1. Lecture 3, February 14

1.1. **Ringed space.** A topological space X and a sheaf of rings $\mathscr{O}_X$ on X is a ringed space. A morphism of ringed spaces $(X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ is a pair $(f, f^\sharp)$ where $f \colon X \longrightarrow Y$ is a continuous map, and where $f^\sharp \colon \mathscr{O}_Y \longrightarrow f_* \mathscr{O}_X$ is a map of sheaves on Y.

1.2. **Locally ringed space.** A ringed space $(X, \mathscr{O}_X)$ is a *locally* ringed space if the stalks $\mathscr{O}_{X,x}$ is a local ring for every point $x \in X$. A map $(f, f^\sharp) \colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ of locally ringed spaces is a map of ringed spaces such that the induced map

$$f^\sharp_x \colon \mathscr{O}_{Y, f(x)} \longrightarrow \mathscr{O}_{X,x}$$

is a local homomorphism of local rings.

Example 1.3. The affine schemes $(\operatorname{Spec}(A), \mathscr{O}_A)$ are locally ringed spaces.

Proposition 1.4. *Let $\varphi \colon A \longrightarrow B$ be a homomorphism of rings. Then we have a natural induced map of affine schemes*

$$(f, f^\sharp) \colon (\operatorname{Spec}(B), \mathscr{O}_B) \longrightarrow (\operatorname{Spec}(A), \mathscr{O}_A).$$

Proof. The map $f \colon \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ sending a prime Q to $\varphi^{-1}(Q)$ is continuous. In fact we have that

$$f^{-1}(D(x)) = D(\varphi(x)),$$

for any element $x \in A$. To describe a map of sheaves it suffices to describe the map on basic open set, as long as these maps are compatible. For any basic open set $D(x)$ in $\operatorname{Spec}(A)$ we have $f_* \mathscr{O}_B(D(x)) = B_{\varphi(x)}$. We therefore obtain the natural map by localization;

$$\begin{array}{ccc} A & \xrightarrow{\ \varphi\ } & B \\ \downarrow & & \downarrow \\ A_x & \longrightarrow & A_x \bigotimes_A B = B_{\varphi(x)} \end{array}$$

These maps defined over basic open sets clearly are compatible, hence give a map of sheaves $\mathscr{O}_A \longrightarrow f_* \mathscr{O}_B$. □

1.5. **Scheme.** A locally ringed space $(X, \mathscr{O}_X)$ is a *scheme* if there exists an open cover $\{U_i\}$ of X such that the locally ringed space $(U_i, \mathscr{O}_{X|U_i})$ is an affine scheme, for each i. That is, there is a homomorphism

$$(f, f^\sharp) \colon (U_i, \mathscr{O}_{X|U_i}) \longrightarrow (Spec(A_i), \mathscr{O}_{A_i})$$

with f a homeomorphism, and $f^\sharp$ an isomorphism of sheaves.

Example 1.6. If $(X, \mathscr{O}_X)$ is a scheme, and $U \subseteq X$ an open subset, then $(U, \mathscr{O}_{X|U})$ is a scheme.

Theorem 1.7. *Let A be a ring, and X a scheme. A morphism of schemes $f\colon X \longrightarrow \operatorname{Spec}(A)$ induces a homomorphism of rings $\varphi\colon A \longrightarrow \Gamma(X, \mathscr{O}_X)$. This assignment gives a bijection*

$$\operatorname{Hom}_{schemes}(X, \operatorname{Spec}(A)) \longleftrightarrow \operatorname{Hom}_{rings}(A, \Gamma(X, \mathscr{O}_X)).$$

Proof. Clearly a morphism of schemes $f\colon X \longrightarrow \operatorname{Spec}(A)$ gives a ring homomorphism, of global sections, $\varphi\colon A \longrightarrow \Gamma(X, \mathscr{O}_X)$. Conversely, given $\varphi\colon A \longrightarrow \Gamma(X, \mathscr{O}_X)$. Let $U = \operatorname{Spec}(B)$ be an open affine in X. The composition of φ with the restriciton map $\rho_{XU}\colon \Gamma(X, \mathscr{O}_X) \longrightarrow \Gamma(U, \mathscr{O}_X) = B$ gives a ring homomorphism $A \longrightarrow B$. As in Proposition (1.4) we get a morphism of affine schemes $\operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$. If U_i and U_j are two open affines in X, let $\{W_{ijk}\}$ be an affine covering of $U_i \cap U_j$. Since the restriction maps are compatible, we get that the map of affine schemes $\operatorname{Spec}(B_i) \longrightarrow \operatorname{Spec}(A)$ agree on overlaps. This gives a map of schemes from the union, which is X, to $\operatorname{Spec}(A)$.

It is clear that if we start with a global section $\varphi\colon A \longrightarrow \Gamma(X, \mathscr{O}_X)$, make the construction above to get a morphism $f\colon X \longrightarrow \operatorname{Spec}(A)$ of schemes, and then take the global sections then we get back φ.

We need to show that if we start with a morphism of schemes $f\colon X \longrightarrow \operatorname{Spec}(A)$, take its global sections, and then make the construction above, we get back f. We do this by showing that two different maps f and g give different map of global sections. We note that difference of a maps of spaces and sheaves can be checked locally. Hence we may assume that $X = \operatorname{Spec}(B)$ is affine.

Let $(f, f^\sharp)\colon (\operatorname{Spec}(B), \mathscr{O}_B) \longrightarrow (\operatorname{Spec}(A), \mathscr{O}_A)$ be a morphism of schemes. Let $\varphi\colon A \longrightarrow B$ be the induced map of global sections, and let $(\varphi_1, \varphi_1^\sharp)$ be the map of schemes $\operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ we get by the construction given in Proposition 1.4.

Let $D(x) \subseteq \operatorname{Spec}(A) = Y$ be a basic open affine, and consider the commutative diagram

$$\begin{array}{ccccc}
\mathscr{O}_A(Y) = A & \xrightarrow{\varphi} & f_*\mathscr{O}_B(Y) = B & & \\
\downarrow & & \downarrow & & \\
A_x = \mathscr{O}_A(D(x)) & \xrightarrow{\varphi\otimes 1} & A_x \bigotimes_A B & \longrightarrow & f_*\mathscr{O}_B(D(x)) = \mathscr{O}_B(f^{-1}(D(x))) \\
\downarrow & & \downarrow & & \downarrow \\
A_P & \longrightarrow & A_P \bigotimes_A B & \longrightarrow & B_Q
\end{array}.$$

The composition of the maps in the second horizontal row is $f_U^\sharp$, with $U = D(x)$. The map $B \longrightarrow \mathscr{O}_B(f^{-1}(D(x)))$ is the restriction map, and it will factorize via $A_x \bigotimes_A B$ by the defining properties of the tensor product. The lower horizontal map we obtain by taking the limit around each basic open $D(x)$ containing the point $P = f(Q)$, with some fixed $Q \in \operatorname{Spec}(B)$. The composition of the lower horizontal map is $f^\sharp_{f(Q)} \colon \mathscr{O}_{A,f(Q)} \longrightarrow \mathscr{O}_{B,Q}$. It follows from the diagram that $\varphi^{-1}(Q) = P$, for all points $Q \in \operatorname{Spec}(B)$ hence $f = \varphi_1$. In particular we have that $f_*\mathscr{O}_B = \varphi_{1*}\mathscr{O}_B$. We have two maps $f^\sharp$ and $\varphi_1^\sharp$ of sheaves $\mathscr{O}_A \longrightarrow f_*\mathscr{O}_B$ on $\operatorname{Spec}(A)$, that agree on stalks (check). They are therefore equal. Hence $(f, f^\sharp) = (\varphi_1, \varphi_1^\sharp)$. □

Corollary 1.8. *The category of commuative rings and ring homomorphisms is equal with the category of affine schemes and morphism of schemes.*

Example 1.9. Consider the open subset $U = \operatorname{Spec}(k[x,y]) \setminus V(x,y)$. We have that $(U, \mathscr{O}_U)$ is a scheme, where $\mathscr{O}_U$ is the restrction of the structure sheaf on $X = \operatorname{Spec}(k[x,y])$ to U. The inclusion map $i \colon U \longrightarrow X$ is in a natural way a morphism of schemes.

Note that we have the open affine cover $D(x) \cup D(y)$ of U. We use this cover to compute that the global sections $\Gamma(U, \mathscr{O}_U) = k[x,y]$. From the Theorem we obtain that the map $i \colon U \longrightarrow X$ is given by the map of global section $k[x,y] \longrightarrow \Gamma(U, \mathscr{O}_U)$. We have that this map is the identity map. In particular, from the Corollary, we see that U is *not* an affine scheme.

1.10. **Exercises.** Hartshorne, chapter 2.2, Exercises 2.2, 2.7, 2.8, 2.12, 2.13, 2.16, 2.17, 2.18

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