1. Lecture 2, February 7

1.1. **Injective and surjective.** A map of sheaves $\varphi\colon \mathscr{F} \longrightarrow \mathscr{G}$ is *injective* if the kernel sheaf is the trivial sheaf. The map is *surjective* if the image sheaf equals $\mathscr{G}$.

Proposition 1.2. *Let $\varphi\colon \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves.*

1. *The map φ is injective if and only if φ_U is injective for all open $U \subseteq X$ if and only if φ_P is injective for all points $P \in X$.*
2. *The map φ is surjective if and only if φ_P is surjective for all points $P \in X$ iff and only if, for any open U, any $t \in \mathscr{G}(U)$ there exists an open cover $\{U_i\}$ of U, and sections $s_i \in \mathscr{F}(U_i)$ such that $\varphi_{U_i}(s_i) = t_{|U_i}$ for all i.*

Proof. We proved this in Proposition 1, last time. □

1.3. **Direct image sheaf.** Let $f\colon X \longrightarrow Y$ be a continuous map of topological spaces, and let $\mathscr{F}$ be a sheaf on X. Then the presheaf $U \mapsto \mathscr{F}(f^{-1}(U))$ is a sheaf on Y. This sheaf is denoted $f_*\mathscr{F}$, and we call it the direct image sheaf.

Example 1.4. The global sections of a sheaf $\mathscr{G}$ on a space X is $\mathscr{G}(X)$, and often denoted $\Gamma(X, \mathscr{G})$. If $f\colon X \longrightarrow Y$ is a continuous map, and $\mathscr{F}$ a sheaf on X, then we have that at the global sections of $f_*\mathscr{F}$ is by definition the global sections of $\mathscr{F}$. In particular, if $Y = \{pt\}$ is a point, we have that the sheaf $f_*\mathscr{F}$ is given by the group $\Gamma(X, \mathscr{F})$.

1.5. **Inverse image sheaf.** Let $\mathscr{G}$ be a sheaf on Y, and $f\colon X \longrightarrow Y$ a continuous map of spaces. Then the sheaf associated to the presheaf

$$U \mapsto \lim \mathscr{G}(V)$$

on X, is called the inverse image sheaf and denoted $f^{-1}\mathscr{G}$.

Example 1.6. Let $X = x \in Y$ be a point, and $\mathscr{G}$ a sheaf on Y. Let $f\colon X \longrightarrow Y$ be the inclusion map. Then the sheaf $f^{-1}\mathscr{G}$ on X is given by the group $\mathscr{G}_x$.

1.7. **Exercises.** Recommended exercises are from Section 2.1 (in [Ha][1]) 1.6, 1.7, 1.8, 1.14, 1.15, 1.18, 1.19.

2. Spectra of rings

Let A be a commutative, unital ring, and let $\mathrm{Spec}(A)$ denote the set of prime ideals in A. For any ideal $\subseteq A$ we let $V(I)$ denote the set of prime ideals in A, containing the ideal I. We have the following

$$V(I \cdot J) = V(I) \cup V(J) \quad \text{and} \quad V(\sum_\alpha I_\alpha) = \cap V(I_\alpha)$$

[1]Robin Hartshorne, Algebraic Geometry, GTM 52.

for all ideal I, J and I_α in A. Moreover we have $V(0) = \mathrm{Spec}(A)$ and $V((1)) = \emptyset$. So $V(I)$ form the closed set of the Zariski topology on $\mathrm{Spec}(A)$. The sets $D(f) = \mathrm{Spec}(A) \setminus V((f))$ form a basis for the topology.

2.1. **Compactness.** The topological space $X = \mathrm{Spec}(A)$ is quasi-compact. To have a covering $X = \{D(f_\alpha)\}_{\alpha \in \mathscr{A}}$ is that the ideal $(f_\alpha)_{\alpha \in \mathscr{A}} = A$ generated by the f_α is the whole ring. In particular $1 = \sum_{i=1}^n x_i f_{\alpha_i}$, which is equivalent with that the finite opens $\{D(f_{\alpha_i}\}_{i=1}^n$ cover X.

2.2. **Maps of spectra.** If $\varphi\colon A \longrightarrow B$ is a homomorphism of rings, then we get an induced map $f\colon X = \mathrm{Spec}(B) \longrightarrow Y = \mathrm{Spec}(A)$ sending a point $x \mapsto \varphi^{-1}(x)$; the inverse image of a prime ideal is a prime ideal. We have that

$$f^{-1}(V(I)) = V(IB),$$

where $IB = \varphi(I)B$ is the ideal generated by the image of I. In particular the map of spectra is continuous. We also have that the inverse image of basic opens is basic open,

$$f^{-1}(D(g)) = D(\varphi(g)).$$

The continuous map $f\colon X \longrightarrow Y$ is not always a closed map, we have

$$\overline{f(V(J))} = V(J \cap A),$$

where $J \cap A$ is the ideal $\varphi^{-1}(J)$.

2.3. **Homeomorphisms.** If $I \subseteq A$ is an ideal we have the induced projection map $\varphi\colon A \longrightarrow A/I$. The induced map of spectra

$$f\colon \mathrm{Spec}(A/I) \longrightarrow \mathrm{Spec}(A)$$

is a homeomorphism onto its image, which is $V(I)$. And similarily, if $A \longrightarrow S^{-1}A$ is a localization, then the induced map $\mathrm{Spec}(S^{-1}A) \longrightarrow \mathrm{Spec}(A)$ is a homeomorphism onto its image. In particular, with $S = \{g^n\}$, then $\mathrm{Spec}(A_g)$ is identified with the basic open set

$$\mathrm{Spec}(A_g) = D(g).$$

Example 2.4. Play around with $K[x, y]$ the polynomial ring in two variables over an algebraically closed field K.

2.5. **Exercises.** The above facts are given by the following exercises (in [AM][2]) Chapter 1, 15, 16, 17, 18, 19, 20, 21.

Lemma 2.6. *Let A be a ring, and assume that $f_1, \dots, f_k$ generate the ring $(f_1, \dots, f_n) = A$. Then the sequence*

$$A \longrightarrow \prod_i A_{f_i} \rightrightarrows \prod_{i,j} A_{f_i f_j}$$

is exact.

[2]Atiyah-MacDonald, Introduction to Commutative Algebra

Proof. Injectivity is clear. If $x \in A$ is mapped to zero in A_{f_i} then $f_i^{n_i} x = 0$ in A, for some n_i. We have that $A_f = A_{f^n}$, so if $\{D(f_i)\}$ cover $X = \mathrm{Spec}(A)$ then so will $\{D(f_i^{n_i})\}$. In particular we have that $1 = \sum_{i=1}^k a_i f_i^{n_i}$, and it follows that $x = 1 \cdot x = 0$. To prove surjectivity, we follow [Mu][3]. Let $B \subseteq \prod A_{f_i}$ denote the subring $\{x_1, \dots, x_k\}$ such that $x_i = x_j$ in $A_{f_i f_j}$, for all i, j. We have $A \subseteq B$, and need to establish surjectivity. For each $i = 1, \dots, k$, we have that $x_i = \frac{a_i'}{f_i^{n_i}}$, with $a_i = \in A$. We can assume that

$$\{x_1, \dots, x_n\} = \{\frac{a_1}{f_1^n}, \dots, \frac{a_k}{f_k^n}\},$$

that is the power of the numerators are the same n. By assumption we have

$$(f_i f_j)^{n_{i,j}} (f_j^n a_i - f_i^n a_j) = 0$$

in A, for all i, j. Let $N = \max\{n_{i,j}\}$, and set $M = N + n$. With $b_i = a_i f_i^N$ we have that $x_i = \frac{b_i}{f_i^M}$, for all $i = 1, \dots, k$. And we have

$$f_j^M b_i = f_i^M b_j$$

in A, for all i, j. By the covering assumption, we have that $1 = \sum_{i=1}^n y_i f_i^M$, with some $y_1, \dots, y_k$ in A. Let $x = \sum_{i=1}^k y_i b_i$. For each $j = 1, \dots, k$ we now have

$$f_j^M x = \sum_{i=1}^k y_i f_j^M b_i = \sum_{i=1}^k y_i f_i^M \cdot b_j = b_j.$$

Thus, the element $x \in A$ is such that $x = x_j$ in A_{f_j}, for all $j = 1, \dots, k$, proving $A = B$. □

Proposition 2.7. *Let A be a ring. There is a unique sheaf $\mathscr{O}_A$ on $X = \mathrm{Spec}(A)$ such that for any basic open set $U = D(f)$ we have*

$$\mathscr{O}_A(U) = A_f.$$

We have, moreover that $\mathscr{O}_{A,P} = A_P$, for any point $P \in X$.

Proof. For any open U we can find some covering $\{D(f_i)\}$ by basic opens. For basic opens $D(f)$ we assign the value of $\mathscr{O}_A$ as A_f. We will define $\mathscr{O}_A(U)$ as the kernel of

$$\prod_i A_{f_i} \rightrightarrows \prod_{i,j} A_{f_i f_j}$$

Note first that if $U = D(f)$ was a basic open set then using Lemma 2.6 with A_f replacing A, we see that the assigment $U \to \mathscr{O}_A(U)$ is well-defined for basic opens. Using the lemma again, one establishes that $\mathscr{O}_A(U)$ is well-defined (independent of the covering) for arbitrary open U. This proves the first part of the proposition. The second part follows since we may restrict ourselves to basic opens when considering the direct limit. □

[3] David Mumford, The Red Book of Varities and Schemes, LNM 1358

2.7.1. Let U_i be a basis for the topology on a space X, and assume that we have a functorial assignment $U_i \mapsto \mathscr{F}(U_i)$ of abelian groups, for each basis. Assume furthermore that this assignment satisfies the sheaf axiom for basic opens. Then $\mathscr{F}$ extends uniquely to a sheaf on X. See e.g. Proposition 1.12 in [EH][4]. In the situation above, we are in a slightly more restricted situation since the intersection of the basic open sets $D(f_i)$ on $X = \mathrm{Spec}(A)$ are again a basic open set.

Department of Mathematics, KTH, Stockholm, Sweden
E-mail address: skjelnes@kth.se

[4]David Eisenbud and Joe Harris, The Geometry of Schemes, GTM 197