allows fibre bridging with the highest possible stresses without causing significant fibre failure, leading to maximum toughening.

### 4.4. Comments regarding the model assumptions

The present model is planar, so that the only way to distinguish between single fibre bridging and ligament bridging is through the parameter \( G_c/Eh \); a ligament consisting of several fibres has a larger \( h \) than a single fibre. In reality, however, there is also a geometrical difference. Ligaments are expected to possess approximately rectangular cross-sections, while fibres have a circular cross-section. This is not accounted for in the model.

The present model predicts the fracture resistance under the assumption of a constant ligament density, \( \eta \). However, the number of bridging ligaments (per unit area) loaded in tension may differ from the number of bridging ligaments loaded in compression. Moreover, the ratio between the numbers of the two types of ligaments may vary as a function of the ratio between the tangential and normal crack opening displacements.

In the present model, the fracture resistance is predicted using a constant ligament height, \( h \). In reality \( h \) may be a distribution, ranging from single fibres to thicker multi-fibre ligaments. A more refined model could account for the variation in the height of ligaments. This is under investigation in another study (Østergaard and Sørensen, in preparation).

In practice, the deflection of a bridging ligament can be larger than the ligament thickness \( h \), so that the problem becomes a large displacement problem. Our model does not account for large displacements and large rotations. However, since the model predicts that \( \sigma_t \) approaches the Mode II value for \( \delta_t/2h \approx 0.1 \), i.e., within the range of small displacements, it is anticipated that this prediction may also be approximately correct under large displacements. On the other hand, the beam model neglects shear deformation, which may be of importance for very small values of \( \ell/h \). Nevertheless, being based on classic beam theory, our model is expected to be accurate at intermediate crack opening displacements where the majority of the energy dissipation occurs.

Another issue related to large displacements is that large deflections of a bridging ligament result in a shortening (Bisshopp and Drucker, 1945). Thus, under prescribed displacements, a large deflection will induce a tensile stress in the ligament. This stress is not predicted by the present model. A more detailed analysis is required to estimate how large this effect can be.

As mentioned, the model is built upon the assumption that the ligaments peel off at a constant (mode independent) fracture energy, \( G_c \). An approximate analysis (Appendix B) shows that the change
in the mode mixity for ligament bridging is less than
15°, which typically only results in a change of a few
percentages in the macroscopic fracture energy of
interfaces for moderate mode mixities, |ψ| < 60°.
(Cao and Evans, 1989; Wang and Suo, 1990; Liechti
and Chai, 1992).

Another assumption is the use of fracture energy
as the criterion for the propagation of the cracks.
Fracture energy is a linear elastic fracture mechanics
concept. In reality, the fracture process zone at the
interface may not be sufficiently small to justify
the application of linear elasticity. A more advanced
analysis could model the interface fracture process
zone by a cohesive zone.

Finally, a cautious note. Under the assumption
made in the present paper, the bridging laws were
opening path-independent and could thus be
derived from a potential function. However, this
finding may not hold true for more realistic,
advanced models accounting for ligament splitting
and failure, plasticity, etc.

5. Conclusions

A micromechanical model of mixed mode fibre
cross-over bridging predicts coupled mixed mode
bridging laws; both normal and shear stresses
depend on the normal and tangential crack opening
displacements. The normal stress decreases rapidly
towards zero with increasing normal and tangential
crack opening displacements. In contrast, the shear
stress increases with increasing normal and tangen-
tial crack opening displacements, approaching a
constant value, corresponding to the shear stress
under a pure Mode II crack opening displacement.
The toughening due to the cross-over bridging
mechanisms is predicted to be much higher under
Mode II and mixed mode than under pure Mode I.

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Appendix A

The test fixture consists of parts that are made of
non-magnetic materials (e.g. austenitic stainless
steel) in order to reduce the disturbance of the mag-
etic lenses in the ESEM. The fixture consists of
parts that are mounted at the x–y–z table of the
microscope. The specimen and fixture can thus be
moved so that a fairly large area (≈ 50 mm by
50 mm) can be viewed. The opening rate of the fix-
ture is controlled by the operator through the
microscope’s software.

A sketch of the fixture is shown in Fig. A.1. Transverse arms are bonded to the ends of the
beams of the DCB-specimen (1) [numbers in paren-
thesis refer to Fig. A.1]. One of the transverse arms
(2) is held fixed (3); it carries the specimen. The
other arm (4) is loaded by a force acting in the x1-
direction, through a loading arm (5). The loading
arm (5) and the transverse arm (4) are in contact
via a V-notch (6). The loading arm is connected to
a sledge (7), which is mounted by a screw (8) at a
transverse bar (9). The transverse bar (9) is held at
its ends by two 0.2 mm thin steel blades springs
(10) that are mounted at a fixation plate (11) that
can move (by a motor) in the x1-direction. The load-
ing arm (5) is equipped with four strain gauges (12),
which are connected as a Wheatstone Bridge and
used as load cell. The steel blades (10) enable the

Fig. A.1. Schematics drawing of the loading fixture for the
ESEM. (1) test specimen, (2) transverse arm, (3) stationary basis
of fixture, (4) free transverse arm, (5) loading arm, (6) tip of
loading arm, (7) adjustable sledge, (8) fixation screw, (9)
transverse bar, (10) blade springs, (11) moveable basis of fixture,
(12) strain gauges, (13) additional fixation thread holes, (14)
adjustable part of the transverse arm, (15) additional V-notches.
load cell and the transverse bar (9) to flex such that it follows the deflection of the specimen in the \( x_2 \)-direction without inducing a significant force in the \( x_2 \)-direction (the compliance of the fixture towards sideways displacement (i.e. in the \( x_2 \)-direction) was estimated to 2.5 mm/N). The moment arm \( D \) is changed by adjusting the sledge (7) in the \( x_2 \)-direction – using fixation point (8) or one of the two additional fixation points (13) and moving the adjustable part of the transverse arm (14), using the V-notch (6) or one of the two other V-notches (15). The maximum allowable force is 250 N.

### Appendix B

In this Appendix, we estimate the mode mixity of a ligament being peeled off. We utilize a solution given by Suo and Hutchinson (1990). Their problem consists of asymmetric bimaterial specimens loaded with an axial force and a bending moment (the same loading as in Fig. 2). Our problem consists of an axial force, \( p_x \), and a transverse force, \( p_y \). The latter creates a bending moment at the crack tip. We consider two cases. In the first case, the ligament length, \( 2\ell \), is very small. In the second case, \( \ell \) is large. When \( \ell \) is small, then, according to our model, \( p_x \ll p_y \), so that the moment is the dominating crack tip loading. When \( \ell \) is large, our model predicts \( p_x \gg p_y \), i.e., the axial force dominates.

Now, we apply the solution given by Suo and Hutchinson (1990). When the moment dominates the crack tip loading (short \( \ell \)) we obtain (assuming \( h \ll H \) and assuming the same elastic properties of the ligament as that of the thick part)

\[
\psi = \tan^{-1} \left( \frac{1}{\tan \omega} \right),
\]  
(B.1)

where \( \omega \) is a phase angle tabulated by Suo and Hutchinson (1990). For large \( \ell \), we simply have

\[
\psi = \omega.
\]  
(B.2)

From Table 1 in Suo and Hutchinson (1990) we obtain \( \omega = 52.1^\circ \). Then, by (B.1) and (B.2) we find \( \psi = 37.9^\circ \) for short \( \ell \), and \( \psi = 52.1^\circ \) for large \( \ell \).

### References


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