1. Lecture 6, March 6

1.1. **Notation.** Let A be a ring, and let $R = A[x_0, \ldots, x_n]$ denote the graded polynomial ring where $\deg(x_i) = 1$. We let $\mathbf{P}^n_A = \mathrm{Proj}(R)$. If $Z \subseteq \mathbf{P}^n_A$ is a closed subscheme, then for each $i = 0, \ldots, n$ we have that $Z \cap D_+(x_i)$ is a closed subscheme of $D_+(x_i)$. We let $I_i = I(Z \cap D_+(x_i)$ be the ideal in $R_{(x_i)} = A[t_1, \ldots, t_n]$ that defines $Z \cap D_+(x_i)$.

Proposition 1.2. *Let $Z \subseteq \mathbf{P}^n_A$ be a closed subscheme. Let $g \in I_0 = I(Z \cap D_+(x_0))$. Then there exists a homomogeneous element $G \in R = A[x_0, \ldots, x_n]$, say of degree d, such that*

$$G/x_0^d = g \quad \text{and} \quad G/x_i^d \in I_i \quad \text{all} \quad i = 0, \ldots, n.$$

Proof. Clearly we have that $x_0^d g = f$ is homomogeneous, where $d = \deg(g)$. We have furthermore that $f/x_0^d = g$, and that $f/x_0^d \in I_{0,i} = I(Z \cap D_+(x_0 x_i))$, for all $i = 1, \ldots, n$. It could be that $f/x_i^d \in R_{(x_i)}$ was not in the ideal I_i. However, since $f/x_i^d \in I_{0,i}$, and the ring $R_{(x_0 x_i)}$ is obtained by localization of $R_{(x_i)}$ by localization at x_0/x_i, it follows that

$$(\frac{x_0}{x_i})^m_i \frac{f}{x_i^d} \in I_i \subseteq R_{(x_i)},$$

for some m_i. Let m be the maximum of $\{m_1, \ldots, m_n\}$, and we have that $x_0^{m+d} g = G$ is a homogeneous element with the desired property. □

Corollary 1.3. *In partiuclar we have that any closed subscheme $Z \subseteq \mathbf{P}^n_A$ is given by some homogenous ideal $I \subseteq R = A[x_0, \ldots, x_n]$.*

Definition 1.4. A morphism $f \colon X \longrightarrow S$ is *universally closed* if for all morphism $T \longrightarrow S$ the induced map $f_T \colon X \times_S T \longrightarrow T$ is a closed map. A morphism $f \colon X \longrightarrow S$ is a *closed* map if $|f(Z)|$ is a closed subset in $|S|$ for any closed subset Z in $|X|$.

Example 1.5. The map $\mathbf{A}^1_k \longrightarrow \mathrm{Spec}(k)$, where k is a field is a closed map. Since the map $\mathbf{A}^1 \times_k \mathbf{A}^1 \longrightarrow \mathbf{A}^1$ is not closed, we have that the first map is not universally closed.

Lemma 1.6. *A morphism $f \colon X \longrightarrow S$ is universally closed if and only if $f_A \colon X \times_S \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A)$ is closed, for any ring A, any morphism $\mathrm{Spec}(A) \longrightarrow S$.*

Proof. Closedness is a local property. □

Proposition 1.7. *The map $\pi \colon \mathbf{P}^n_A \longrightarrow \mathrm{Spec}(A)$ is universally closed.*

Proof. We have that $\mathbf{P}^n_A \times_A \mathrm{Spec}(B) = \mathbf{P}^n_B$, so it suffices to show that π is a closed map, for arbitrary A. Let Z be a closed subset of $\mathbf{P}^n_A$, and let $I \subseteq R$ be a homogeneous ideal that defines the closed subset. We will show that the set $U = \mathrm{Spec}(A) \setminus \pi(Z)$ is open. Let $p \in U$ be a point. Then p is a prime ideal in A, such that $Z \times_A \mathrm{Spec}(\kappa(p))$ is empty. Consider the basic open affines $D_+(x_i)$ of $\mathbf{P}^n_A$. And let

$I_i \subseteq R_{(x_i)}$ be the ideal defining $Z \cap D_+(x_i)$. By assumption we have that $\kappa(p) \otimes_A R_{(x_i)}/I_i R_{(x_i)} = 0$. Which means that $1 \in I_i R_{(x_i)}$. If we let A_p denote the local ring of p in A, then we have that there is an element $f = \sum_\alpha f_\alpha \otimes_A \frac{a'_\alpha}{s_\alpha}$ in $I_i \otimes_A A_p$ that is mapped to 1 in $A_p[t_1, \dots, t_n] = A_p \otimes_A R_{(x_i)}$. We can write $f = \sum f_\alpha \otimes \frac{a_\alpha}{s}$, with $s \in A \backslash p$. Thus $f \in I \otimes_A A_s$ is mapped to $\sum a_\alpha f_\alpha = 1$ in $A_s[t_1, \dots, t_n]$. By Proposition 1.2 we can find a homogeneous element $F \in A_s[x_0, \dots, x_n]$ such that $F/x_j^d \in I_j$, for all $j = 0, \dots, n$, and where $F/x_i^d = 1$ in $A_s \otimes_A R_{(x_i)}$. For each i we get such an element F_i, and an open $\operatorname{Spec}(A_{s_i})$ around the point p. By taking their open, non-empty, intersection we get an open $\operatorname{Spec}(A_{s_0 \cdot s_n})$ around p, where $\pi^{-1}(\operatorname{Spec}(A_{s_0 \cdot s_n})) = \emptyset$. □

1.8. Any scheme X comes equipped with a morphism $f\colon X \longrightarrow \operatorname{Spec}(\mathbf{Z})$. It is natural to always consider a morphism of schemes $f\colon X \longrightarrow S$. A morphism of schemes $f\colon X \longrightarrow S$ is *separated* if the diagonal map $\Delta\colon X \longrightarrow X \times_S X$ is a closed immersion. Note that the diagonal map always is a homeomorphism onto its image.

Proposition 1.9. *A morphism of affine schemes is separated.*

Proof. The diagonal map of a homomorphism of rings $A \longrightarrow B$ is the homomorphism $B \otimes_A B \longrightarrow B$ that sends $b \otimes c \mapsto bc$. This is surjective, hence a closed immersion. □

1.10. If $\{U_i\}$ is an open covering of X, then $U_i \times_S U_j$ is an open cover of $X \times_S X$. Note that $\Delta_X^{-1}(U_i \times_S U_j) = U_i \cap U_j$. Therefore the restriction of $\Delta_X\colon X \longrightarrow X \times_S X$ to the open $U_i \times_S U_j$ in $X \times_S X$ becomes

$$U_i \cap U_j \longrightarrow U_i \times_S U_j. \tag{1.10.1}$$

That the map $X \longrightarrow X \times_S X$ is a closed immersion is equivalent with the maps 1.10.1 being closed immersions, for all i, j.

Example 1.11. The affine line with one point doubled is not separated. We can cover this scheme with two open affines U and V, both being $\operatorname{Spec}(k[x])$. Their intersection is $U \cap V = \operatorname{Spec}(k[x, x^{-1}]$, with the canonical maps. The map

$$U \cap V \longrightarrow U \times_k V$$

is given by the morphism $k[x] \otimes_k k[x] \longrightarrow k[x, x^{-1}]$ that sends $x \otimes 1$ and $1 \otimes x$ to x. This is not surjective, and we have that the scheme is not separated.

Example 1.12. The projective line is separated. We can cover this scheme with two open affines $U = \operatorname{Spec}(k[x])$ and $V = \operatorname{Spec}(k[y])$. Their intersection $U \cap V = \operatorname{Spec}(k[x, y]/(xy - 1))$, and the inclusion maps are the canonical ones. Now we have that the map

$$U \cap V \longrightarrow U \times_k V$$

is given by the morphism $k[x] \otimes_k k[y] \longrightarrow k[x,y]/(xy-1)$ that sends $x \otimes 1$ to x and $1 \otimes y$ to y. This is surjective, and it follows that the projective line is separated.

1.13. Exercises. Hartshorne, Chapter 2.4: 4.1, 4.2, 4.3, 4.4.

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