Investigation of Submarine Roll Behaviour

Performed at ASC Pty Ltd, Adelaide

Sara Hedberg

Adelaide December 2006
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Foreword

This report is a Masters Thesis at the Royal Institute of Technology, KTH, The School of Science, Centre for Naval Architecture. It was performed at ASC Pty Ltd in Adelaide, Australia between July and December 2006.

The author would like to thank:
ASC for giving the author this once in a life time opportunity to work and live in a foreign country while performing her Masters Thesis.
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Sara Hedberg
Summary. The Thesis purpose is to investigate how to simulate the roll motion of a submarine. The Australian Collins Class submarine have been used as a benchmark submarine, but as the data for it is confidential a Collins Class like mock-up submarine have been made to perform the investigations. The Equation of Motion is analysed and solved for the artificial submarine. Two investigation techniques are used and compared, linear analytical in the frequency domain and nonlinear numerical in the time domain. The two methods are used to study how different sea states, adding appendages and varying the transverse metacentre height will affect the roll angle of the submarine whilst surfaced. MATLAB \cite{7} is the main tool for the simulations.

When going from the linear to the nonlinear approach the biggest difference is found in the stationary case where the linear case under predicts the damping of the system, but besides that no significant advantage by using the time consuming nonlinear time approach was found.

Den största skillnaden som fanns vid övergång från linjär till olinjär metod fanns i det stationära fallet där den linjära metoden underestimerar dämpningen av systemet. Förutom den skillnaden fanns ingen större fördel med att använda den tidskrävande olinjära tidsdomänmetoden.
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<th>Description</th>
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</tr>
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<td>$A_{44}$</td>
<td>$m$</td>
<td>Added mass in roll</td>
</tr>
<tr>
<td>$B_{44}$</td>
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<td>$B_{442}$</td>
<td>$m$</td>
<td>Nonlinear damping in roll, viscous forces</td>
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<td>Distance between metacentre and centre of buoyancy</td>
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<td>$m$</td>
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<td>$\bar{c}$</td>
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<td>Moment of inertia in roll</td>
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### Nomenclature

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<th>Unit</th>
<th>Description</th>
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<td>1/m</td>
<td>Wave number</td>
</tr>
<tr>
<td>$K_{GT}$</td>
<td>m</td>
<td>Distance between centre of buoyancy and keel</td>
</tr>
<tr>
<td>$l_0$</td>
<td>m</td>
<td>Length of appendage under surface</td>
</tr>
<tr>
<td>$l_c$</td>
<td>m</td>
<td>Length between appendages</td>
</tr>
<tr>
<td>$l_d$</td>
<td>m</td>
<td>Distance from sub. centre to surface</td>
</tr>
<tr>
<td>$m_0$</td>
<td>rad$^2$</td>
<td>Variance</td>
</tr>
<tr>
<td>$\dot{\omega}_i$</td>
<td>rad/sec</td>
<td>Frequency step</td>
</tr>
<tr>
<td>$\rho$</td>
<td>kg/m$^3$</td>
<td>Water density</td>
</tr>
<tr>
<td>$r_A$</td>
<td>m</td>
<td>Length from centre of rotation to centre of appendage</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>rad</td>
<td>Root Mean Square (RMS)</td>
</tr>
<tr>
<td>$T_0$</td>
<td>sec</td>
<td>Modal period</td>
</tr>
<tr>
<td>$\bar{T}$</td>
<td>sec</td>
<td>Mean wave period</td>
</tr>
<tr>
<td>$T_z$</td>
<td>sec</td>
<td>Mean zero crossing period</td>
</tr>
<tr>
<td>$U$</td>
<td>m/sec</td>
<td>Ship speed</td>
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<tr>
<td>$\zeta_0$</td>
<td>m</td>
<td>Wave amplitude</td>
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Introduction

1.1 Submarines in Australia

In 1914 the Australian navy commissioned two submarines, AE1 and AE2. They were used during WW1 but were both lost during combat. In 1919 they acquired two J class submarines from the British navy, these submarines were in such bad condition they had to be refurbished for a considerable sum of money at their arrival in Sydney. Even though a lot of money were spent on them, they were not used for more than a few missions and all of them except J7 were decommissioned in 1924. After WW1 The Australian Navy had a development program where in 1929 two submarines were ordered from the British Navy, HMAS Otway and HMAS Oxley, but due to the depression and maintenance problems they where returned to Britain in 1931. The ex-Dutch submarine K9 was bought during WW2 to train The Australian Navy in submarine detection. The beginning of The Australian Royal Navy submarine fleet did not start until the four Oberon class submarines were commissioned between 1967 and 1969. Even though the Oberon subs never went in to combat they were a big asset as their presence had a deterring effect on the enemies of the state. In the end of the 70's two more submarines where commissioned. In 1982 The Royal Australian Navy initiated the Collins Class project. These submarines were to be custom made to fit the Australian navy and the geographical demands of Australia. The decision was made that the submarines would be large conventional submarines, and they are now the second largest non-nuclear submarines in the world. The first parts of the first boat were made in 1990 and the sixth and final boat was delivered in 2003 [1].

The current task for ASC is to for the next 23 years to design, maintain and enhance the submarines during their operational life time.
1.2 Task

The overall task with this thesis is to examine how to model the roll characteristics of a surfaced submarine, through both linear and nonlinear approach, and how these are affected by different sea states, adding appendages and varying the transverse metacentric height.

Task background

The Collins class submarines are diesel-electric submarines, and one of the biggest of their kind. To design a submarine it is important to calculate the wave forces on the hull, not just the strength of the hull but also how the submarine behaves under certain wave conditions. For any submarine it is not desired to be on the surface often due to the risk for detection by enemy vessels. Whether wanted or not, they do have to surface from time to time and it is during a surfacing most important that the submarine can cope with the environment [10]. One issue is how the submarine behaves in roll motion, and what can be done to dampen these motions if they grow too big. If a submarine rolls to large angles, which is not unusual, it is not just uncomfortable for the crew, but equipment can break loose and cause big damage on board.

1.3 Method

The 6 degree of freedom Equation of Motion is studied and simplified to suit the case investigated. All non-linear coefficients \(^1\) are at first linearised so the equation can be studied in the frequency domain. As a comparison it is solved for in the time domain. Based on the result for the linearised case some coefficients are allowed to become non-linear once again and the equation is purely investigated in the time domain. Coefficients are also modified by introducing bilge keels and varying the \(GM_T\). To perform numerical calculation the matrix based mathematical program MATLAB \([7]\) is used. MathCAD \([6]\) is utilised for symbolic calculations and some formatting. Before any large calculations were made, rough estimates were done by hand.

1.4 Boundaries

The projects physical boundaries are the roll motions of the submarine in water on the surface. The roll is solely due to waves and not the snap-roll

\(^1\) The non-linearity is introduced as the coefficients change with wave frequency and sea state
that occurs when submarine is turning. Neither parametric rolling nor active control is examined or calculated.
The Physical Model

In order to investigate the roll motions, physical properties of the submarine such as dimensions, mass properties, metacentric height are required. For the Collins Class a number of these values are sensitive so a benchmark submarine was developed with dimensions and appendage similar to the Collins Class.

2.1 Benchmark

The outer dimensions of the Collins class submarines are 78 m long, a diameter of 8 metres and a displacement of 3000 tonnes. They have four aft control surfaces in an X-configuration (see left Figure 2.1), which is one of the most common ways to assemble the planes. Another common configuration is the Cross configurations which some of the American submarines have (See right Figure 2.1). The advantage with the X-configuration is that when the angle between two actuators at each side is approximately 90 degrees, the angling makes it possible to get a much larger span for each surface while not exceeding the limits of the box defined by the maximum beam and draft of the hull. One of the drawbacks with the configuration is that the forces generated in both vertical and horizontal directions are symmetric and there is little scope for independent configuration for either of them to enhance the steering [10].
2.2 Dimensions

Assuming the length of 100 m and a diameter of 10 m for simplicity, the other properties were developed by looking at the benchmark submarine, and the final dimensions of the imaginary submarine are as in Figure 2.3, Figure 2.4 and Figure 2.2. Through these the displacement, the centre of buoyancy, $BM$ and $KBT$ are calculated. Assuming a natural roll period of 10 seconds $G$ is positioned, see Appendix A. 10 seconds were chosen since it is a plausible natural roll period for submarines, the normal period for a submarine is between 10-16 seconds but there are some submarines that have periods significantly larger.

The actuator planes are assumed to have a NACA0012 shape (Figure 2.6), and the fin a NACA 0020 shape (Figure 2.7). Assuming these values for the surfaced ship the calculated figures are as in Table 2.1 and Figure 2.3. To see the calculation of these values see Appendix A.

<table>
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<tr>
<td>$VCB$</td>
<td>7m fore of $L/2$</td>
</tr>
<tr>
<td>$LCB$</td>
<td>0.41m under CL</td>
</tr>
<tr>
<td>$KB$</td>
<td>4.59m</td>
</tr>
<tr>
<td>$GM$</td>
<td>0.67m</td>
</tr>
<tr>
<td>$BM$</td>
<td>0.35m</td>
</tr>
</tbody>
</table>
2.2 Dimensions

Fig. 2.2. Distances for Table 2.1

Fig. 2.3. Submarine dimensions

Fig. 2.4. Aft actuator dimensions
**Fig. 2.5.** Actuator size

**Fig. 2.6.** The NACA0012 shape
Fig. 2.7. The NACA0020 shape
The Mathematical Model

As for most engineering problems the mathematical model is just an idealised version of the real world. A range of simplifications are made which reduces the problem but also makes it less complex. The general approach for the model is; do we know that a property is hard to calculate; we make the model so that the property is negligible.

A real submarine has got a rounded hull with a tapered end where the propeller, the aft rudders and stabilisers are. On top of the circular section there is a superstructure where sonar and pipe systems are fitted and there is also a fin. The submarine is starboard port symmetric (It is normal to neglect the asymmetry created by the propeller rotation [5]) but not forward aft symmetric like a canoe. See Figure 3.1. The submarine is modelled as a simple cylinder with constant cross section, though it keeps the properties of the tapered submarine such as buoyancy, weight and centre of gravity. For calculations of added masses there are no appendices, but for calculations of the damping the appendices are present. The fin can be neglected when the submarine is at the surface and the rolling angles are small enough so that the fin is not breaching the water. Normally the boat fin does not breach the water; and the goal is to have smaller roll angles. If the roll angles are so large that the fin breaches the water the effects are considered small enough to be neglected. See Figure 3.2 and Figure 3.3.
3.1 Theory

A submarine can experience large rolling angles in severe sea states and the submarine can handle it if it is fully functional, but mostly it dives to get out of similar surroundings. The speed is from zero when hovering, around ten knots when snorkeling and up to 20 knots when submerged. The forward speed has some influence on the roll damping depending on lift from appendages and skin friction. The directions of motions for a submarine are described in Figure 3.4. Note that it is different from the directions for a standard ship; the vertical axis is directed down into the sea instead of up.

The equation of motion for ships is based on Newton’s second law:

\[ F = m \ddot{\eta} \] (3.1)

The righthand side is the inertia force and the force at the lefthand side is made up by the properties of the ship, such as hydrostatic forces, wave
3.1 Theory

![Coordinate system](image)

**Fig. 3.4. Coordinate system**

excitation forces, added inertia forces and hydrodynamic damping forces, see Equation 3.2

\[ F = F_{\text{static}} + F_{\text{excitation}} + F_{\text{added}} + F_{\text{damping}} \]

These forces are evaluated for the ship and they can be both linear and nonlinear\(^1\). These dependencies make the equation of motion nonlinear, and there have been many empirical and analytical analysis to find the proper relations. The most common way to express the equation of motion is through equation 3.3, where \( M \) is the mass or moment of inertia, \( A \) is the added mass, \( B \) is a damping coefficient and \( C \) is a hydrostatic coefficient.

\[
(M_{jk} + A_{jk}) \ddot{\eta}_k + B_{jk} \dot{\eta}_k + C_{jk} \eta_k = F_{0j}(\omega t) \quad \text{where} \quad j, k = 1 \ldots 6 \quad (3.3)
\]

A ship with starboard/port symmetry has no coupling between the lateral and vertical movements, so surge (\( \eta_1 \)), heave (\( \eta_2 \)) pitch (\( \eta_5 \)) is decoupled from sway (\( \eta_2 \)), roll (\( \eta_4 \)) and yaw (\( \eta_6 \)). These can be treated as two separate systems of equations, which is the case for a submarine.

As roll is the contemplated direction of motion we are most interested in the sway, roll and yaw changes; which are lightly dynamically connected [5].

Parametric roll is the roll that a ship can experience when the waves are of the same size as the ship, the trough and the wave crest are alternating in such a way so the metacentre height is changing for the ship, and this creates

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\(^1\) Dependent of the wave frequency and boat speed squared
a rolling movement. This type of roll investigation needs a nonlinear approach where $GM$ is dependent on the wave frequency. Parametric roll is not further investigated in this project.

A number of common assumptions have been employed. One is that all free surfaces are neglected. Free surfaces in a submarine can occur when it is surfacing and there is water pouring out of the casing, or when the tanks are filling up or emptying. It will also be assumed that the waves the submarine is exposed to are linear, even though normal waves are quite nonlinear. The submarine is also understood to have a slender body, where the beam is much smaller than the length; the slender-body approximation states that the derivatives in the longitudinal direction are small compared to the derivatives in the transverse direction [4] which is the same as that the slope of the boat hull is less in the longitudinal than in the transverse direction (see Figure 3.5) which implies that the structure can be split up into strips where every slice is solved for in two dimensions. The body is also said to be rigid, it will not change shape as the wave forces strike. Effects to wind have also been neglected. The coupling between sway, roll and yaw is weak and dynamic and for the most part all coupling terms are neglected. An explanation to this is that for example in beam waves there is practically no yaw motion. As for sway, even though there is a significant coupling, it does not alter the basic second order spring mass system characteristics of the roll motion [5].

The model geometry is starboard/port symmetric and front aft symmetric, not considering the actuator planes. A fore aft symmetry will be considered, and even though this is not true for a submarine, the assumption will not cause significant errors due to that the effect of the asymmetry mostly is negligible [5].

For the part of the equation of motion where roll is involved, there are three coupled equations (see Equations 3.4 to 3.6) for which in the ensuing section the coefficients are reviewed.
\[ F_{20}(\omega t) = (M_{22} + A_{22})\ddot{\eta}_2 + (M_{24} + A_{24})\ddot{\eta}_4 + (M_{26} + A_{26})\ddot{\eta}_6 + B_{22}\ddot{\eta}_2 + \\
+ B_{24}\dot{\eta}_4 + B_{26}\dot{\eta}_6 + C_{22}\ddot{\eta}_2 + C_{24}\ddot{\eta}_4 + C_{26}\ddot{\eta}_6 \] (3.4)

\[ F_{40}(\omega t) = (M_{42} + A_{42})\ddot{\eta}_2 + (M_{44} + A_{44})\ddot{\eta}_4 + (M_{46} + A_{46})\ddot{\eta}_6 + B_{42}\ddot{\eta}_2 + \\
+ B_{44}\dot{\eta}_4 + B_{46}\dot{\eta}_6 + C_{42}\ddot{\eta}_2 + C_{44}\ddot{\eta}_4 + C_{46}\ddot{\eta}_6 \] (3.5)

\[ F_{60}(\omega t) = (M_{62} + A_{62})\ddot{\eta}_2 + (M_{64} + A_{64})\ddot{\eta}_4 + (M_{66} + A_{66})\ddot{\eta}_6 + B_{62}\ddot{\eta}_2 + \\
+ B_{64}\dot{\eta}_4 + B_{66}\dot{\eta}_6 + C_{62}\ddot{\eta}_2 + C_{64}\ddot{\eta}_4 + C_{66}\ddot{\eta}_6 \] (3.6)

As roll is the motion to be investigated Equation 3.5 (the second of the three equations) is where most information will be found. As discussed previously all coupling has been neglected since the objective is to study roll motions. Calculating cross coupling terms would be too time consuming to be performed for this Thesis.

### 3.1.1 Restoring Forces

Coefficient \( C_{42} \) in Equation 3.5 is immediately set to zero due to geography, as there is no restoring forces in sway, yaw and surge and no connection between roll and sway. The coefficient \( C_{46} \) connects roll with yaw, which is a weak connection for a fore/aft symmetric ship and consequently it is omitted.

The hydrostatic coefficient is found by geometry. A ship’s static floating height is decided by the principle of Archimedes. In roll the ship is pushed further into the water by a lever, which is related to the initial metacentre.

Through contemplating Figure 3.6 Equation 3.7 is found.

\[ F_{stat} = -\rho g GZ \nabla \] (3.7)

The hydrostatic lever is found by assuming that the angle is small.

\[ \sin \eta = \frac{GZ}{GM_0} \rightarrow GZ = GM_0 \sin \eta \rightarrow \{ \eta \approx \text{small} \} \rightarrow GZ = GM_0 \eta \] (3.8)

Combining Equations 3.8 and 3.7 and using the notation from Figure 3.4 we get the relationship:

\[ F_{stat} = -\rho g \nabla GM_0 \eta_4 \] (3.9)

Comparing this to Equation 3.5 the \( C_{44} \) coefficient can be set to be as in Equation 3.10

\[ C_{44} = \rho g \nabla GM_0 \] (3.10)
When the nonlinear case is considered the angle $\eta_4$ is not said to be small and the $\sin(\eta_4)$ is kept. In a normal ship this would not give the proper $GZ$ curve, but for a submarine with a circular hull this is applicable (see Figure 3.7). This is true because the $GM_T$ does not change drastically when the submarine leans over as it does when a ship with a square, or non-circular, hull tips over.
3.1.2 Excitation Forces

There are two kinds of excitation forces, first order and second order. The first order excitation forces are made up by the zero mean oscillatory forces caused by the waves which can divided up into two parts. The first one is called Froude-Kriloff forces which are forces due to the incident waves under the assumption that the hull is restrained from moving and that the presence of the hull does not disturb the flow field. The second one is called diffraction forces and is a correction for the modification on the flow field due to the hull, without it there would be a mass transfer through the hull. The second order forces include wave-drift loads, slowly varying and rapidly varying wave loads [8]. In this thesis only forces due to incident waves are treated. This is possible due to three facts; for low frequencies the diffraction part is very small, for high frequencies the wave force amplitude remains almost equal to the amplitude for incident waves [3] and the other forces does not alter the investigation technique significantly.

When the equation of motion is linearised the wave excitations can be allowed to be considered independent of any ship motions and can be expressed as a function of the wave amplitude alone. The wave depression can at any point be expressed as

\[ \zeta = \zeta_0 \sin(\omega_e t - kx \cos(\mu) + ky \sin(\mu)) \quad m \]  \hspace{1cm} (3.11)

Where \( \omega_e \) is the frequency of encounter, but for a submarine rolling in beam seas the frequency of encounter is the same as the wave frequency and the submarine does not translate either forward or sideways so it can be expressed simply as

\[ \zeta = \zeta_0 \sin(\omega t) \quad m \]  \hspace{1cm} (3.12)

When the submarine submerges the Equation 3.12 needs to be related to the depth for which the submarine is travelling. To find that expression it is convenient to study the surface profile in Figure 3.8. By using the Bernoulli equation and assuming that the wave velocity is small (find derivation in Appendix C) the expression in Equation 3.13 for the compression of the contour is found

\[ \zeta_p = \zeta_0 e^{-kz_p} \sin(kx - \omega t) \quad m \]  \hspace{1cm} (3.13)

By simply assuming \( z_p = 0 \) and no translation \( x = 0 \) Equation 3.13 becomes Equation 3.12. The actual force \( F_{40} \) is expressed by assuming that for a long wave the static forces dominate. The slope of the wave can be compared to the angle of the boat as seen in Figure 3.9. The rolling moment can then be expressed as in Equation 3.14.
Fig. 3.8. Constant pressure beneath a regular wave.

Fig. 3.9. Schematic illustration of a section of a ship in fixed sea for long and short waves.
Calculating the differential of $\zeta$ from Equation 3.13 and doing this in the ships centerline ($x = 0$) we get Equation 3.15
\[
\frac{d\zeta}{dx} = k\zeta_0 \cos(\omega t)
\] (3.15)

Equations 3.14 and 3.15 put together results in Equation 3.16.
\[
F_{40}^{ex} = C_{44} k \zeta_0 \cos(\omega t)
\] (3.16)

### 3.1.3 Inertia forces

The added mass for a cylinder, totally submerged with the centre of gravity in the centre is zero, and since $GM_0$ for a submarine is small, so $A_{44}$ can be considered zero. The other added masses in the roll equation are small from the start and hence neglected.

The products of inertia $I_{42}$ and $I_{46}$ in Equation 3.5 are also set to zero due to that connection to sway and yaw is weak as the mass is evenly spread throughout the vehicle. Due to $I_{44}$’s size it is of importance to the equation of motion. For most calculations it can be set to:
\[
I_{44} = k^2 M
\] (3.17)

Where it is common to set [2]
\[
k = \frac{\sqrt{B^2 + D^2}}{2\sqrt{3}}
\] (3.18)

Where $B$ in this case is the width of the ship, which for the cylinder is the diameter, $D$ is the depth of the ship and $M$ is the mass in kilograms of the submarine. This depth is not the same thing as the draught for a submarine, but rather a measure of how deep it can go and to produce the right figure the diameter $D$ is used.

The damping of the system will be very small compared to the moment of inertia so it is quite important to get the inertia right. It can take a lot of time to calculate the proper moment of inertia for a submarine, as the book of weights is a very large book; therefore the moment of inertia is approximated as shown above.
3.1.4 Damping forces

Coefficients $B_{42}$ and $B_{46}$ in Equation 3.5 are neglected as they are not possible to calculate for this Thesis.

Damping is what makes an oscillating system to slow down and maybe eventually stop. Without damping a spring system would after being disturbed by some force oscillate for infinitely long time. For a ship roll damping can be induced by four categories; wave making damping, eddy roll damping, skin friction damping and appendage roll damping. Normally the damping is nonlinear and can be expressed as in Equation 3.19 [3]:

$$M = B_{441} \dot{\eta}_4 + B_{442} | \dot{\eta}_4 | \dot{\eta}_4 + B_{443} \dot{\eta}_4^3$$  \hspace{1cm} (3.19)

The third order term will not be used as the method used in this Thesis based on [5] has not got a third order term in it, this is due to that the third order terms that could be included have been simplified down to second order terms. The third order terms could be present if the motions are very nonlinear and could occur due to the speed dependance for Reynolds numbers. Equation 3.19 will then express $B_{44} \dot{\eta}_4$ from Equation 3.5 as two parts: $B_{441} \dot{\eta}_4 + B_{442} | \dot{\eta}_4 | \dot{\eta}_4$.

Wave making roll damping and the damping due to appendage forces at high forward speed are linear, but viscous damping is not. The viscous damping at zero speed is caused by skin friction, eddies and appendage forces. This means that $B_{441}$ is made up by the wave making and roll damping forces and $B_{442}$ takes care of the viscous forces. The damping can be approximately calculated in a linear way if it is considered as related to the dissipated energy it causes [5]. This dissipated energy is mostly generated at the natural roll frequency $\omega_{4r}$, and hence the calculations are based on that particular frequency and damping is considered to be constant for all frequencies. The damping coefficient $B_{44}$ in Equation 3.5 is calculated separately for each of the different damping categories and later added together as in Equation 3.20.

$$B_{44} = B_{44wm} + B_{44er} + B_{44sf} + B_{44ad} + B_{44al}$$  \hspace{1cm} (3.20)

When the nonlinear solutions is calculated the equation of motion is kept as it is and it will still be considered decoupled from the other lateral motions but with the nonlinear components as in Equation 3.19.

The damping for the linear case is accordingly based upon the dissipated energy around the natural roll frequency [5]. The natural roll frequency can be found by using the coefficients from the simplified undamped equation of motion seen in Equation 3.21.
\[ \ddot{\eta}_4 + \omega_{4e}^2 \eta_4 = \frac{F_{40}}{I_{44} + A_{44}} \cos(\omega t) \quad (3.21) \]

where
\[ \omega_{4e} = \sqrt{\frac{C_{44}}{I_{44} + A_{44}}} \quad (3.22) \]

It is supposed that the rolling motion is given by
\[ \eta_4 = \eta_{40} \sin(\omega_{4e} t) \quad (3.23) \]

Where the amplitude \( \eta_{40} \) is based upon the solution and is hence slightly non-linear.\(^2\)

The roll damping moment around the centre of gravity is \( B_{44} \dot{\eta}_4 \), and in one roll period the work done and the energy dissipated by this linearised damping term will be the integral of the moments times the angular distance moved.
\[ E = 4 \int_{0}^{\eta_{40}} B_{44} \dot{\eta}_4 d\eta_4 \quad (3.24) \]

Equation 3.24 is transformed to the time domain through Equation 3.23
\[ \begin{cases} \eta_4 : 0 \rightarrow \eta_{40} \\ t : 0 \rightarrow \frac{\pi}{2\omega_{4e}} \end{cases} \text{ and } \dot{\eta}_4 = \frac{d\eta_4}{dt} = \eta_{40} \omega_{4e} \cos(\omega_{4e} t) \rightarrow \\
\rightarrow d\eta_4 = \eta_{40} \omega_{4e} \cos(\omega_{4e} t) dt \quad (3.25) \]

And the integral simplifies to
\[ E = 4 \int_{0}^{\frac{\pi}{2\omega_{4e}}} B_{44} \omega_{4e}^2 \eta_{40}^2 \cos^2(\omega_{4e} t) dt = 4B_{44} \omega_{4e}^2 \eta_{40}^2 \frac{\pi}{2\omega_{4e}} = B_{44} \omega_{4e} \eta_{40}^2 \frac{\pi}{2} \quad (3.26) \]

\[ \Rightarrow B_{44} = \frac{E}{\omega_{4e} \eta_{40}^2 \frac{\pi}{2}} \quad (3.27) \]

The following section will explain how this dissipated energy \( E \) is calculated for the different damping sources, and how the corresponding damping coefficient \( B_{44} \) is expressed.

**Wave making damping**

The wave making damping predicted by the potential flow calculations around a cylindrical hull form is only a small fraction of the total roll damping. And as it contributes so little it has been neglected.

\(^2\) This may show in the results but the difference is so small that it is not further investigated.
\[ B_{44\text{ew}} = 0 \] (3.28)

**Eddy roll damping**

Eddy roll damping is due to the formation of eddies from the shape of the hull and from the additions to the hull as struts, docking keels etc. For a round bilge section the eddy shedding roll damping is negligible, and hence that part of the damping can be set to zero [5].

\[ B_{44\text{er}} = 0 \] (3.29)

**Skin friction roll damping**

When water flows past a ship hull it creates frictional forces on the surface, and it is common to express these forces with a non dimensional local skin friction coefficient defined as

\[ C_F = \frac{F}{\frac{1}{2} \rho U^2 A} \] (3.30)

Where \( F \) is the frictional force on the element, \( \rho \) is the density of water, \( U \) is the local velocity and \( A \) is the area of the element. If the hull surface is divided into elements \( \delta s \) the friction force acting on it can be expressed as

\[ \delta F = C_F \frac{1}{2} \rho (r\dot{\theta}_1)^2 \delta s \delta \eta_1 \] (3.31)

And the moment about the centre of gravity is (we assume that the centre of gravity is in the centre of the circle for simplicity. We also assume that the submarine rotates around the centre of gravity though this is not completely true; it actually rotates around a point between the metacentric height and the centre of gravity which is frequency dependant, but this is a reasonable assumption as the distances are small)

\[ \delta F_4 = C_F \frac{1}{2} \rho r^3 \dot{\theta}_1^2 \delta s \delta \eta_1 \] (3.32)

For the nonlinear case this would for our simplified model become:

\[ F_4 = C_F \frac{1}{2} \rho r^3 A L S \dot{\eta}_1^2 \] (3.33)

where \( L \) is the length of the submarine and \( S \) is the wetted surface. The values in front of \( \dot{\eta}_1 \) can be identified as a part of \( B_{442} \) in Equation 3.19.

To linearise Equation 3.32, consider the work done by the moment in a complete roll cycle
3.1 Theory

\[ E = 4 \int_0^{\eta_0} \delta F \delta \eta_4 \]  
(3.34)

Using Equations 3.23, 3.32 and 3.34 this becomes

\[ E = 4 \int_0^{\eta_0} C_F \frac{1}{2} \rho r^3 \eta_2^2 \delta \tilde{s} \delta \eta_4 \]  
(3.35)

Which transformed in the same way as Equation 3.26 becomes

\[ E = 4 \int_0^{\pi / 2} C_F \frac{1}{2} \rho r^3 \omega_4^2 \eta_4^3 \cos^2 (\omega_4 t) \delta \tilde{s} \delta \eta_1 \, dt \Rightarrow \]
\[ \Rightarrow E = 4C_F \frac{1}{2} \rho r^3 \frac{2}{3} \omega_4^2 \delta \tilde{s} \delta \eta_1 \]  
(3.36)

As the submarine is assumed to have the shape of a cylinder, and the centre of gravity is close to the cylinder centre, the \( \delta \tilde{s} \delta \eta_1 \) part is just the surface area of the cylinder under the water surface. The hull surface area is calculated by looking at Figure 3.10 and the knowledge that \( s = \alpha r \). The angle \( \beta \) is calculated by using the triangle with sides 3.5m and 5m, and to get the angle \( \alpha \) it is a simple matter by multiplying by two and adding \( \pi \). Then;

\[ \delta \tilde{s} \delta \eta_1 = r \left( \pi + 2 \arcsin \left( \frac{3.5}{5} \right) \right) L \]  
(3.37)

Where \( L \) is the length of the submarine. If the submarine model had not been cylindrical we had had to integrate over the submarine in slices instead of the multiplication with \( L \). This finally simplifies to

\[ B_{44f} = \frac{E}{\pi \omega_4 \eta_4^4} = C_F \frac{4\pi}{3} \rho r^4 \omega_4 \eta_4 L \left( \pi + 2 \arcsin \left( \frac{3.5}{5} \right) \right) \]  
(3.38)
Appendage roll damping

The submarine has got four aft actuator planes, in the shape as in Figure 2.2. These will give two types of damping. The first one is drag induced by the rolling movement when the boat has no speed and the other is the lift that occurs when the ship has forward speed.

If the submarine is surfaced, some of the two upper appendages will be above the surface, as shown in Figure 3.11 with dimensions in Table 3.1. The distance $l_0$ is calculated by

$$\sin(\phi) = \frac{l_d}{l_c/2 + l_0} \Rightarrow l_0 = 3.5\sqrt{2} - 1.25 \approx 3.7m$$

For simplicity it is assumed that the appendage surface above the water is constant even though the submarine is rolling and hence going in and out of the water. It is also assumed that the submarine rotates about the centre of gravity which is assumed to be in the centre of the cylinder. The roll motion will induce a transverse velocity to the appendage, which creates a drag force. This drag force is based on the lever from the centre of rotation times the roll velocity:

$$F_D = C_D \frac{1}{2} \rho (r_A \dot{\eta}_k)^2 A_A$$

![Diagram of submarine appendages with dimensions](image-url)
3.1 Theory

Where $C_D$ is the non-dimensional drag coefficient (can for appendages as rudders be set to 1.17 [5]), $r_A$ is defined as in Figure 3.12 and $\rho$ is the density of the surrounding water. The drag force creates a roll damping moment as

\[
F_4 = C_D \frac{1}{2} \rho r_A^3 \dot{\eta}_4^2 A_A
\]  (3.41)

Which is a part of $B_{442}$ in Equation 3.19. The energy dissipated by one roll cycle is

\[
E = 4 \int_{0}^{\eta_{40}} F_4 d\eta_4
\]  (3.42)

Using Equation 3.41
\[
E = \frac{4}{3} C_D \rho A_A r_A^3 \dot{\eta}_{40}^2 \omega^2 e
\]  (3.43)

Together with Equation 3.27 this becomes
\[
B_{44ad} = \frac{4}{3\pi} C_D \rho \eta_{40} \omega^2 e \sum A_A r_A^3
\]  (3.44)

The sum in Equation 3.44 is made up by the four appendages, where two of them are partly above the surface and hence have both smaller areas and levers $r_A$. To calculate the values of $r_A$ and $A_A$ (See Equations 3.45-3.48) see Figures 3.13 and 3.14, with the lengths defined as on Table 3.2.

**Table 3.2. Dimensions**

<table>
<thead>
<tr>
<th>$c_t$</th>
<th>$3m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_r$</td>
<td>$4m$</td>
</tr>
<tr>
<td>$l_A$</td>
<td>$5.5m$</td>
</tr>
<tr>
<td>$l_e$</td>
<td>$2.5m$</td>
</tr>
</tbody>
</table>
Fig. 3.13. Appendage at surface

Fig. 3.14. Dimensions at the aft end of submarine

\[ r_{Aupper} = \frac{l_c}{2} + \frac{l_0}{2} = 1.25 + \frac{3.5}{2} = 3.1 \text{ m} \quad (3.45) \]

\[ r_{Alower} = \frac{l_c}{2} + \frac{l_A}{2} = 1.25 + \frac{5.5}{2} = 4 \text{ m} \quad (3.46) \]

\[ A_{Aupper} = \frac{c + c_r}{2} l_0 = \frac{\left( \frac{c_r - c}{c} \left( l_A - l_0 \right) 2 + c_t \right)}{2} l_0 = \]
\[ = \frac{0.5 (5.5 - l_0) 2 + 3}{2} + 4 = 13.6 \text{ m}^2 \quad (3.47) \]

\[ A_{Alower} = \frac{c_t + c_r}{2} l_A = \frac{3 + 4}{2} 5.5 = 19.25 \text{ m}^2 \quad (3.48) \]

So the sum turns out to be

\[ \sum A_A r_A^3 = 2A_{Aupper} r_{Aupper}^3 + 2A_{Alower} r_{Alower}^3 \approx 3274.3 \text{ m}^5 \quad (3.49) \]
If the ship is non-stationary we need to use a different approach [5]. The speed of the water hitting the appendages will start coming from the front as well as perpendicular as shown in Figure 3.15. The resulting force $F$ is a combination of the drag and lift forces, but as the lift force is so much bigger the drag force is neglected and the assumption seen in Equation 3.50 is employed.

$$F = F_L \cos(\alpha) + F_D \sin(\alpha) \approx F_L$$

(3.50)

From Figure 3.15 it is seen that the angle of incidence on each appendage can be expressed as:

$$\alpha = \arctan \left( \frac{r_A \dot{h}_4}{U} \right) \approx \frac{r_A \dot{h}_4}{U} \quad \text{if} \quad r_A \dot{h}_4 \ll U$$

(3.51)

The total velocity experienced by each appendage is

$$q = \sqrt{U^2 + r_A^2 \dot{h}_4^2} \approx U \quad \text{if} \quad r_A \dot{h}_4 \ll U$$

(3.52)

So if the angle $\alpha$ is small the total roll moment experienced by the ship from the appendages is

$$F_4 = F_{r_A} \approx \frac{1}{2} \rho \frac{dC_L}{d\alpha} r_A \dot{h}_4 U^2 A_A r_A =$$

$$= \frac{dC_L}{d\alpha} \frac{1}{2} \rho U A_A r_A^2 \dot{h}_4^2$$

(3.53)

The roll damping coefficient simply becomes

$$B_{44el} \approx \frac{1}{2} \rho U \sum A_A r_A^2$$

(3.54)

Where no linearisation techniques are required. This is the same coefficient used for $B_{441}$ in the nonlinear approach. The lift curve slope is an empirical value that can be expressed as [5]:

![Fig. 3.15. roll damping due to lifting surface](image)
\[ \frac{dC_L}{d\alpha} = \frac{1.8\pi a}{1.8 + \sqrt{a^2 + 4}} \]  
(3.55)

Where the equivalent aspect ratio of the appendages is

\[ a = \frac{2b}{\bar{c}} = \frac{4b}{c_r + c_t} \]  
(3.56)

and \( b \) is the outreach, \( \bar{c} \) is the mean chord, \( c_r \) is the root chord and \( c_t \) is the tip chord as seen in Figure 3.13.

### 3.2 Conclusion and discussion

The calculation of the restoring force produces the hydrostatic coefficient \( C_{44} \), and it is expressed as:

\[ C_{44} = \rho g \nabla GM_0 \]  
(3.57)

The investigation of the inertia form the following expression for \( I_{44} \)

\[ I_{44} = k^2_i M \]  
(3.58)

The damping in the nonlinear case is made up by the following parts:

**The appendage damping:**

When the speed is zero

\[ F_4 = C_D \frac{1}{2} \rho \pi a^2 \eta_i^2 A_A \]  
(3.59)

When the ship has got speed and the drag is neglected.

\[ F_4 = \frac{dC_L}{d\alpha} \frac{1}{2} \rho U A_A r^2 \dot{\eta}_i \]  
(3.60)

If the drag would not be neglected it would look like:

\[
F = \frac{1}{2} \rho (U^2 + r^2 \dot{\eta}_i^2) \left( \cos \left( \arctan \left( \frac{r A H_i}{U} \right) \right) C_L + \right. \\
+ \sin \left( \arctan \left( \frac{r A H_i}{U} \right) \right) C_D \right) 
\]  
(3.61)

Which would make the equation nonlinear and hard to solve. The contribution from the drag is very small so it is not completely wrong to neglect it. This may have impact on the solution for very low speeds, where the speed of the ship and the roll speed are of the same size.

**The skin friction damping:**

\[ F_4 = C_F \frac{1}{2} \rho r^3 A_A \bar{S} \eta_i^2 \]  
(3.62)
Where \( S \) is the wetted surface, \( C_D \) is the drag coefficient, \( \rho \) is the density of water, \( r_A \) is the lever from the centre of gravity to the middle of the appendage, \( A_A \) is the area of the appendage and \( \eta_4 \) is the roll angle, \( U \) is the boat speed and \( \frac{dC_L}{d\alpha} \) is a lift coefficient.

So the nonlinear coefficients are summarised as:

\[
B_{441} = \frac{1}{2} \rho U \sum \frac{dC_L}{d\alpha} A_A r_A^2 \tag{3.63}
\]
\[
B_{442} = C_F \frac{1}{2} \rho r^3 A_A LS + C_D \frac{1}{2} \rho \sum r_A^3 A_A \tag{3.64}
\]

And in the linear case:

\[
B_{44\text{stationary}} = B_{44s} + B_{44ad} = C_F \frac{4\pi}{3} \rho r^4 \omega_4 \eta_4 L \left( \pi + 2 \arcsin \left( \frac{l_d}{r} \right) \right) + \frac{4}{3\pi} C_D \rho \eta_4 \omega_4 \sum A_A r_A^3 \tag{3.65}
\]
\[
B_{44\text{speed}} = B_{44s} + B_{44ad} = C_F \frac{4\pi}{3} \rho r^4 \omega_4 \eta_4 L \left( \pi + 2 \arcsin \left( \frac{l_d}{r} \right) \right) + \frac{1}{2} \rho U \sum \frac{dC_L}{d\alpha} A_A r_A^2 \tag{3.66}
\]

The above Equations are found by linearising the nonlinear coefficients in Equations 3.63 and 3.64 through using Equation 3.27.
Sea States and Wave Spectra

When looking at the sea it seems as if it is a chaotic mass of waves, and to try to represent all these waves with some simple mathematical expressions can appear impervious. To represent the waves it has been necessary to turn to statistics and Fourier series, which has made the modelling of sea states a simple task. Just by measuring wave heights and periods for a limited time and taking the averages of that we can create an ergodic time history.

4.1 Theory

The following explanation is based on Seakeeping by A.R.J.M Lloyd [5].

The waves can be represented by an infinite Fourier series for any given time history $T_H$ with:

$$\zeta(t) = \bar{\zeta} + \sum_{n=1}^{\infty} A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$$  \hspace{1cm} (4.1)

Where the frequencies are given by:

$$\omega_n = \frac{2\pi n}{T_H} \text{ with } n = 1, 2, 3 \ldots \infty$$  \hspace{1cm} (4.2)

And the coefficients are expressed as:

$$A_n = \frac{2}{T_H} \int_0^{T_H} \zeta(t) \cos(\omega_n t)$$  \hspace{1cm} (4.3)

$$B_n = \frac{2}{T_H} \int_0^{T_H} \zeta(t) \sin(\omega_n t)$$  \hspace{1cm} (4.4)

Equation 4.1 can also be expressed as:
\[ \zeta(t) = \bar{\zeta} + \sum_{n=1}^{\infty} \zeta_{n0} \cos(\omega_n t + \varepsilon_n) \quad (4.5) \]

Where the coefficients are:

\[ \zeta_{n0} = \sqrt{A_n^2 + B_n^2} \quad (4.6) \]

With the phase angles given as:

\[ \varepsilon_n = -\frac{B_n}{A_n} \quad (4.7) \]

The Equation 4.5 may be interpreted as a sum of an infinite number of sine waves of amplitude \( \zeta_{n0} \) and the frequency \( \omega_n \). For a wave representation the constant \( \bar{\zeta} \) is zero.

It is common to quantify the sine waves making up an irregular wave time history by using a wave amplitude energy density spectrum, or more commonly known as the Wave energy spectrum, see Figure 4.1. The energy per square metre of the sea surface for the \( n \):th wave component can be expressed as \( \rho g \zeta_{n0}^2 / 2 \), and it is proportional to the total energy of all wave components within that range of frequencies.

One of the most important terms is the Significant wave height, \( H_{1/3} \), which is an average of the highest third of the wave heights, this measure is well corresponding to the perceived highest wave height. Based on the significant wave height and the average wave period a wave energy spectrum is created.
4.2 Selected Wave spectra and Sea States

An ITTC two parameter spectra was chosen

\[ S(\omega) = \frac{A}{\omega^5} e^{-B \omega^4} \]  \hspace{1cm} (4.8)

Where

\[ A = 172.75 \frac{H_{1/3}^2}{T^4} \]  \hspace{1cm} (4.9)
\[ B = \frac{691}{T^4} \]  \hspace{1cm} (4.10)

The two parameters are the significant wave height and the average period, which is based on the mean frequency which is found by taking moments about the spectral ordinate axis and determining the centre of area of the spectrum.

A set of sea states where chosen as in Table 4.1.

<table>
<thead>
<tr>
<th>Sea state</th>
<th>$H_{1/3}$ [m]</th>
<th>$T$ [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.3</td>
<td>2.9329</td>
</tr>
<tr>
<td>3</td>
<td>0.9</td>
<td>3.9751</td>
</tr>
<tr>
<td>4</td>
<td>1.9</td>
<td>5.5660</td>
</tr>
<tr>
<td>5</td>
<td>3.3</td>
<td>7.4865</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>9.3377</td>
</tr>
<tr>
<td>7</td>
<td>7.5</td>
<td>11.1017</td>
</tr>
</tbody>
</table>

The figures are approximal mean values of the span that sea states originally are made up of, this is just to have values to compare with. 1. Values ranging from low to high sea states are chosen as the influence from a wide range of values on the model must be examined. In Figure 4.2 a wave spectra for significant wave height 0.9 metres and mean period 3.9751 sec is shown.

The aim is to vary the sea state for the submarine to see how it affects the roll angle and period. To also isolate the influence from changes in solely significant wave height or mean period one of the parameters is held constant whilst the other is varied.

---

1 These figures should not be copied or used by others as they do not follow the WMO scale standard.
Fig. 4.2. Example wave spectra
Solution in the Frequency domain

In order to interpret the equation of motion it has to be solved in some way. To start with it is easiest to solve it for one simple sinusoidal wave fed with a range of frequencies. If solved for a set of irregular waves there will be no simple solution. In the latter case it is possible to view the system as a linear operator and then superpose the signals. This approach is based on the methods used when filtering in electronics and telecommunications [11]. The calculations will produce a transfer function which is a type of operator that transfers the wave input to the movements of the ship.

5.1 Theory

To find the transfer function the uncoupled equation of motion for roll is studied, and it is defined as

\[(M_{44} + A_{44})\ddot{\eta}_4 + B_{44}\dot{\eta}_4 + C_{44}\eta_4 = F_{40}\cos(\omega t)\]  

(5.1)

To simplify the equation it is denominated by \((A_{44} + I_{44})\), and the new coefficients are defined as;

\[\omega_0 = \sqrt{\frac{C_{44}}{A_{44} + I_{44}}}\]  

(5.2)

\[\delta = \frac{B_{44}}{2(A_{44} + I_{44})}\]  

(5.3)

Where \(\omega_0\) is the eigenfrequency of the system, and \(\delta\) is the damping of the system. Equation 5.1 becomes

\[\ddot{\eta}_4 + 2\delta\dot{\eta}_4 + \omega_0^2\eta_4 = \frac{F_{40}}{A_{44} + I_{44}}\cos(\omega t)\]  

(5.4)
To solve this second order differential equation we need to make an initial guess, an ansatz. With the help from the Euler formula we can formulate the ansatz:

\[ \eta_4(t) = \hat{\eta}_4 e^{i \omega t} \tag{5.5} \]

This differentiated twice become

\[ \dot{\eta}_4 = i \omega \hat{\eta}_4 e^{i \omega t} \tag{5.6} \]
\[ \ddot{\eta}_4 = -\omega^2 \hat{\eta}_4 e^{i \omega t} \tag{5.7} \]

These put into equation 5.4 and the knowledge that \( \cos(\omega t) = \Re(e^{i \omega t}) \) gives

\[ -\omega^2 \hat{\eta}_4 e^{i \omega t} + 2 \delta \omega \hat{\eta}_4 e^{i \omega t} + \omega_0^2 \hat{\eta}_4 e^{i \omega t} = \frac{F_{40}}{A_{44} + I_{44}} e^{i \omega t} \tag{5.8} \]

Collect \( \hat{\eta}_4 \) and divide by \( e^{i \omega t} \)

\[ \hat{\eta}_4 = \frac{F_{40}}{I_{44} + A_{44}} - \omega^2 + 2 \delta \omega + \omega_0^2 \tag{5.9} \]

The wave roll amplitude is found by taking the absolute value of Equation 5.9

\[ \eta_{40} = |\hat{\eta}_4| = \sqrt{a^2 + b^2} = \frac{F_{40}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4 \delta^2 \omega^2}} \tag{5.10} \]

\( F_{40} \) is the force on the ship because of the waves. It is found by approximating the rolling moment as the wave slope times the restoring lever and results in the following relation:

\[ F_{40} = C_{44} k \zeta_0 \tag{5.11} \]

Where \( k \) is the wave number and \( \zeta_0 \) is the wave amplitude. Put 5.11 into 5.10 and realise that you have the factor \( \omega_0 \) in the numerator

\[ \eta_{40} = \frac{k \zeta_0 \omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4 \delta^2 \omega^2}} \tag{5.12} \]

What is sought for is the relationship between the rolling of the ship and the induced rolling from the waves. This can be expressed either as just \( \eta_0 / \zeta_0 \), or another common way of expressing it is the dimensionless ratio, \( \eta_0 / k \zeta_0 \). The second way is chosen which makes the transfer function approach 1 when the frequency approaches zero \(^1\). That means that the ship will have the same roll amplitude as the waves when the waves are very long. The transfer function is now formulated as

\(^1\) It never becomes zero, the spectra is not defined there
\[ Y(\omega) = \frac{\eta_{10}}{K_{0}} = \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}} \]  

(5.13)

It is often common to show the phase angle when showing the transfer function. The phase angle shows the lag between the fed wave and the ship's response. The expression for the phase angle is the argument of Equation 5.9.

\[ \varepsilon = \arg(\tilde{\eta}_4) = \arctan \left( \frac{\Im(\tilde{\eta}_4)}{\Re(\tilde{\eta}_4)} \right) = \arctan \left( \frac{-2\delta\omega}{\omega_0^2 - \omega^2} \right) \]  

(5.14)

To plot this function in MATLAB [7] it is important to use the function `atan2(x,y)` which is defined in all 4 quadrants. If the normal `atan(x/y)` is used it does not heed to the fact that the sign change between the quadrants.

### 5.1.1 The Response Spectra

As an irregular wave spectra can be expressed by plotting \( \frac{1}{2}\zeta_2^2 \) with each frequency, the response spectra can be plotted in the same way. The ratio between the wave spectra and the response spectra is easily translated to the square of the transfer function, with the wave number appearing as we chose to represent the transfer function as the slope:

\[ \frac{S_0(\omega_n)}{S_\zeta(\omega_n)} = \frac{1}{2}\frac{\eta_{10}}{\zeta_{0n}} = \left( \frac{\eta_{10}}{\zeta_{0n}} \right)^2 = Y^2(\omega_n)k_n^2 \Rightarrow S_\eta(\omega_n) = Y^2(\omega_n)k_n^2S_\zeta(\omega_n) \]  

(5.15)

### 5.2 Simulations

The simulations were made for a surfaced submarine with and without speed. The with speed case was examined with \( U = 5 \) knots as this is a plausible speed for a submarine at the surface. The eigenperiod of the system was set to 10 seconds, so the eigenfrequency \( \omega_0 \) is determined from that and the Equation 5.2 would give the same value in the linear case.

#### 5.2.1 Without speed

The damping is very low when the ship has no forward speed, and by looking at the transfer function plot it becomes apparent that this affects the transfer function, as the peak is very high. In figure 5.1 that is calculated with the values for the made-up submarine and Equation 5.13 the peak is prominent as expected. The wave spectra and the transfer function peaks does not coincide for low sea states so the response spectra does not get the same pointy
shape for low sea states, but as the significant wave height (and mean period) increases the response spectra gets a more peaked look. This becomes obvious when comparing Figure 5.2 and Figure 5.3.

**Fig. 5.1.** Transfer function for no speed

**Fig. 5.2.** Response spectra for no speed, short waves
5.2 Simulations

5.2.2 With speed

As speed is introduced the transfer function gets a less pointy peak, see Figure 5.4. The response spectra for the low sea states are still very unaffected by the pointedness of the transfer function (see Figure 5.5), but for higher sea states it is still predominant, but in a less palpable way (see Figure 5.6).

![Transfer function for 5 knots](image)

**Fig. 5.4.** Transfer function for 5 knots

The plot of the phase angle for a set of speeds are shown below in Figure 5.7. The plot is looking good and nothing unusual is showing. It is also seen
that the faster the boat travels the bigger the lag between the waves and the boat displacement, the steepest curve is the one for zero speed.
5.3 Conclusion and discussion

The roll motion of the submarine is strongly dependent on whether the boat is stationary or not, and if the ship has a problem with large rolling motions it should increase speed. This is old knowledge but it is good that the calculations show the same thing; otherwise the model would be useless.
Solution in the Time Domain

The equation of motion can be solved in the time domain. This requires more computer power and time than the frequency domain approach to do in a reasonable time span. The equation is solved numerically as no plausible ansatz is at hand. This methodology is more rarely used than the frequency domain approach as it is much more time demanding, and the frequency approach is often sufficiently accurate for initial investigations. In this part it is done to compare with the frequency domain calculations and prepare for the nonlinear calculations where a frequency domain approach is not possible.

6.1 Theory

The uncoupled equation of motion in roll is defined as:

\[(M_{44} + A_{44})\ddot{\eta}_4 + B_{44}\dot{\eta}_4 + C_{44}\eta_4 = F_{40}(\omega t)\]  \hspace{1cm} (6.1)

Where \(F_{40}(\omega t)\) for a simple sinusoidal wave is \(C_{44}k\zeta_0\sin(\omega t)\) where \(k\) is the wave number. Solving Equation 6.1 with the sinusoidal excitation force for a range of frequencies will produce data which is comparable with the transfer function and phase angle. The answer settles to oscillate between plus and minus a constant value, as seen in Figure 6.1. This value is saved for each plot, and hence each frequency. If these values are plotted against each other a familiar graph (the transfer function) will appear. The phase angle is found by calculating the time difference between the excitation force maxima and the response maxima, as shown by Equation 6.2 and the example plot in Figure 6.2.

\[\epsilon = -2\pi \frac{\Delta t}{T} = -\omega \Delta t\]  \hspace{1cm} (6.2)
Fig. 6.1. Example of time domain solution in MathCAD

In the case of an irregular set of waves $F_{40}(\omega t)$ is $C_{44}k\zeta(t)$ where

$$\zeta(t) = \bar{\zeta} + \sum_{n=1}^{\infty} \zeta_0 \cos(\omega_n t + \epsilon_n)$$  \hspace{1cm} (6.3)

where $\bar{\zeta}$ is zero and

$$\zeta_0 = \sqrt{2\zeta \omega_n}$$  \hspace{1cm} (6.4)

where $S_n(\omega_n)$ is the wave spectra component.

To prevent the resulting synthesised time history to repeat itself at intervals of $T$ seconds the frequency step must be non constant. One way is to randomly choose frequencies, but as it is important that the same random fig-
ures are chosen for each simulation so that the values are comparable, another technique is used. This procedure divides the spectrum into n segments with equal areas [12], which results in n sine waves with all equal amplitudes.

The area of the spectrum is calculated with the integral:

\[ \int_0^\infty S_\zeta(\omega) d\omega = \int_0^\infty \frac{A}{\omega^6} e^{-B/\omega^4} d\omega = \frac{A}{4B} \quad (6.5) \]

But the spectrum extends to infinity so it must be truncated at some fraction C of the total area. To find the frequency "bars" the following equation is evaluated

\[ \int_0^{\hat{\omega}_i} S_\eta(\omega) d\omega = \int_0^\infty S_\eta(\omega) \frac{i}{N} C \quad (6.6) \]

Where i is the bar number (see Figure 6.3) going from 1 to N, N is the number of bars and C is the fraction of the total area. Solving this equation yields

\[ \hat{\omega}_i = \left[ -B \ln \left( \frac{i}{N} C \right) \right]^{1/4} \quad (6.7) \]

To set a lower limit we assume that the spectral level is insignificant below 60% of the spectral eigenfrequency (The frequency at the maxima of the spectrum), so the frequency of the first bar is defined as follows:

\[ \omega_1 = \frac{0.6 \omega_{\text{max}} + \hat{\omega}_1}{2} \quad (6.8) \]

The remainders of the frequencies are calculated by finding the midpoint of

\[ \text{Fig. 6.3. 1 Spectra area division} \]
the bars
\[
\omega_i = \frac{\dot{\omega}_{i-1} + \dot{\omega}_i}{2} \quad \text{for} \quad i = 2 \ldots N \tag{6.9}
\]
And the phase angles are randomly chosen.

6.2 Simulations

The equations were first solved for a simple sinusoidal wave to investigate the transfer functions to see a first indication on if the methods were accurate. Secondly they were solved for a series of waves but the solutions in the time domain do not give us an energy spectra as the frequency domain calculations, one can be found but it is not in our interest to do that. The solutions are further investigated and compared in Chapter 7.

6.2.1 Linear Equation

The linear equation was first solved in MATLAB [7] with the ordinary differential equation solver ode15s, this solver was chosen as the ode45 solver was too slow and the problem is stiff, more about the solver is found in Appendix B. As a comparison it was also solved in MathCAD [6] with a stiff solver, which gave almost identical results. For a set of values of the equation 6.1 was solved, the oscillating answer look as in Figure 6.1, this plot is for \( \omega = 1.1 \) rad/sec. The transfer function will look as seen in Figure 6.4, which was generated in MATLAB.

6.2.2 Nonlinear Equation

In the nonlinear case it is only possible to find the solution to the Equation of Motion in the time domain. The solution converges much faster when the nonlinear terms are introduced. Some values which are conformed to be used for simplified linear calculations such as Reynolds numbers and friction coefficients are taken from the frequency calculations, this is done when no better options were found.

6.3 Results

In this part the “transfer functions” are scrutinised and no response spectra will be calculated. The solutions are calculated but no further investigations are made.
6.3.1 Linear Equation

If Figure 6.4 is compared with Figure 5.1, which should be the same, it is seen that the peaks are not of equal height. They do however agree well enough to be confident of good results in the ensuing investigations. The peak height is dependent on how many frequency steps are taken and the size of them. This is predominantly a problem in the zero speed case as the damping is larger for the case where the submarine is moving. The values for low frequencies are somewhat fluctuating as the solver has got difficulties with the solution as values tend to be big when dividing with small numbers. The phase angles

are shown in Figure 6.6 for some speeds. They are not very smooth as the programming method to find them is crude, but they still show a very good resemblance to the phase angles found through frequency domain calculations, see Figure 5.7.

Figures 6.7 and 6.8 show an example of the response with a significant wave height of 1.9m.
6.3.2 Nonlinear Equation

If the nonlinear "transfer function" for zero speed is compared with the linear transfer function (See Figure 6.9), the reason for the fast converging becomes apparent. The peak is a lot less pointy and this simplifies the work for the solver. This bluntness is due to the nonlinear damping which probably is more accurate than the simplified linear version. Surveying the next figure, Figure 6.10, another property becomes more apparent; the eigenfrequency for the system is slightly lower than in the linear case. This is due to the introduction of the nonlinear hydrostatic restoring force, where the angles are not considered small and the \(\sin(\eta_k)\) is kept. It is also seen that the damping
Fig. 6.7. Linear solution for $H_{1/3} = 1.9\text{m}$ and $t = 1000\text{sec}$ zero speed

Fig. 6.8. Linear solution for $H_{1/3} = 1.9\text{m}$ and $t = 1000\text{sec}$ 5 knots

is slightly lower, this is owing to that the linear simplification slightly over predicts the damping.
To compare the responses from the linear case with the nonlinear case the figures in Figure 6.7 and 6.8 are compared with the plots in Figures 6.11 and 6.12. There is not much that can be seen just through looking at the solutions and comparing them. They look quite alike but the linear solution has slightly bigger displacements, which is consistent with the conclusions drawn from the transfer functions.
6.3 Results

Fig. 6.11. Nonlinear solution for $H_{1/3} = 1.9\text{ m}$ and $t = 1000 \text{ sec}$ zero speed

Fig. 6.12. Nonlinear solution for $H_{1/3} = 1.9\text{ m}$ and $t = 1000 \text{ sec}$ 5 knots
6.4 Conclusion and Discussion

The linear transfer functions found in the time domain compare well with the frequency domain calculations, which shows that the time domain calculations are good and should be alright to use for the nonlinear case. It could be argued that it is no guarantee that the solver can handle the nonlinear just because it could handle the linear case, but it was decided to continue with the same solver too see if it managed, and it did.

If the transfer functions (Figures 6.9 and 6.10) are compared it shows that the linear method over predicts damping when the boat is moving and under predicts it for zero speed. It is also found that the eigenfrequency changes and becomes slightly lower. The difference in damping can also be seen by just looking at the solutions (Figures 6.7, 6.11, 6.8 and 6.12).
The standard deviation or root mean square of the surface depression relative to the mean is a common figure to use when characterising ship movements. It is a good measure of how the ship moves in certain conditions. Often it is used as a control figure, and if it exceeds some criteria, the design will need to be modified to reduce motions.

7.1 Theory

When a response is measured it is measured at discrete time points with constant time increment $\delta t$, as shown in Figure 7.1. The variance of the response relative to the mean is defined as

$$m_0 = \sum_{n=1}^{N} \frac{(\zeta_n + \bar{\zeta})^2}{N}$$  \hspace{1cm} (7.1)$$

where $\bar{\zeta}$ is the mean response. $N$ is the number of measured points. The standard deviation or the root mean square (RMS) of the response relative to the mean

$$\sigma_0 = \sqrt{m_0}$$  \hspace{1cm} (7.2)$$
7.2 Simulations

To compare the solution methods and check if they agree, the RMS values for their responses are calculated as described above. The six sea states shown in Table 4.1 are investigated with no speed and with a speed of 5 knots. Just to check what shape the plot really has got the simulation was run for when $N = 509$ in Equation 6.7 and a large number of “sea states”, this is solely done for the linear time domain as it only is to give as an idea of the shape.

To isolate the influence from the mean period from the significant wave height the simulations were also run with either parameter constant. The spectrum is divided into parts as described in Chapter 4, but the number of frequency steps is varied to check if it influences the solution. $N$ is evaluated for 19, 59, 509 and 1009 steps. The time for the solver was set to $t = 10000$ seconds as this was judged a sufficiently long time.

As some parts of the damping coefficients are dependant on the rms the solutions had to be recalculated with the calculated value until it converged.

7.2.1 Linear Equation

The graph in Figure 7.2 plots the roll RMS against the significant wave height (chosen as Table 4.1) provided to the energy spectrum for zero speed, and although the points are situated close to each other they visibly differ. The plot of the RMS values for speed equal to 5 knots in Figure 7.3 shows a much better coincidence between the simulations. Figure 7.4 shows a plot of the RMS values for $N = 509$ and both zero speed and a speed of 5 knots. To create the plots in Figure 7.5 and 7.6, which shows smooth curves reminding us of
the first half of a spectrum plot, a dense number of “sea states” are used. If the period is allowed to vary from say 0.5 sec to 25 sec and the Significant wave height is held constant the shape for 5 knots is shown in Figure 7.7 and for zero knots in Figure 7.8. If the mean period is kept constant while the significant wave height is varied we see a linear correlation for the 5-knots case (see Figure 7.9), while the plot for the zero speed case is slightly bent (see Figure 7.10). To inspect how the dependence on speed express itself the frequency calculations where run for a few different speeds, the plot is shown in Figure 7.11. It is noteworthy that the calculations for speed must be of a magnitude larger than about 2 knots (ca 4 m/sec) otherwise the solutions
Fig. 7.4. RMS values for N=509

Fig. 7.5. RMS values for N=509 and zero speed

saves that the damping will be larger for the stationary ship than for a ship travelling with low speed. This is due to omitting drag in the calculations of appendage damping, which was mentioned in Chapter 3.1.4. It becomes obvious that speed is important to reduce roll, but after the initial significant increase in damping more speed does not increase the damping radically; the increase is not linearly dependant on speed.
7.2 Simulations

7.2.2 Nonlinear Equation

The nonlinear simulations for zero speed (See Figure 7.12) gives as scattered results as the linear case. The results for the linear and the nonlinear case are plotted together in Figure 7.13, where it is visible that the values for the nonlinear case are slightly lower than for the linear case. The nonlinear results for a speed of 5 knots (See Figure 7.14) gives results showing the same collected values as the linear case, this becomes apparent when studying the plot in Figure 7.15.

**Fig. 7.6.** RMS values for N=509 and 5 knots

**Fig. 7.7.** The period is allowed to be varied while the Significant wave height is kept constant, speed=5 knots
If the mean period is varied while the significant wave height is kept constant and the speed is set to 5 knots the plot received is as in Figure 7.16, and in the zero speed case Figure 7.17. They are uneven plots but with the same overall shape as for the linear case. The slightly lower values in the zero speed case are also consistent with earlier conclusions. Why the look like this could be due to the way the Reynolds number is gathered from the analytic calculations, or the solver could have problems. This could be a detail that can be further investigated.
7.3 Conclusion and discussion

It is clear that the speed of the submarine has a dampening effect on the rolling, which is not surprising as it is usual to increase speed to prevent roll, this is clearly seen in plot Figure 7.4. For small significant wave height the speed has no major impact, but as the waves get bigger speed grows more important for the damping. The values that differ in Figure 7.2 are most certainly due to that for zero speed the transfer function gets a very pointy peak (study for example Figure 5.1), and if the chosen frequency for a bar misses the peak the value will be lower. To prevent this from occurring more
It can be argued that the RMS responses for the lower sea states are of small importance as they are small and seem to follow the waves. The rest of the figures are of the expected size and follow an expected increasing shape.

By varying the Significant wave height and keeping the Mean period constant a linear correlation is expected and found, see Figure 7.10 and Figure 7.9. The linearity is a measure on how accurate the method is, if it is not...
linear the method is not flawless. When varying the Mean period and keeping the Significant wave height constant the plot is nonlinear and shaped like a spectrum with the peak close to the eigenperiod of the submarine, see Figure 7.8 and Figure 7.7. These plots shows us that the nonlinearity that occur for the plots where the sea states are changed derives from the change in period.

The similarities between the nonlinear and the linear values tell us that at least the linear model for a non stationary boat is a satisfactory good model, and that a nonlinear model is not necessary, at least for the hitherto calculations, even though the non-linear calculation gives slightly lower results.
Fig. 7.16. 5 knots

Fig. 7.17. 0 knots

The zero speed case is visibly sensitive due to the pointy peak in the transfer function.
Bilge Keels

In general it is not usual to provide a submarine with roll controlling devices such as bilge keels or stabiliser fins. Unlike pitch or other degrees of freedom, the forces required to create a stabilising effect in roll are relatively small. One normally relies on the hydrostatic restoring moment due to the centre of gravity being below the centre of buoyancy to counteract any asymmetric moments tending to cause roll. \[10\]

Bilge keels are the simplest form of roll stabilisation device, they are very effective and work well especially at low speeds, and hence if they make a big difference for the submarine it can be worth evaluating if they should be included in the design. The problem with bilge keels is that they tend to create vortices and eddies that propagate into the propulsor and cause the propeller to make noise, which is not good for a submarine where stealth is one of the main goals. They also increase the resistance of the ship, but this is handled by aligning the keels so they have as little drag as possible for cruise speed.

8.1 Theory

The mechanism for the bilge keels is similar to that of the appendages, but the drag coefficient is of another magnitude. According to Seakeeping by ARJM Lloyd [5] the drag coefficient used for the no speed case can be calculated from experimental data as follows:

\[
C_D = 0.849\left(\frac{I}{r'}[1 - e^{-Kr'}] + J\right) \tag{8.1}
\]

\[
J = 2.37 - 5.33a_{BK} + 10.35a_{BK}^2 \tag{8.2}
\]

\[
I = \frac{14.66 - J}{K} \tag{8.3}
\]

\[
K = \frac{1}{\sqrt{a_{BK}(14.66 - J)(0.0386 - 0.0735A_{BK})}} \tag{8.4}
\]
Where the equivalent aspect ratio for the bilge keel is

\[ a_{BK} = \frac{2b_{BK}}{c_{BK}} \quad (8.5) \]

With lengths defined in Figure 8.1, and the non dimensional keel radius parameter is

\[ r' = \frac{r_{BK}}{\bar{\eta} \sqrt{A_{BK}}} \quad (8.6) \]

With length also defined in Figure 8.1.

The drag coefficient plotted for some different aspect ratios is found in 8.2.

The values for \( C_D \) calculated in this manner are a bit high for a submarine as the bilge keels will look a bit different to those fitted to either merchant ships or war ships, see Figure 8.3. The bilge has got a smoother shape to avoid creating the most unwanted disturbances in the water. The lift created by the
Fig. 8.3. Merchant ship, warship and submarine bilge keel configuration

Bilge keels is very small compared to the lift created by the appendages, so the drag is big in comparison. That is why the drag part of the bilge keel calculations still is present when the submarine moves, and not just when it is stationary. The Equation 8.8 is present for both the speed and non speed case whereas in the nonlinear case Equation 8.7 is included in $B_{44}$ for both speed and non speed case.

\[ B_{44} = C_D \frac{1}{2} \rho \sum r_{BK}^3 A_{BK} \quad (8.7) \]

\[ B_{44} = \frac{4}{3\pi} C_D \rho \eta_0 \omega_0 \sum r_{BK}^3 A_{BK} \quad (8.8) \]

8.2 Simulations

The simulations were made in a similar manner to the RMS simulations, this to provide comparable data. The size of the bilge keels is chosen to be in the size of commonly used bilge keels, i.e. $b_{BK} = 0.25m$ and $c_{BK} = 0.7L = 70m$, as of Figure 8.1.

8.3 Results

Figures 8.4 to 8.7 were found with the above simulations. It was chosen to plot them in the fashion they are plotted to be able to compare them in a good way.

Studying Figure 8.4 and comparing it to Figure 8.5 the most distinguishable effect is that as soon as some damping is added, both speed and bilge keels, the linear and the nonlinear calculations start to correspond more. By studying Figure 8.6 and Figure 8.7 separately the addition of bilge keel has got the largest effect for the non speed case.
Fig. 8.4. Comparing RMS values for 0 knots with and without bilge keels linear and nonlinear calculations

Fig. 8.5. Comparing RMS values for 5 knots with and without bilge keels linear and nonlinear calculations

8.4 Conclusion and discussion

The change in \( RMS \) is just a couple of degrees at the most when with speed. For zero speed the bilge keels are quite effective, as was anticipated. This can be an advantage for a submarine if it has got engine failure or can't set speed for some other reason.

The model predicts the effect of adding bilge keels well as it predicts the movements in a way that resembles reality.
Fig. 8.6. Comparing linear RMS values for the linear case

Fig. 8.7. Comparing nonlinear RMS values for the nonlinear case
Varying $GM_T$

The $GM_T$ is the transverse distance from the centre of gravity to the metacentre. This is one of the most important values for a ship and tells us much about the stability of the ship. It can be slightly altered by filling and emptying the water tanks, but here it is mostly considered as a design parameter.

### 9.1 Theory

The biggest effect on the model simulations when changing the $GM_T$ is the change of the hydrostatic forces. It is directly proportional to the metacentre height for both the linear and nonlinear approach. The easiest way to change the models metacentre is through changing the eigen roll period. In appendix A the $GM_T$ is found with equation A.1 and is here reprinted in Equation 9.1.

$$T_\phi = \frac{2\pi}{\omega_c} = \frac{2\pi r}{\sqrt{gGM_T}} \implies GM_T = \left(\frac{2\pi r}{T_\phi}\right)^2 \frac{1}{g} \tag{9.1}$$

As mentioned before, 10 seconds is a plausible period for a submarine of the size and shape investigated. To find what happens if the $GM_T$ is varied this figure is varied from 8 to 16 seconds, which for a set of data produces the values in Table 9.1.

<table>
<thead>
<tr>
<th>Period [sec]</th>
<th>$GM_T$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.048m</td>
</tr>
<tr>
<td>10</td>
<td>0.671m</td>
</tr>
<tr>
<td>12</td>
<td>0.466m</td>
</tr>
<tr>
<td>14</td>
<td>0.342m</td>
</tr>
<tr>
<td>16</td>
<td>0.262m</td>
</tr>
</tbody>
</table>
Not only is the inclination of the boat important for comfort but also the lateral and vertical accelerations. For the analytical frequency domain calculations the RMS for the accelerations ($\sigma_4$) can be calculated through spectral moment $m_4$ with the relations in Equations 9.2 [5].

$$\sigma_4 = \sqrt{m_4}$$

$$m_4 = \int_0^\infty \omega^4 S_\eta(\omega) d\omega \quad (9.2)$$

For the nonlinear time domain calculations $\sigma_4$ is found by taking the variance of the derivative of $\dot{\eta}_4$, as it is provided by the solver. To compare the accelerations with demands on acceleration for normal ships the vertical must be separated from the lateral accelerations, which is done by simple trigonometry where Figure 9.1 is used.

![Fig. 9.1. Accelerations of the submarine](image)

The accelerations can then be expressed as in Equations 9.3.

$$a_z = r | \cos(\sigma_0) | \sigma_4$$

$$a_y = r | \sin(\sigma_0) | \sigma_4 \quad (9.3)$$

### 9.2 Simulations

To examine the effect of changing $GM_T$ it is decided just to use one sea state as the change of $GM_T$ probably will have the same effect on all of the higher ones where the change is most severe. The chosen sea state is $H_{1/3} = 5m$ and $T = 9.3377$ sec. Speed is still considered but the effects from change of $GM_T$ is isolated from the effect from adding bilge keels.
The value of $r$ in Equations 9.3 is the distance from the centre of rotation to the thing where we want to know the acceleration. If this would be a person he or she would not be working as far out as the hull but a distance from it, say 1.5 metres. Assuming this the accelerations are calculated at a distance of 3.5 metres from the centre of rotation. To know how bad or good the results are they need to be compared with some Seakeeping Performance Criteria. In the lecture notes from Anders Rosen [11] in Table 5.2 some examples are found, and it was chosen to use the lateral criteria $0.1\, g$ and vertical criteria $0.2\, g$.

### 9.3 Results

The roll inclination increase as $GM_T$ increases, for the linear case as in Figure 9.2 and in the nonlinear case as in Figure 9.3. The accelerations do also increase as $GM_T$ increases, for the linear case as in Figure 9.4 and for the nonlinear as in Figure 9.5. The nonlinear approach results have a less linear shape than the linear approach plots, which results in bigger accelerations for smaller $GM_T$ in the linear case.

![Fig. 9.2. Linear RMS change depending on $GM_T$](image)

### 9.4 Conclusion and discussion

The increase of the accelerations and roll angles with $GM_T$ could easily lead to the conclusion that by just setting a low $GM_T$ value we get low accelerations,
Fig. 9.3. Nonlinear RMS change depending on $GM_T$

Fig. 9.4. Acceleration at $r=3.5m$

but with a too low $GM_T$ the submarine will be unstable. Unstable submarines should be avoided, in case it would get damaged and further change the $GM_T$. The accelerations at a distance of 3.5$m$ from the centre of rotation the lateral accelerations are acceptable up to a $GM_T$ of about 0.95$m$ in the linear case and up to 1.1$m$ in the nonlinear case. As the submarine has got a $GM_T$ on 0.67$m$ the accelerations do not exceeded the chosen lateral standards. The vertical acceleration criteria is not close to being exceeded. However, the difference between 0.95$m$ and 0.67$m$ is not overly big so if something would happen to the boat it could pass the criteria.
Fig. 9.5. Acceleration at $r=3.5\text{m}$
Conclusion and Discussion

The ultimate goal with this Thesis has been to create a method to simulate the roll motion of a submarine in a linear way in the frequency domain and in nonlinear way in the time domain of a surfaced submarine. Some simplifications were required to develop the model within the time available. Despite these simplifications good results have been obtained showing the expected trends.

The only likeness this submarine has got with the Collins Class submarine is that is is the same order of size and the use of X-configuration for the aft actuators. It is just a method for predicting the behaviour for such a submarine, not the actual Collins Class, many of the parameters can be adjusted to fit various configurations. As the author of this thesis is not a submarine expert most of the received values have been judged by the supervisor who works with submarines at a daily basis.

Agreement between the linear and the nonlinear model is good except for the zero speed case which indicates that roll damping is under predicted by the linear model. In all cases roll motion is reduced by including nonlinear terms with a larger reduction for the zero speed case. Roll motion is also reduced significantly with forward speed. The linear approach has got the advantage that it is time saving, so if just brief investigations are needed it is adequate.

The submarine reacts to change in sea states as suspected. The model do have difficulties with very small and very big sea states, but for the sea states in the submarines operability area the simulations behaves good.

Adding bilge keels will increase damping and are most effective at zero speed. They are predicted to reduce $RMS$ roll angles by 10 to 15 degrees (nonlinear case) in higher sea states when stationary. When the submarine is under way (has forward speed) bilge keels are less effective. Therefore bilge keels would be most beneficial in reducing roll motion for a disabled submarine.
When varying the roll period the $GM_T$ will change and this affects the $RMS$ roll angle so that with increasing $GM_T$ the roll angle will also increase. This can seem a bit counterintuitive but it is due to the tuning between the wave spectrum and the submarine, where it is less pronounced for lower $GM_T$ values (longer roll periods). The roll accelerations are also lower for low $GM_T$ and therefore less risk of injury to personnel. Low $GM_T$ values are however detrimental to stability hence a balance is required that provides acceptable stability performance while minimising roll motions.

10.1 Future work

The following subjects are things that has been neglected to investigate as the time has been restricted.

- Simulation of parametric rolling
- Remove some simplifications to get the model more accurate, e.g. add the connection to sway.
- Investigate further a better way of solving the equation of motion in the time domain
- Investigate further how the nonlinear model reacts to change in spectra parameters
- What happens if the submarine is under the surface
- Experimental roll damping coefficients
References

Submarine dimension calculations

The following appendix present the calculations made to create the mock-up submarine used in the thesis.

$GM_T$

The $GM$ value for the submarine is calculated by setting the eigen roll period to 10 sec and evaluating equation A.1 [2]

$$T_\phi = \frac{2\pi}{\omega_c} = \frac{2\pi r}{\sqrt{gGM}} \Rightarrow GM = \left(\frac{2\pi r}{T_\phi}\right)^2 \frac{1}{g}$$  \hspace{1cm} (A.1)

where $r$ is defined as A.2

$$r = \frac{\sqrt{B^2 + D^2}}{2\sqrt{3}}$$  \hspace{1cm} (A.2)

in the same way as in section 3.1.3. So when $T_\phi = 10s$, $r = 4.0825m$ and $g = 9.81$ we get a $GM = 0.6707m$

Displacement, $\nabla$

The submarine is cut up into pieces as seen in Figure A.1

Sphere part

The half sphere is handled in two parts, $V_1$ and $V_2$ shown in Figure A.2, where $V_1$ is simply calculated as a quarter of a sphere:

$$V_1 = \frac{1}{4} V_{sphere} = \frac{\pi r^3}{3} = 524m^3$$  \hspace{1cm} (A.3)

The $V_2$ part is a sphere segment calculated with equation A.4 [9].
Fig. A.1. How the submarine is cut up into pieces to calculate displacement and centre of buoyancy

Table A.1. Sphere dimensions

<table>
<thead>
<tr>
<th>Property</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>r</td>
<td>5m</td>
</tr>
<tr>
<td>h</td>
<td>3.5m</td>
</tr>
<tr>
<td>b</td>
<td>$\sqrt{51/2}m$</td>
</tr>
</tbody>
</table>

Fig. A.2. Values for calculations of half-sphere volume

\[ V_2 = \frac{\pi}{6} h(3r^2 + 3b^2 + h^2) = 230m^3 \]  \hspace{1cm} (A.4)

The total volume of the half sphere becomes:

\[ V_{tot} = 754m^3 \]  \hspace{1cm} (A.5)

Cylinder part

The volume created by the middle part of the submarine is a cylinder with the top part cut off by the sea surface. The easiest way to calculate the volume is to evaluate the area of the circle that makes the shape of the cylinder and then multiply that with the length.
Table A.2. Cylinder dimensions

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>5m</td>
</tr>
<tr>
<td>$h$</td>
<td>3.5m</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\pi - 2\beta$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\arcsin\left(\frac{h}{r}\right)$</td>
</tr>
</tbody>
</table>

Fig. A.3. dimensions to calculate the area of a cut-off circle

The area $A_E$ is first calculated and then removed from the area of a circle seen below.

$$A_E = \frac{r^2}{2}(\alpha - \sin(\alpha)) = \frac{r^3}{2}(\pi - 2\arcsin\left(\frac{h}{r}\right) - \sin(\pi - 2\arcsin\left(\frac{h}{r}\right))) = 7.387471005m^2$$  \hspace{1cm} (A.6)

$$A_{TOT} = \pi r^2 = 78.53981634m^2$$  \hspace{1cm} (A.7)

$$A = A_{TOT} - A_E = 71.15234533m^2$$  \hspace{1cm} (A.8)

$$V = A \cdot L = 5692.18762m^3$$  \hspace{1cm} (A.9)

**Cone part**

The cone part is split into two parts, one part that has got some over the surface and one part that is fully submerged, as seen in Figure A.4.

**First part**

Figure A.5 shows the distances in Table A.3. First the surface $A_0$ is expressed as a function of $x$ as it changes along the length of the cone in Equation A.10.

$$A_0 = \int \sqrt{f(x)^2 - y^2} - hdz \hspace{1cm} (A.10)$$

where $f(x) = r - \left(\frac{r}{l_{full}} + \frac{r}{l_{xt}} \frac{x}{l_{xt}} \right) x$ and $y = \sqrt{f(x)^2 - h^2}$
A Submarine dimension calculations

Fig. A.4. The cone part of the submarine is divided into two parts

Table A.3. Cone part one dimensions

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l$</td>
<td>8.5m</td>
</tr>
<tr>
<td>$l_1$</td>
<td>10m</td>
</tr>
<tr>
<td>$l_{full}$</td>
<td>16.67m</td>
</tr>
<tr>
<td>$h$</td>
<td>3.5m</td>
</tr>
<tr>
<td>$r$</td>
<td>5m</td>
</tr>
</tbody>
</table>

Fig. A.5. First part of the cone

\[
A(x, y) = \pi f(x)^2 - \int \sqrt{f(x)^2 - y^2} - h dz \quad (A.11)
\]

The Equation A.11 is integrated numerically along the length of the cone-part with Simpson’s rule (Equation A.12) [9] and the volume turns out to have the value $V = 182m^3$.

\[
\int_a^b f(x) dx = \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})] + RT
\]

\[
h = \frac{b - a}{2n}
\]

\[
x_k = a + hk
\]

\[
RT = -\frac{(b - a)h^4}{180} f^{(4)}(\xi), \quad a < \xi < b \quad (A.12)
\]
Second part

The second cone part has got the values as of Figure A.6 and Table A.4. The

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_h$</td>
<td>7m</td>
</tr>
<tr>
<td>$l_2$</td>
<td>5m</td>
</tr>
<tr>
<td>$l_{cone}$</td>
<td>6.67m</td>
</tr>
<tr>
<td>$l_{tip}$</td>
<td>1m</td>
</tr>
</tbody>
</table>

volume of a circular cone is calculated with Equation A.13

$$V = \frac{1}{3}bh \quad (A.13)$$

where $b$ is the the base area and $h$ is the height of the cone. The cone to be

![Fig. A.6. Second part of the cone](image)

evaluated for has got the top cut off so the volume is calculated as:

$$V_{tot} = V + V_0 = \frac{1}{3}A \cdot l_{cone}$$

$$V_0 = \frac{1}{3} \cdot \pi \left( \frac{l_{tip}}{2} \right)^2$$

$$V = \frac{1}{3}(A \cdot l_{cone} - \pi \cdot \left( \frac{l_{tip}}{2} \right)^2 \cdot (l_{cone} - l_2) =$$

$$= \frac{1}{3}(\pi \cdot \left( \frac{l_h}{2} \right)^2 \cdot l_{cone} - \pi \cdot \left( \frac{l_{tip}}{2} \right)^2 \cdot (l_{cone} - l_2)) \approx 213 \text{m}^3 \quad (A.14)$$
**Pieces put together**

\[ V_{\text{subtot}} = 754 + 5692 + 182 + 213 \, m^3 = 6841 \, m^3 \]

**BM**

The \( BM_T \) is calculated as in Equation A.15

\[
BM_T = \frac{I_{WAx}}{\nabla} = \frac{1}{\nabla} \int_{L_{WL}} \frac{swl^3}{12} \, dx \quad (A.15)
\]

Where \( swl \) is calculated along the ship as of Figure A.7 and Equations A.16-A.18.

![Fig. A.7. dimensions](image)

\[
swl_1 = \sqrt{3.5^2 - (x - 3.5)^2} \times 2 \quad 0 < x < 3.5 \quad (A.16)
\]

\[
swl_2 = 3.5 \times 2 \quad 3.5 < x < 83.5 \quad (A.17)
\]

\[
swl_3 = (61.95 - 0.7x) \times 2 \quad 83.5 < x < 88.5 \quad (A.18)
\]

\[
\int_{L_{WL}} \frac{swl^3}{12} \, dx = 58.93 + 2286.7 + 35.729m^4 = 2381m^4 \quad (A.19)
\]

\[
BM_T = \frac{2381}{6841} = 0.348m \quad (A.20)
\]

**CB**

To calculate the centre of buoyancy the submarine is cut up into pieces as in Figure A.1.

**Sphere part**

The front part of the submarine is represented by a half a sphere, to calculate the centre of gravity of it we look at it from the side as in Figure A.8.
A Submarine dimension calculations

Fig. A.8. The half sphere from the side

\[ A = \int_{-5}^{3.5} xdz = \int_{-5}^{3.5} \sqrt{r^2 - z^2} = 35.576219m^2 \] \text{(A.21)}

\[ T_x = \frac{1}{2A} \int_{-5}^{3.5} x^2dz = \frac{1}{2A} \int_{-5}^{3.5} \left(\sqrt{r^2 - z^2}\right)^2dz = 2.2m \] \text{(A.22)}

As we know that the submarine is symmetric the height wise centre will be on the same place no matter if we look at half the sphere or the whole sphere.

\[ T_z = \frac{1}{A} \int_{-5}^{3.5} zydz = \frac{1}{A} \int_{-5}^{3.5} z\sqrt{r^2 - z^2}dz = -0.426565662m \] \text{(A.23)}

which result in \( c_B = (0.0, 2.2, -0.43)m \)

\textbf{Cylinder part}

The cylinder centre of gravity is calculated with help from Figure A.9

\[ A = \int_{-5}^{3.5} (\sqrt{r^2 - z^2} + \sqrt{r^2 - z^2})dz = 71.152378m^2 \] \text{(A.24)}

\[ T_z = \frac{1}{A} \int_{-5}^{3.5} z(\sqrt{r^2 - z^2} - (\sqrt{r^2 - z^2}))dz = -0.426564661m \] \text{(A.25)}

\[ T_z = \frac{1}{2A} \int_{-5}^{3.5} \left(\sqrt{r^2 - z^2}\right)^2 - \left(-\sqrt{r^2 - z^2}\right)^2dz = 0m \] \text{(A.26)}

\( c_B = (40, 0, -0.427)m \)
Cone part

The cone part is split into two parts, one part that has got some over the surface and one part that is fully submerged, as of Figure A.4.

First part

\[ A = \int_{0}^{5} [3.5 - (-5 + 0.3x)]dx = 38.75m^2 \quad (A.27) \]

\[ T_x = \frac{1}{A} \int_{0}^{5} x(3.5 - (-5 + 0.3x))dx = 2.41935m \quad (A.28) \]

\[ T_z = \frac{1}{2A} \int_{0}^{5} (3.5^2 - (-5 + 0.3x)^2)dx \approx -0.39m \quad (A.29) \]

\[ c_B = (2.42, 0, -0.39)m \]

Second part

\[ A = \int_{0}^{10} [(3.5 - 0.3x) - (-3.5 + 0.3x)]dx = 40m^2 \quad (A.30) \]
Fig. A.11. Coordinate system to find the centre of gravity for the second cone part

\[
T_x = \frac{1}{A} \int_0^{10} x(7 - 0.6x)\,dx = 3.75m \quad (A.31)
\]

\[c_B = (3.75, 0, 0)\,m\]

**Pieces put together**

\[
x_B = \frac{\sum (V_{part}x_{gpart})}{\sum V_{part}} = 57m \quad (A.32)
\]

\[
z_B = \frac{\sum (V_{part}z_{gpart})}{\sum V_{part}} = -0.4127m \quad (A.33)
\]

And because it is symmetric we have know that \(y_B = 0\) so the centre of buoyancy is situated in \((57, 0, -0.4127)\,m\) with the same coordinate system as in Figure A.7.
B

Usage of Numeric Solver ode15s

ode15s is a multistep and variable order solver based on the numerical differentiation formulas (NDFs). Optionally, it uses the backward differentiation formulas (BDFs, also known as Gear’s method) that are usually less efficient. It is recommended by the Matlab Help documentation to try it when ode45 fails, is very inefficient, if suspected that the problem is stiff, or when solving a differential-algebraic problem [7].

To use ode15s we need to reduce the problem to set of first order differential equations. We begin with the decoupled equation of motion, and solve out $\ddot{\eta}_4$

$$\ddot{\eta}_4 = \frac{F_{40}(\omega t) - B_{44}\dot{\eta}_4 - C_{44}\eta_4}{(M_{44} + A_{44})} \quad (B.1)$$

And then we define the components of of a vector $\vec{\eta}_4 = [\eta_1^4 \eta_2^4]^T$ as follows

$$\eta_1^4 = \eta_4 \quad (B.2)$$
$$\eta_2^4 = \dot{\eta}_4 \quad (B.3)$$

After that we form the first derivatives of $\vec{\eta}_4$

$$\dot{\eta}_1^4 = \dot{\eta}_4 = \eta_2^4 \quad (B.4)$$
$$\dot{\eta}_2^4 = \ddot{\eta}_4 \quad (B.5)$$

Now we need to cast the problem needed to use ode15s

$$\vec{\eta}_4 = \frac{d\eta_4}{dt} = \frac{d}{dt} \begin{bmatrix} \eta_1^4 \\ \eta_2^4 \end{bmatrix} = \begin{bmatrix} \eta_2^4 \\ \frac{F_{40}(\omega t) - B_{44}\eta_4 - C_{44}\eta_4}{(M_{44} + A_{44})} \end{bmatrix} \quad (B.6)$$

With the initial values chosen arbitrary $\eta_4^0$
\[ \eta_4(0) = \eta_4^0 = \begin{bmatrix} \eta_4^1(0) \\ \eta_4^2(0) \end{bmatrix} = \begin{bmatrix} \eta_4(0) \\ \dot{\eta}_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]  \hspace{1cm} (B.7)

The syntax used for the solver is

\[ [T,Y] = \text{solver}(odefun,tspan,y0) \]

where `odefun` is an (or a handle for an) m-file containing the previous reduced function, `tspan` is for how long the solver should solve and `y0` is the initial values. T will hold an array with the time values and Y is a matrix with \( \eta_4 \) and \( \dot{\eta}_4 \).

Below an example code is shown for a single sinus wave.

```matlab
function phi_dot = difeq_3(t,phi,A,B,C,D,w)
    k = w.^2/9.81; %the wave number
    g = D*k*sin(w*t); %The input force
    phi_dot = zeros(size(phi)) ; % Must return a column vector
    phi_dot(1) = phi(2) ; phi_dot(2) = (g-B*phi(2) - C*(phi(1))) ;
end
```

The above function is called in the main program by

```matlab
dq_fun = @difeq_3 ;
tspan = [0:dt:simtime] ;
x0 = [0,0] ;
[t,phi] = ode15s(dq_fun, tspan, x0,[],A,B,C,D,w) ;
```
Surface Profile

This chapter is based on information taken from Seakeeping by ARJM Lloyd [5].

One of the most fundamental equations in the classical fluid dynamics is the Laplace equation.

\[ \Delta \Phi = 0 \]  

(C.1)

The velocity potential must satisfy the Laplace equation to represent a valid fluid flow. In a 2D case the velocities are defined by the equations

\[ \frac{\partial \Phi}{\partial x} = u, \quad \frac{\partial \Phi}{\partial z} = w \]  

(C.2)

The velocity profile \( \Phi \) describes the fluid flow associated with a regular wave and can according to Seakeeping [5], with \( d \) as the depth which is approaching infinity, be expressed as

\[ \Phi = \frac{g\zeta_0}{\omega} \frac{\cosh[k(d-z)]}{\cosh(kd)} \cos(kx - \omega t) \approx \frac{g\zeta_0}{\omega} e^{-kz} \cos(kx - \omega t) \]  

(C.3)

To express the surface profile the pressure contour is examined through the Bernoulli equation.

\[ \frac{q^2}{2} + \frac{\partial \Phi}{\partial t} - \Omega + \frac{P}{\rho} = 0 \]  

(C.4)

Where

\[ q^2 = u^2 + w^2 \]  

(C.5)

And \( \Omega \) is the force potential defined by

\[ \frac{\partial \Omega}{\partial x} = F_x; \quad \frac{\partial \Omega}{\partial z} = F_z \]  

(C.6)

To find the surface profile associated with the velocity potential given by C.3 the Bernoulli equation must be applicable everywhere. As the only force ap-
plied externally to any fluid is the gravity the following values can be assigned

\[ F_x = \frac{\partial \Omega}{\partial x} = 0; \quad F_z = \frac{\partial \Omega}{\partial z} = g; \]  

(C.7)

Which gives us

\[ \Omega = gz \]  

(C.8)

So the Equation C.4 becomes

\[ \frac{q^2}{2} + \frac{\partial \Phi}{\partial t} - g z + \frac{P}{\rho} = 0 \]  

(C.9)

With values and distances as of figure C.1.

The calm water pressure at depth \( z_p \) is expressed by Equation C.10.

\[ P = \rho g z_p \]  

(C.10)

If the surface is distorted by regular waves the depth of a point on this contour is

\[ z = z_p + \zeta_p \]  

(C.11)

The following expression is found as both the pressure everywhere along a contour and the depth \( z_p \) are constant and will be a constant on the contour.
for any given time $t$.

$$\int_{0}^{t} \left( \frac{P}{\rho} - g \zeta_p \right) dt = \text{constant} \quad (C.12)$$

Define a new velocity potential where Equation C.12 is added, this is possible because the velocities are functions of the potential gradients and not the potential itself

$$\phi' = \phi + \int_{0}^{t} \left( \frac{P}{\rho} - g \zeta_p \right) dt \quad (C.13)$$

Differentiate on both sides so that

$$\frac{\partial \phi'}{\partial t} = \frac{\partial \phi}{\partial t} + \frac{P}{\rho} - g \zeta_p \quad (C.14)$$

Put Equation C.14 into Equation C.9 so that it becomes

$$\frac{q^2}{2} + \frac{\partial \phi'}{\partial t} - g \zeta = 0 \quad (C.15)$$

Where the prime can be omitted in further calculations. To go further with the investigations assume that the velocity is small, which is comparable to assuming that the wave amplitude $\zeta_0$ is small compared to the wave length. This gives us opportunity to neglect $q^2$ so that the depression of the constant pressure surface is

$$\zeta_p = \frac{1}{g} \left( \frac{\partial \phi}{\partial t} \right)_{z=z_p+\zeta_p} \quad (C.16)$$

But since $\zeta_p$ is small it can be expressed as

$$\zeta_p \approx \frac{1}{g} \left( \frac{\partial \phi}{\partial t} \right)_{z=z_p} \quad (C.17)$$

Putting equation C.17 into Equation C.3 the equation for the constant pressure contour at depth $z_p$ is expressed as

$$\zeta_p = \zeta_0 e^{-kz_p} \sin(kx - \omega t) \quad (C.18)$$