1. Lecture 8, March 20

Proposition 1.1. *Let $A \longrightarrow B$ be a morphism of rings, and let $f\colon X = \mathrm{Spec}(B) \longrightarrow Y = \mathrm{Spec}(A)$ be the corresponding morphism of schemes.*

(1) *The map $M \mapsto \tilde{M}$ is an exact, fully faithful functor from the category of A-modules to the category of $\mathscr{O}_X$-modules.*
(2) *For any A-modules M_1 and M_2 we have that*
$$\widetilde{(M_1 \otimes_A M_2)} = \tilde{M}_1 \otimes_{\mathscr{O}_X} \tilde{M}_2.$$
(3) *For any collection of A-modules M_i we have*
$$\widetilde{(\oplus_i M_i)} = \oplus_i \tilde{M}_i.$$
(4) *For any B-module N, let N_A denote the group N considered as an A-module via the ring homomorphism $A \longrightarrow B$. Then we have that*
$$f_*(\tilde{N}) = \tilde{N}_A.$$
(5) *For any A-module M we have that*
$$f^*\tilde{M} = \widetilde{(M \otimes_A B)}.$$

Proof. We show that the functor is full: Given a map of $\mathscr{O}_X$-modules $\varphi\colon \tilde{M} \longrightarrow \tilde{N}$. We let $F = \varphi(X)$ denote the induced map of global sections $M \longrightarrow N$. The claim is that $\tilde{F} = \varphi$. We have that these two maps agree on global sections. We will show that they agree on any open $U \subseteq X$.

Let $\{U_i\}$ be an affine open cover of X. We may assume that the cover is finite, and that each open $U_i = D(g_i)$ is a basic open affine. Let $\varphi_{U_i}\colon \tilde{M}_{|U_i} \longrightarrow \tilde{N}_{|U_i}$. And let $\varphi_i = \varphi_{|U_i}(U_i)$. We have that the different φ_i agrees on intersections.

We let H denote the A-module $H = \mathrm{Hom}_A(M, N)$, and we have the exact sequence
$$H \longrightarrow \prod_{i=1}^n H_{g_i} \rightrightarrows \prod_{i,j} H_{g_i g_j}.$$
Note that
$$H_g = \mathrm{Hom}_A(M, N) \otimes_A A_g = \mathrm{Hom}_A(M, N_g) = \mathrm{Hom}_{A_g}(M_g, N_g).$$
Consequently our $\varphi_i \in H_{g_i}$, for all i. By the exactness there exists a unique element in H that restricts to φ_i. That element is $\varphi(X) = F$. We also have that the image of $F \in H$ in $H_g = H \otimes_A A_g$ is φ_i. We then have that $\tilde{F}$ and φ agree on every basic open. Then it follows that $\tilde{F} = \varphi$, and the functor is full. □

Proposition 1.2. *Let X be an affine scheme, and assume that we have a short exact sequence of $\mathscr{O}_X$-modules*
$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0,$$

where $\mathscr{F}'$ is quasi-coherent. Then the corresponding sequence of global sections

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0,$$

is exact.

Proof. We only need to show surjectivity of the rightmost map. Let $s \in \Gamma(X, \mathscr{F}")$ be a global section of $\mathscr{F}"$. Then there exists a covering $\{U_i = D(g_i)\}_{i=0}^n$ of X, and elements $t_i \in \mathscr{F}(U_i)$ such that $\psi(t_i) = s_{|U_i}$. let $f = g_0$, and let $t = t_0$. We first show that $f^d s$ has a (global) lift, some integer d.

On $U_0 \cap U_i$ the restriction of t and t_i both are liftings of $s_{|U_0 \cap U_i}$. Therefore

$$t_{|U_{0,i}} - t_{i|U_{0,i}} = \varphi(x_i),$$

for some $x \in M_{fg_i}$. Where M is the A-module such that $\tilde{M} = \mathscr{F}'$. The restricion map is the localization map

$$M_{g_i} \longrightarrow M_{fg_i}.$$

Hence, $f^{m_i} x_i \in M_{g_i}$, some m_i. Let m be an integer that works for all $i = 1, \dots, n$. Then $f^m x_i \in M_{g_i}$, for all $i = 1, \dots, n$. Consider now

$$u_i = \varphi(f^m x_i) + f^m t_i \in \mathscr{F}(U_i),$$

for all $i = 0, \dots, n$. Then each u_i is a lift of $f^m s_{|U_i}$. On intersections $U_i \cap U_j$ the two different u_i and u_j give the same lift. So $u_{i|U_{i,j}} - u_{|U_{i,j}} = \varphi(y_{i,j})$, with $y_{i,j} \in M_{g_i g_j}$. However, on tripple intersection $D(f) \cap D(g_i) \cap D(g_j)$ we have that the the restricitons of u_i and u_j agree:

$$\begin{aligned} u_{i|U} - u_{j|U} &= f^m \big(\varphi(x_i) + t_i - \varphi(x_j) - t_j\big) \\ &= f^m (t_{|U} - t_{i|U} + t_{i|U} - t_{|U} + t_{j|U} - t_{j|U}) = 0. \end{aligned}$$

So, $f_{i,j}^M u_i = f^{M_{i,j}} u_j$ on $D(g_i g_j)$, some $M_{i,j}$. Then we can find an integer M that holds for all $D(g_i g_j)$. We obtain that $f^M u_i$ is a lifting of $f^{m+M} s_{|U_i}$, and that these sections agree on intersections. Hence there exists $t \in \Gamma(X, \mathscr{F})$ such that $\psi(t) = f^{m+M} s$.

We have then proven that given $s \in \Gamma(X, \mathscr{F}")$, there exists a covering $\{D(g_i)\}$ of X, and global sections $t_i \in \Gamma(X, \mathscr{F})$ such that $\psi(t_i) = g_i^{d_i} s$, for each $i = 0, \dots, n$. We can assume that we have the same degree $d_i = d$. As $(g_0, \dots, g_n) = A$ we can write $1 = \sum_{i=0}^n a_i g_i^d$, with $a_i \in A$. Then

$$t = \sum_{i=0}^n a_i t_i \in \Gamma(X, \mathscr{F})$$

is a global section that is mapped to s. □

Proposition 1.3. *Let $f \colon X \longrightarrow Y$ be a morphism of schemes.*

(1) *If $\mathscr{G}$ is quasi-coherent on Y, then $f^*\mathscr{G}$ is quasi-coherent.*

(2) *If X and Y are Noetherian, and $\mathscr{G}$ is coherent on Y. Then $f^*\mathscr{G}$ is coherent.*
(3) *Assume that X is Noetherian, or that f is quasi-compact and separated. If $\mathscr{F}$ is quasi-coherent, then $f_*\mathscr{F}$ is quasi-coherent.*

Proof. The two first statements are clear. To prove the third, we may assume that Y is affine. Let $U \subseteq X$ be an open affine. Then $f_*\mathscr{F}_U$ is quasi-coherent, where f is the composite map $U \subseteq X \longrightarrow Y$. If $\{U_i\}$ is an open affine cover of X, then we have the following exact sequence of sheaves on Y,

$$f_*\mathscr{F} \longrightarrow \prod_i f_*(\mathscr{F}_{|U_i}) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} f_*(\mathscr{F}_{|U_i \cap U_j})$$

By assumption X is compact, so we can take a finite open cover $\{U_i\}$ of X. A finite direct product is a fintie sum, and we have that direct sums of quasi-coherent are quasi-coherent. Thus the middle term is quasi-coherent. By assumption we can find a finite affine open cover $\{V_{i,j,k}\}$ of $U_i \cap U_j$. In the Noetherian situation we use compactness, in the separated situation we use that the intersection of affines is affine. So $f_*\mathscr{F}_{|U_i \cap U_j}$ sits inside a finite product $\prod f_*\mathscr{F}_{|V_{i,j,k}}$. We can then assume that $f_*\mathscr{F}$ is the kernel of a map of quasi-coherent sheaves. But, then $f_*\mathscr{F}$ itself is quasi-coherent. □

Example 1.4. Let $X = \sqcup \operatorname{Spec}(\mathbf{Z})$ be the infinite disjoint union. The map $f\colon X \longrightarrow \operatorname{Spec}(\mathbf{Z})$ is the identity on each component, and corresponds to the map $\mathbf{Z} \longrightarrow \prod \mathbf{Z} = \Gamma(X, \mathscr{O}_X)$. Let $U = \operatorname{Spec}(\mathbf{Z}_2)$ be the open subscheme we get by inverting 2. We get an induced map $f^{-1}(U) = \sqcup U \longrightarrow U$, and we have that $\prod \mathbf{Z}_2 = \Gamma(U, \mathscr{O}_{X|U})$. If $f_*\mathscr{O}_X$ were quasi-coherent, we would have $f_*\mathscr{O}_X = \tilde{M}$ where $M = \prod \mathbf{Z}$. Then $\tilde{M}(U) = M \otimes_{\mathbf{Z}} \mathbf{Z}_2$. However, we have that

$$M \otimes_{\mathbf{Z}} \mathbf{Z}_2 \neq \prod \mathbf{Z}_2 = \Gamma(U, \mathscr{O}_{X|U}),$$

and consequently it is not quasi-coherent.

1.5. **Ideal sheaves.** For any closed immersion of schemes $i\colon Y \longrightarrow X$, we let $\mathscr{I}_Y$ denote its associated sheaf of ideals, given as the kernel

$$\mathscr{O}_X \longrightarrow i_*\mathscr{O}_Y.$$

The $\mathscr{I}_Y$ is the kernel of a map of $\mathscr{O}_X$-modules, so in particular the ideal sheaf is an $\mathscr{O}_X$-module. Any $\mathscr{O}_X$-module, being a submodule of $\mathscr{O}_X$ is a sheaf of ideals.

As the closed immersion $i\colon Y \longrightarrow X$ is both quasi-compact and separated, we have that $\mathscr{I}_Y$ is a quasi-coherent sheaf of ideals.

1.6. When $X = \operatorname{Spec}(A)$ is affine, any quasi-coherent sheaf of ideals $\mathscr{I}$ on X is the tilde of an A-module I. Moreover, since the ˜ operation is exact, the A-moldule I has to be an A-submodule of A. That is an ideal $I \subseteq A$. Any ideal $I \subseteq A$ corresponds to a closed subscheme of X.

1.7. Closed subschemes. Let X be a scheme, and let $\mathscr{I}$ be a quasi-coherent sheaf of ideals on X. We have the global section 1 of $\mathscr{O}_X/\mathscr{I}$, and its support is a closed subset $Y \subseteq X$. Then $(Y, \mathscr{O}_X/\mathscr{I})$ is a closed subscheme of X, and any closed subscheme of X arises in this way.

1.8. Exercises. Hartshorne, Chapter 2.5: 5.1, 5.4, 5.5, 5.15, 5.16, 5.18.

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