1. Lecture 4, February 21

1.1. **Open immersion.** Let $(X, \mathscr{O}_X)$ be a scheme. If $U \subseteq X$ is an open subset then $(U, \mathscr{O}_{X|U})$ is a scheme, and we have a natural map of schemes $(i, i^\sharp)\colon (U, \mathscr{O}_{X|U}) \longrightarrow (X, \mathscr{O}_X)$. We say that U is an open subscheme of X.

A morphism of schemes $f\colon Y \longrightarrow X$ is an *open immersion* if f induces an isomorphism with Y and an open subscheme $U \subseteq X$.

1.2. **Glueing.** Let U_1 and U_2 be two schemes, and let $f_i\colon W \longrightarrow U_i$ be two open immersions. Then the topological space $XU_1 \cup_W U_2$ obtained by glueing the disjoint union $U_1 \sqcup U_2$ along the W, becomes a scheme in the natural way. Over U_i we set $\mathscr{O}_{X|U_i} = \mathscr{O}_{U_i}$, and then we use the isomorphism $f_i^{\sharp^{-1}}$ to pass over to the structure sheaf on W. Concretely, we have the isomorphism

$$(f_i, f_i^\sharp)\colon (W, \mathscr{O}_W) \longrightarrow (W_i, \mathscr{O}_{X|W_i}),$$

where $W_i = f_i(W)$ is the open subset of U_i. Use these isomorphisms to identify the sections over W_1 with sections over W_2.

Example 1.3. Consider $U = \operatorname{Spec}(k[x])$ the affine line over a field k. The localization map $k[x] \longrightarrow k[x, x^{-1}]$ gives an open immersion of $W = \operatorname{Spec}(k[x, x^{-1}])$ into U. The scheme $U \cup_W U$ is the line with a double point.

Example 1.4. We have that $W = k[t, t^{-1}] = k[x, y]/(xy - 1)$ which gives us two open immersions of W into the affine line $\operatorname{Spec}(k[t])$ - depending on letting t be the x or the y variable. Now the glueing $U \cup_W U$ is a scheme that we usually denote by $\mathbf{P}^1_k$.

1.5. **Proj of graded rings.** Let $S = \bigoplus_{d \geq 0} S_d$ be a (positively) graded ring. For any homogenous element $f \in S$ we let $S_{(f)}$ denote the subring of S_f consisting of degree zero elements.

Let $\operatorname{Proj}(S)$ denote the set of homogeneous prime ideals $P \subseteq S$ that do not contain the irrelevant ideal $S_+ = \bigoplus_{d>0} S_d$.

Lemma 1.6. *Let $I \subseteq S$ be a homogeneous ideal, and let $V(I)$ denote the set $\{P \in \operatorname{Proj}(S) \mid I \subseteq P\}$. Then we have that*

1. $V(IJ) = V(I) \cup V(J)$,
2. $V(\sum_\alpha I_\alpha) = \cap V(I_\alpha)$.

Proof. Exercise. $\square$

As a consequence of the lemma, we have that the sets $V(I)$ form the closed sets for the Zariski topology on $\operatorname{Proj}(S)$. In particular we have that the open sets $D_+(f) = \operatorname{Proj}(S) \setminus V(f)$, with homogeneous elements f, form a basis for the topology on $\operatorname{Proj}(S)$.

Lemma 1.7. *Let $f \in S_+$ be a homogeneous element. Then the set $D_+(f)$ is, naturally, homeomorphic to $|\operatorname{Spec}(S_{(f)})|$.*

Proof. For any homogeneous ideal $I \subseteq S$, let $\varphi(I) = IS_f \cap s_{(f)}$. We then have the map $\tilde{\varphi}\colon D_+(f) \longrightarrow |\operatorname{Spec}(S_{(f)})|$ that sends a homogenous prime ideal P not containg (f), to the prime ideal $\varphi(P)$ in $S_{(f)}$. One check is continuous (e.g. inverse image of closed are closed). By using the properties of localization one can verify that the map is both injective and bijective. Finally, one has that $\tilde{\varphi}(V(I)) = V(\varphi(I))$, hence the map is a closed map. Thus the map $\tilde{\varphi}\colon D_+(f) \longrightarrow |\operatorname{Spec}(S_{(f)})|$ is a homeomorphism. □

Lemma 1.8. *Let $f \in S_+$, and let $\{f_i\}$ be a collection of homogeneous elements such that $\{D_+(ff_i)\}$ is a covering of $D_+(f)$. Then we have exact sequence*

$$S_{(f)} \longrightarrow \prod_i S_{(ff_i)} \rightrightarrows \prod_{i,j} S_{(ff_if_j)}$$

Proof. Exercise: For instance, use that $S(g)$ is a subring of S_g, and that the sequence sits naturally in the sequence of localized rings. The sequence of localized rings is exact by Lemma 1.7 above, and Lemma 2.6 of Lecture 2. Then exactness follows by the diagram. □

1.9. **The scheme** Proj(S)**.** A basis for the topological space $X = \operatorname{Proj}(S)$ are the open sets $D_+(f)$, with $f \in S_+$. On each open we define the ring $S_{(f)}$. By Lemma 1.8 we have exactness on basic opens, and therefore there is a unique sheaf of rings $\mathscr{O}_X$ on X, such that $\mathscr{O}_X(D_+(f)) = S_{(f)}$. Now we have that $X = \operatorname{Proj}(S)$ is a scheme, where the structure sheaf $\mathscr{O}_X(D_+(f)) = S_{(f)}$ for any basic open $D_+(f)$, with homogeneous $f \in S_+$.

Example 1.10. Let $S = A[x, y, z]$ the polynomial ring in three variables over a ring $A = S_0$, and where the variables all have degree 1. We have that $D_+(x), D_+(y)$ and $D_+(z)$ cover Proj(S), since they generate the irrelevant ideal. We compute that

$$S_{(x)} = A[\frac{y}{x}, \frac{z}{x}] = A[t_1, t_2],$$

and that

$$S_{(xy)} = A[\frac{y}{x}, \frac{z}{x}\frac{x}{y}, \frac{z}{y}] = A[t_1, t_1^{-1}, t_2].$$

If we similarly write $S_{(y)} = A[u_1, u_2]$, we get that the identification of affine planes $D_+(x)$ and $D_+(y)$ is done by the ring isomorphism

$$A[t_1, t_1^{-1}, t_2] \longrightarrow A[u_1, u_1^{-1}, u_2]$$

that sends $t_1 \mapsto u_1^{-1}$ and $t_2 \mapsto u_2u_1^{-1}$. This scheme is usually denoted $\mathbf{P}^2_A$, the projective plane over A.

1.11. **Closed immersion.** A morphism $f\colon X \longrightarrow Y$ of schemes is a *closed immersion* if f induces a homeomorphism of the topological space $|X|$ with a closed subset of Y, and the induced map of sheaves $f^\sharp\colon \mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X$ is surjective.

Example 1.12. Let $I \subseteq A$ be an ideal, and consider the projection map $\varphi\colon A \longrightarrow A/I = B$. We have that the induced map of schemes $\mathrm{Spec}(A/I) \longrightarrow \mathrm{Spec}(A)$ identifies the space $|\mathrm{Spec}(A/I)|$ with the closed set $V(I)$. The map of sheaves is, for any basic open set $D(x) \subseteq \mathrm{Spec}(A)$ the following

$$A_x \longrightarrow A_x \otimes_A A/I = A_x/IA_x = B_{\varphi(x)} = f_*\mathscr{O}_B(D(x)).$$

That is simply the map of global sections $A \longrightarrow B = A/I$, tensored with A_x. It is surjective, hence the map of sheaves is surjective. In fact, it turns out that any closed subscheme of an affine scheme is given by an ideal in this way, see Exercise 3.11, Chapter 2.3.[1]

Example 1.13. Consider the map of schemes given by the ring homomorphisms

$$k[x,y] \longrightarrow k[x,y]/(x^2, xy) \longrightarrow k[x,y]/(x) = k[y].$$

Note also that for any ring A the map of spaces

$$|\mathrm{Spec}(A)| \longrightarrow |\mathrm{Spec}(A/\sqrt{0})|$$

is the identity.

1.14. **Some words.** A scheme $(X, \mathscr{O}_X)$ is *connected* if the underlying topological spaces $|X|$ is connected. The scheme is *irreducible* if the underlying space $|X|$ is irreducible. The scheme is *reduced* if the ring $\mathscr{O}_X(U)$ is reduced, for any open $U \subseteq X$. The scheme is *integral* if the ring $\mathscr{O}_X(U)$ is integral domain, for any open $U \subseteq X$.

1.15. **Home assignment.** Exercise 3.11 (closed subschemes), Chapter 3.2, is to be handed in not later than March 20, nicely in Tex.

1.16. **Exercises.** Hartshorne, Chapter 3.2: 3.6, 3.18.

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[1]R. Hartshorne, Algebraic Geometry, GTM 52.