1. Lecture 1, January 31

1.1. **Presheaf.** Let X be a topological space. A *presheaf* $\mathscr{F}$ of abelian groups on X is a collection

(1) For any open $U \subseteq X$, an abelian group $\mathscr{F}(U)$.
(2) For any inclusin of opens $V \subseteq U$, a homomorphism of abelian groups $\rho_{U,V} \colon \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$

satisfying the following $\mathscr{F}(\emptyset) = 0$, $\rho_{UU} = \mathrm{id}$, and for any open inclusions $W \subseteq V \subseteq U$ we have

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV}.$$

1.2. **Alternatively.** Another way to phrase the presheaf axioms is to consider the topological space X as a category $C(X)$, where the objects are open subsets of X, and the morphisms are inclusions. Then a presheaf $\mathscr{F}$ on X is a contra variant functor $\mathscr{F} \colon C(X) \longrightarrow \mathrm{Ab.\,grp}$ from $C(X)$ to the category of abelian groups.

1.2.1. We will also consider presheaves of rings, and sets.

1.3. **Notation.**

- Elements $s \in \mathscr{F}(U)$ are called sections (over U).
- The homomorphisms ρ_{UV} are called restriction maps.
- We will often write $s_{|V}$ instead of $\rho_{UV}(s)$.

Example 1.4. Holomorphic functions on a complex manifold, differentiable functions on a manifold.

1.5. **Sheaf.** A presheaf $\mathscr{F}$ on X is a *sheaf* if for any open cover $\{V_i\}$ of an open U, the following holds

(1) If $s \in \mathscr{F}(U)$ is such that $s_{|V_i} = 0$ for all i, then $s = 0$.
(2) Given $s^i \in \mathscr{F}(V_i)$ such that $s^i_{|V_i \cap V_j} = s^j_{|V_i \cap V_j}$ for all i, j, then there exist $s \in \mathscr{F}(U)$ such that $s_{|V_i} = s^i$, for all i.

1.6. **Alternatively.** A presheaf $\mathscr{F} \colon C(X) \longrightarrow \mathrm{Ab.\,grps}$ is a sheaf, if for any open covering $\{V_i\}$ of any open U, the following sequence

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(V_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathscr{F}(V_i \cap V_j)$$

is exact. Exactness means that the leftmost group is identified with those elements x in the middle group such that $p(x) = q(x)$.

Example 1.7. The presheaf examples considere above, are sheaves.

1.8. **Stalk.** If $\mathscr{F}$ is a presheaf on X, and $P \in X$ a point, then we have the notion of the stalk of $\mathscr{F}$ at P. Note that if U and V are two open sets containing P, then the open set $U \cap V$ also contains P. Thus the opens sets containing P form a directed set, and we can form the direct limit

$$\mathscr{F}_P = \lim \mathscr{F}(U).$$

A property of direct limits we will use often is that if $s \in \mathscr{F}(U)$ is mapped to zero in the direct limit, then $s_{|V}$ in $\mathscr{F}(V)$, for some open $V \subseteq U$.

1.9. **Morphism of sheaves.** A morphism $\varphi \colon \mathscr{F} \longrightarrow \mathscr{G}$ of presheaves is, for any open $U \subseteq X$ a homomorphism of groups $\varphi_U \colon \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$, making the commutative diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\rho_{VU}"'] & \mathscr{G}(U) \arrow[d, "\rho_{VU}"] \\ \mathscr{F}(V) \arrow[r, "\varphi_V"] & \mathscr{G}(V), \end{tikzcd}$$

for all open $V \subseteq U$. A morphism $\varphi \colon \mathscr{F} \longrightarrow \mathscr{G}$ of sheaves is a morphism of $\mathscr{F}$ and $\mathscr{G}$ considered as presheaves.

Proposition 1.10. *Let $\varphi \colon \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves on a topological space X. Then φ is an isomorphism if and only if the induced map of stalks $\varphi_P \colon \mathscr{F}_P \longrightarrow \mathscr{G}_P$ is an isomorphism for all points $P \in X$.*

Proof. We proved this as in [Ha][1] Proposition 1.1. □

1.11. **Kernel, image and cokernel.** The kernel, image and cokernel of a map of presheaves is defined in the obvious way. The kernel of a map of sheaves is a sheaf, but the image and cokernel is not always a sheaf.

Example 1.12. Let $X = \mathbf{R}^2 \setminus (0,0)$, and let $\mathscr{F} = C^\infty$ be the sheaf of differentiable functions on X. Consider

$$\mathscr{G}(U) = \{(g_1, g_2) \in \mathscr{F}(U) \times \mathscr{F}(U) \mid \frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial x}\}.$$

It is readily checked that $\mathscr{G}$ is a sheaf. We have furthermore the map of sheaves $\varphi \colon \mathscr{F} \longrightarrow \mathscr{G}$, that for any open U sends $f \in \mathscr{F}(U)$ to

$$\varphi_U(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}).$$

We have that the image presheaf $\mathrm{Im}(\varphi) \subseteq \mathscr{G}$. On simply connected domains U we know that $(g_1, g_2) = \varphi_U(f)$ if and only if $\frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial x}$. As a consequence we have that for any $P \in X$ the stalk of $\mathrm{Im}(\varphi)$ equals the

[1] Robin Hartshorne, Algebraic Geometry, GTM 52

stalk of $\mathscr{G}$, at P. If $\mathrm{Im}(\varphi)$ was a sheaf, we would by Proposition 1.10 have equality of sections as well. We have however the global section

$$(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}) \in \mathscr{G}(X)$$

which we know does not have a potential function on X. Therefore $\mathrm{Im}(\varphi)$ is not a sheaf.

1.13. **Sheafification.** Let $\mathscr{F}$ be a presheaf on X. Then there exists a sheaf $\mathscr{F}^+$, and a map of presheaves $\Phi\colon \mathscr{F} \longrightarrow \mathscr{F}^+$ having the following, universal, property. Any morphism $\varphi\colon \mathscr{F} \longrightarrow \mathscr{G}$ to a sheaf $\mathscr{G}$ will have a unique factorization via $\Phi\colon \mathscr{F} \longrightarrow \mathscr{F}^+$.

- The sheaf $\mathscr{F}^+$ is called the associated sheaf to $\mathscr{F}$.
- If $\mathscr{F}$ was a sheaf then $\Phi\colon \mathscr{F} = \mathscr{F}^+$.
- For any point $P \in X$ we have $\Phi_P\colon \mathscr{F}_P = \mathscr{F}^+_P$.

1.14. **Image and cokernel.** If φ is a morphism of sheaves, then the image sheaf and cokernel sheaf always means the sheaf associated to the image presheaf and cokernel presheaf.

Example 1.15. In the Example 1.12 the image sheaf of the morphism of sheaves $\varphi\colon \mathscr{F} \longrightarrow \mathscr{G}$ considered there, is $\mathscr{G}$.

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