1. Lecture 5, February 28

Proposition 1.1. *Let X be a scheme. Then X is integral if and only if X is reduced and irreducible.*

Proof. We proved this as in [Ha][1] Proposition 3.1. □

Definition 1.2. A scheme X is *locally Noetherian* if there exists an open affine covering $\{\mathrm{Spec}(A_i)\}_i$ of X, with B_i Noetherian ring, for all i. If furthermore the space $|X|$ is quasi-compact, then we say that the scheme X is *Noetherian.*

Remark 1.3. If X is a Noetherian scheme then its underlying space $|X|$ satisfies the descending chain condition for closed subsets. Construct an example showing the converse is not true, a scheme X does not have to be Noetherian, even if it satisfies the descending chain condition for closed subsets.

Proposition 1.4. *A scheme X is locally Noetherian if and only if for every open $U = \mathrm{Spec}(A)$ in X, we have that A is Noetherian.*

Proof. Note that if A is Noetherian, then so is any localization A_f since $A_f = A[x]/(xf-1)$. Thus, if $U = \mathrm{Spec}(B)$ is an open affine in X, we can assume that we have an open, finite, covering $\{\mathrm{Spec}(A_i)\}_{i=1}^n$ of U, with Noetherian rings A_i. Let $I \subseteq B$ be an ideal. We have the exact sequence

$$\begin{array}{ccccc} B & \longrightarrow & \prod_{i=1}^n A_i & \rightrightarrows & \prod_{i,j} A_i \bigotimes_B A_j \\ \downarrow & & \downarrow & & \downarrow \\ B/I & \longrightarrow & \prod_{i=1}^n A_i/IA_i & \rightrightarrows & \prod_{i,j} A_i/IA_i \bigotimes_B A_j/IA_j \end{array}$$

Exactness of both sequences follows from Lemma 2.6, Lecture 2. The kernel of the leftmost vertical map is I, and the kernel of the middle vertical map is $\prod IA_i$. Since the left horizontal maps are injective, we get that I is the intersection of B with $\prod IA_i$. If we let the open immersion $\mathrm{Spec}(A_i) \subseteq \mathrm{Spec}(B)$ be given by the ring homomorphism $\varphi_i \colon B \longrightarrow A_i$, we have that $I = \cap_{i=1}^n \varphi_i^{-1}(\varphi_i(I)A_i)$. If we now are given an ascending chain of ideals $I_1 \subseteq \cdots I_j \subseteq I_{j+1}$ in B, we get by extension to A_i, an ascending chain in A_i. Since A_i is Noetherina, the chain stabilizes at n_i. Let $N = \max\{n_1, \ldots, n_n\}$. Then the extension of the chain of ideals stabilizes at N, at each extension A_i. Since $I = \cap_{i=1}^n \varphi_i^{-1}(\varphi_i(I)A_i)$ it follows that the chain is stabilized at N in the ring B, and we have that B is Noetherian. □

Definition 1.5. A morphism of schemes $f \colon X \longrightarrow Y$ is *locally of finite type* if there exists an open affine covering $\{\mathrm{Spec}(A_i) = U_i\}_{i \in \mathscr{I}}$ of Y, and for each i, and open affine covering $\{\mathrm{Spec}(B_{i,\alpha})\}_{\alpha \in \mathscr{A}}$ of the scheme

[1] R. Hartshorne, Algebraic Geometry, GTM 52

$f^{-1}(U_i)$ such that under the induced map $A_i \longrightarrow B_{i,\alpha}$ we have that $B_{i,\alpha}$ is a finitely generated A_i-algebra, for all $\alpha \in \mathscr{A}$, all $i \in \mathscr{I}$

Definition 1.6. A morphism of schemes $f\colon X \longrightarrow Y$ is *quasi-compact*, if there exists an affine cover $\{U_i\}$ of Y such that the underlying space $|f^{-1}(U)|$ is quasi-compact. A morphism of schemes $f\colon X \longrightarrow Y$ that is both locally of finite type and quasi-compact is of *finite type*

Remark 1.7. Show that $f\colon X \longrightarrow Y$ is quasi-compact if and only if the inverse image of any open affine $U \subseteq Y$ is quasi-compact. Show that $f\colon \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ is of finite type if and only if B is a finitely generated A-algebra.

1.8. **Fiber product.** Let $f\colon X \longrightarrow S$ and $g\colon Y \longrightarrow S$ be morphism of schemes. The fiber product is a triple $(X \times_S Y, p, q)$ where $X \times_S Y$ is a scheme and p and q are morphisms of schemes making the diagram

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{q} & Y \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

commutative. AND, having the following universal defining property. For any triple (Z, F, G) where $F\colon Z \longrightarrow X$ and $G\colon Z \longrightarrow Y$ are morphism of schemes such that $f \circ F = g \circ G$, there exists a unique morphism $\varphi\colon Z \longrightarrow X \times_S Y$ such that $F = p \circ \varphi$ and $G = q \circ \varphi$.

Note that if the fiber product exists it will be unique.

Theorem 1.9. *The fiber product exists.*

Proof. We followed the proof in [Ha], Theorem 3.3. □

1.10. **Fiber.** Let $f\colon X \longrightarrow S$ be a morphism of schemes. For any $g\colon T \longrightarrow S$ we will with the fiber of f over T, mean the fiber product $X \times_S T$.

In particular if $U \subseteq S$ is an open subscheme, then $U \times_S X$ equals the scheme $f^{-1}(U) \subseteq X$. Show that.

Moreover, if $t \in T$ is a point, then $f^{-1}(t)$ - which is only a set - always means the following. The point $t \in T$ should always mean a map $\operatorname{Spec}(k) \longrightarrow T$, with k a field, and the fiber means $X \times_T \operatorname{Spec}(k)$.

Example 1.11. Consider $f\colon X = \operatorname{Spec}(R) \longrightarrow S = \operatorname{Spec}(A)$, and let $T = \operatorname{Spec}(A/I)$ be a closed subscheme given by an ideal $I \subseteq A$. Then the fiber of f over T is the affine scheme given by

$$R \bigotimes_A A/I = R/IR\,.$$

Example 1.12. We pictured $\operatorname{Spec}(\mathbf{Z}[\mathbf{t}]) \longrightarrow \operatorname{Spec}(\mathbf{Z})$.

Example 1.13. And $\operatorname{Spec}(A[t]/(t^3 + at^2 + bt + c)) \longrightarrow \operatorname{Spec}(A)$.

Example 1.14. And we draw $\operatorname{Spec}(k[t, x, y]/(ty - x^2)) \longrightarrow \operatorname{Spec}(k[t])$.

1.15. **Exercises.** Hartshorne, Chapter 3.2: 3.3, 3.4, 3.5, 3.9, 3.10, 3.10, 3.11.

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