1. Lecture 7, March 13

1.1. **Separatedness.** Recall that a map $X \longrightarrow S$ of schemes is separated if the diagonal map $\Delta\colon X \longrightarrow X \times_S X$ is a closed immersion.

Lemma 1.2. *We have that*

1. *Open and closed immersions are separated.*
2. *Compositions separated maps are separated.*
3. *Separatedness is stable under base change.*

Proof. Exercise. □

1.3. **Properness.** A morphism $f\colon X \longrightarrow S$ of schemes is *proper* if f is separated, finite type, and universally closed.

Lemma 1.4. *We have that*

1. *Closed immersions are proper.*
2. *Compositions of proper maps are proper.*
3. *Properness is stable under base change.*

Proof. Exercise. □

Proposition 1.5. *The map* $\mathbf{P}^n_A \longrightarrow \operatorname{Spec}(A)$ *is proper.*

Proof. We have earlier shown that the map is universally closed, and we have that the map is of finite type. We need to show separatedness. Let $x_0, \dots, x_n$ be variables over the ring A. Then we have that $\{D_+(x_i)\}_{i=0}^n$ is an open affine cover of $\mathbf{P}^n_A$. To check closedness of the diagonal map Δ, we need to see that the map

$$D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j) \longrightarrow D_+(x_i) \times_A D_+(x_j)$$

is a closed immerison, for every i, j. This map of affine rings, given by

$$A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \otimes_A A[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}] \longrightarrow A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_0}{x_j}, \dots, \frac{x_0}{x_j}],$$

where the map is the obvious one. Since

$$A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_0}{x_j}, \dots, \frac{x_0}{x_j}] = A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}],$$

it is clear that the map is surjective, hence a closed immersion. □

2. Sheaves of modules

We start by listing a collection of words that we will look closer at thereafter.

2.1. **$\mathscr{O}_X$-modules.** Let $(X, \mathscr{O}_X)$ be a scheme, or more generally a ringed space. A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is a sheaf $\mathscr{F}$ that is an $\mathscr{O}_X(U)$-module for each open $U \subseteq X$, and for each inclusion of opens $V \subseteq U$ the restriction map $\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ is an $\mathscr{O}_X(U)$-module map, compatible with the restrcition map $\mathscr{O}_X(U) \longrightarrow \mathscr{O}_X(V)$.

A morphism $\varphi \colon \mathscr{F} \longrightarrow \mathscr{G}$ is a morhpism of sheaves, that is an $\mathscr{O}_X(U)$-module map over each open $U \subseteq X$.

We have the notion of kernel, image and cokernel of a morphism of $\mathscr{O}_X$-modules, and these sheaves are $\mathscr{O}_X$-modules. We have the tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$, which is the sheaf associated to the presheaf $U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$.

2.2. **Notation.** An $\mathscr{O}_X$-module $\mathscr{F}$ is *free* if $\mathscr{F}$ is isomorphic to a direct sum $\oplus_{i \in \mathscr{I}} \mathscr{O}_X$. The $\mathscr{O}_X$-module $\mathscr{F}$ is *locally free* if there exists an open cover $\{U_i\}$ of X such that

$$\mathscr{F}_{|U_i} \backsimeq \oplus_{i \in \mathscr{I}} \mathscr{O}_{U_i},$$

as sheaves on U_i.

2.3. **Inverse and direct image.** Let $(f, f^\sharp) \colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ be a morphism of schemes, or ringed spaces. If $\mathscr{F}$ is an $\mathscr{O}_X$-module, then $f_*\mathscr{F}$ will be an $f_*\mathscr{O}_X$-module. And, then via the map $f^\sharp \colon \mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X$, we have that $f_*\mathscr{F}$ is an $\mathscr{O}_Y$-module.

If $\mathscr{G}$ is an $\mathscr{O}_Y$-module, then $f^{-1}\mathscr{G}$ is an $f^{-1}\mathscr{O}_Y$-module. Via the adjoint property, we have that $f^\sharp$ corresponds to a morphism $f^b \colon f^{-1}\mathscr{O}_Y \longrightarrow \mathscr{O}_X$. We define

$$f^*\mathscr{G} := \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{G},$$

which is an $\mathscr{O}_X$-module.

2.4. **Quasi-coherent sheaves.** Now, it is time to get a bit more concrete.

Lemma 2.5. *Let A be a ring, and M an A-module. Assume that we have elements $f_i \in A$ such that $(f_1, \dots, f_n) = A$. Then the sequence*

$$M \longrightarrow \prod_{i=1}^n M_{f_i} \rightrightarrows \prod_{i,j} M_{f_i f_j}$$

is exact.

Proof. This is proved in the same way as we did when we proved the analogous statement with $M = A$. □

Proposition 2.6. *Let A be a ring, and M an A-module. On the affine scheme $X = \operatorname{Spec}(A)$ there is a unique sheaf $\tilde{M}$ of $\mathscr{O}_X$-modules such that for any basic open set $D(f)$ we have the $A_f = \mathscr{O}_X(D(f))$-module*

$$\tilde{M}(D(f)) = M_f = M \otimes_A A_f.$$

The restriction maps $\tilde{M}(D(f)) \longrightarrow \tilde{M}(D(fg))$ are the obvious ones. In particular we have that $\tilde{M}(X) = M$, and that for any point $x \in X$ that the stalk $\tilde{M}_x = M_x$.

Proof. Again, having the exactness property over basic opens for the topology, determines a sheaf $\tilde{M}$ with the described properties. It also follows that since the sheaf is an module over the basic opens, the sheaf will be an $\mathscr{O}_X$-module. □

Proposition 2.7. *Let $\varphi\colon A \longrightarrow B$ be a morphism of rings, and let $f\colon X = \operatorname{Spec}(B) \longrightarrow Y = \operatorname{Spec}(A)$ be the corresponding morphism of schemes.*

(1) *The map $M \mapsto \tilde{M}$ is an exact, fully faithful functor from the category of A-modules to the category of $\mathscr{O}_X$-modules.*
(2) *For any A-modules M_1 and M_2 we have that*

$$(M_1 \widetilde{\otimes_A} M_2) = \tilde{M}_1 \otimes_{\mathscr{O}_X} \tilde{M}_2.$$

(3) *For any collection of A-modules M_i we have*

$$(\widetilde{\oplus_i M_i}) = \oplus_i \tilde{M}_i.$$

(4) *For any B-module N, let N_A denote the group N considered as an A-module via the ring homomorphism $\varphi\colon A \longrightarrow B$. Then we have that*

$$f_*(\tilde{N}) = \tilde{N}_A.$$

(5) *For any A-module M we have that*

$$f^*\tilde{M} = (M \widetilde{\otimes_A} B).$$

Proof. To see that two sheaves are equal, or isomorphic, it suffices to check this on basic opens. For any element $f \in A$ we have that $M_f \otimes_A N_f = M \otimes_{A_f} \otimes_A N \otimes_A A_f$, and then the second assertion follows. Since tensor product commutes with direct sums, the third assertion follows as well. The thw last assertions follows from the definitions. What one has to prove is the first assertion.

First one proves that sending $M \mapsto \tilde{M}$ is functorial. In particular any A-module morphism $f\colon M \longrightarrow N$ gives a map $F\colon \tilde{M} \longrightarrow \tilde{N}$ of $\mathscr{O}_X$-modules. Then a short exact sequence of A-modules is sent to a short sequence of $\mathscr{O}_X$-modules, which one will have to check is exact. But as exactness can be checked on stalks the result follows from Proposition 2.6.

Let M and N be two A-modules. One checks that two different A-module maps $M \longrightarrow N$ give two different maps of $\mathscr{O}_X$-modules $\tilde{M} \longrightarrow \tilde{N}$. This one can check on stalks. It then follows from Proposition 2.6 that the functor is faithful.

To check that the functor is full, we let H denote the A-module $H = \operatorname{Hom}_A(M, N)$. Note that

$$\operatorname{Hom}_A(M, N) \otimes_A A_f = \operatorname{Hom}_A(M, N_f) = \operatorname{Hom}_{A_f}(M_f, N_f).$$

Then we have that for any basic open U, and then also any open, that we have a natural map of $\mathscr{O}_X(U)$-modules

$$\tilde{H}(U) \longrightarrow \mathrm{Hom}_U(\tilde{M}_{|U}, \tilde{N}_{|U}).$$

Injectivity of this map follows as we have shown faithfullness for $\tilde{(-)}$, over any ring A. Let $U = D(g)$ be a basic open set in $X = \mathrm{Spec}(A)$. For any morphism $F\colon \tilde{M}_{|U} \longrightarrow \tilde{N}_{|U}$ of $\mathscr{O}_U$-modules, we get by taking sections over U, an A_g-module homomorphism

$$f_U\colon \tilde{M}_{|U}(U) = M_g \longrightarrow \tilde{N}_{|U}(U) = N_g.$$

The equalities are given by Proposition 2.6. By doing the $\tilde{(-)}$ construction, over A_g, we get an $\mathscr{O}_U$-module homomorphism $F_1\colon \tilde{M}_{|U} \longrightarrow \tilde{N}_{|U}$. We have as sections over U, both F_1 and F both agree with f_U. But, then we have shown that the two sheaves $\tilde{H}$ and the sheaf $\mathscr{H}om_X(\tilde{M}, \tilde{N})$ are equal. In particular

$$\tilde{H}(X) = \mathrm{Hom}_A(M, N) = \mathrm{Hom}_X(\tilde{M}, \tilde{N}),$$

and we have that the functor $\tilde{(-)}$ is full. □

2.8. **Quasi-coherent sheaves.** Let $(X, \mathscr{O}_X)$ be a scheme. A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is *quasi-coherent* if there exists an open affine covering $\{U_i = \mathrm{Spec}(A_i)\}$ of X such that $\mathscr{F}_{|U_i} = \tilde{M}_i$, for some A_i-module M_i.

If the scheme X is Noetherian then a quasi-coherent sheaf $\mathscr{F}$ is called *coherent*if the modules M_i are finitely generated as A_i-modules.

2.8.1. *Note.* One shows that the existence of an affine cover is equivalent that *any* affine open cover has the properties described above.

Proposition 2.9 (Nakayama's Lemma)**.** *Let X be a Noetherian scheme, and let $\mathscr{F}$ be a coherent sheaf. Assume that we have a morphism $f\colon \mathrm{Spec}(K) \longrightarrow X$ of schemes, with K a field. Then there exists an open $U \subseteq X$ containing the point $f(\mathrm{Spec}(K))$ such that $\mathscr{F}_{|U} = 0$.*

Proof. We may assume $X = \mathrm{Spec}(A)$ is affine since $\mathrm{Spec}(K)$ is a point. Let $\varphi\colon A \longrightarrow K$ be the homomorphism of rings corresponding to f. Let $x = \ker\varphi$, which is a prime ideal. Then the morphism φ factorizes as

$$A \longrightarrow A_x \longrightarrow \kappa(x) \longrightarrow K,$$

where the last map is an inclusion of fields. Let M be the A-module such that $\tilde{M} = \mathscr{F}$. By assumption $f^*(\tilde{M}) = 0$, which by Proposition 2.7 5 and 1, means that $M \otimes_A K = 0$. This is equivalent with $M \otimes_A \kappa(x) = 0$. Nakayama's Lemma now gives that $M \otimes_A A_x = M/xA_xM$ is zero. If we let $x_1, \ldots, x_n$ be elements in M that generates the A-module M. Then $x_1 \otimes 1, \ldots, x_n \otimes 1$ generate $M \otimes_A B$, for any A-algebra B. So in particular the images of $x_1, \ldots, x_n$ generate the stalk M_x, which is zero. Thus, for each i the element x_i is zero in the stalk around the point $x \in X$. Then there is an open U_i on where $x_{i|U_i} = 0$. The finite intersection $U = \cap U_i$ is open, and non-empty. We can assume $U = D(f)$ a basic open. Then $x_i \otimes 1$ are zero in $M \otimes_A A_f$, for all i. Hence $M \otimes_A A_f = 0$. But then we are done since $\tilde{M}(D(f)) = M_f$. $\square$

Proposition 2.10. *Let X be an affine scheme, and assume that we have a short exact sequence of $\mathscr{O}_X$-modules*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0,$$

where $\mathscr{F}'$ is quasi-coherent. Then the corresponding sequence of global sections

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0,$$

is exact.

Proof. We will discuss the proof next time. $\square$

Corollary 2.11. *Kernel, cokernel, and image of a morphism of quasi-coherent sheaves, remains quasi-coherent. Any extension of quasi-coherent sheaves is quasi-coherent.*

Proof. We proved this as in Hartshorne, Propositon 5.7. $\square$

2.12. **Exercises.** Hartshorne, Chapter 2.5: 5.2, 5.3, 5.6, 5.7, 5.8.

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