1. Lecture 9, March 27

Todays material is from [Ma][1]

1.1. **Derivations.** Let A be a ring, and M an A-module. A derivation d is a map of sets $d\colon A \longrightarrow M$ such that

$$d(x+y) = d(x) + d(y) \quad \text{and} \quad d(xy) = xd(y) + yd(x),$$

for all x, y in A. The set of derivations is denoted $\mathrm{Der}(A, M)$, and is in a natural way an A-module.

Note that for any derivation $d\colon A \longrightarrow M$ we have that $d(1) = d(1) + d(1)$. Thus for any integer $n \in \mathbf{Z}$ we have that $d(n) = 0$.

If $a\colon k \longrightarrow A$ is a homomorphism of rings then a derivation $d\colon A \longrightarrow M$ is called *k-linear* if $d \circ a = 0$. The set of k-linear derivations is denoted $\mathrm{Der}_k(A, M)$. In particular we have that $\mathrm{Der}_{\mathbf{Z}}(A, M) = \mathrm{Der}(A, M)$.

1.2. **Liftings.** In the category of k-algebras, consider the following diagram

$$\begin{array}{ccc} D & \xrightarrow{f} & E \\ & & \uparrow{\scriptstyle g} \\ & & A. \end{array} \tag{1.2.1}$$

A ring homomorphism $\gamma\colon A \longrightarrow D$ such that $f \circ \gamma = g$ is called a *lift* of g.

Lemma 1.3. *Consider the situation as above 1.2.1. Assume that* $I = \ker(f)$ *is such that* $I^2 = 0$*. Then we have that*

1. *We have a that I is an A-module.*
2. *For any two liftings γ_1 and γ_2 of g, the map $d = \gamma_2 - \gamma_1$ is a k-linear derivation $A \longrightarrow I$.*
3. *If $d \in \mathrm{Der}_k(A, I)$, and γ a lift of g, then $\gamma + d\colon A \longrightarrow D$ is a lift of g.*

Proof. Check this. □

1.4. **Kähler differentials.** Let I denote the kernel of the multiplication map $A \otimes_k A \longrightarrow A$, and let $\Omega_{A/k} = I/I^2$. Then we have the exact sequence of $A \otimes_k A$-modules

$$0 \longrightarrow \Omega_{A/k} \longrightarrow A \otimes_k A/I^2 \xrightarrow{\mu} A \longrightarrow 0.$$

We have two natural sections $\gamma_i\colon A \longrightarrow A \otimes_k A$ of μ, where $i = 1, 2$. Namely, $\gamma_1(x) = x \otimes 1$ and $\gamma_2(x) = 1 \otimes x$. By Lemma 1.3 we have that $\Omega_{A/k}$ is an A-module, and that $d = \gamma_2 - \gamma_1$ is a derivation $d \in \mathrm{Der}_k(A, \Omega_{A/k})$.

[1] Hideyuki Matsumura, Commuative ring theory. Cambridge studies in advanced mathematics 8.

Lemma 1.5. *The A-module $\Omega_{A/k}$ is generated by $d(y)$, with $y \in A$.*

Proof. Note that for any elements $x \otimes y \in A \otimes_k A$ we have that

$$x \otimes y = (x \otimes 1)(1 \otimes y - y \otimes 1) + xy \otimes 1.$$

Thus any element in the quotient $A \otimes_A /I^2$ is of the form $(x \otimes 1)d(y)$, from where the lemma follows. □

We also observe that if $\varphi\colon M \longrightarrow N$ is an A-module homomorphism, and $D \in Der_k(A, M)$ a derivation. Then the composition $\varphi \circ d$ is a k-linear derivation from $A \longrightarrow N$. It follows that $\mathrm{Der}_k(A, -)$ is a covariant functor from the category of A-modules to sets. If M is a given A-module, and $D \in \mathrm{Der}_k(A, M)$ a derivation, then we get an induced map of functors

$$D_*\colon \mathrm{Hom}_A(M, -) \longrightarrow \mathrm{Der}_k(A, -),$$

by composition. If that functor is a bijection of sets, for any A-module N, then we say that the pair (M, D) represents the functor. Such representing pairs are unique (up to unique isomorphism).

Proposition 1.6. *The functor $Der_k(A, -)$ is represented by the pair $(\Omega_{A/k}, d)$.*

Proof. We need to show that the induced map of functors

$$d_*\colon \mathrm{Hom}_A(\Omega_{A/k}, -) \longrightarrow \mathrm{Der}_k(A, -),$$

is an isomorphism. Injectivity follows readily from Lemma 1.5. We will show surjectivity. Let $D \in \mathrm{Der}_k(A, M)$ be a derivation, where M is some A-module. We define the ring $A * M$ by putting following ring structure on the A-module $A \oplus M$:

$$(x, m) * (y, n) := (xy, xn + ym).$$

This is well-defined, and note that we have $M^2 = 0$. Consider the map $\varphi_1\colon A \longrightarrow A * M$ that sends any element $x \mapsto (x, D(x))$. This is well-defined, and we have clearly that $\varphi_1(x + y) = \varphi_1(x) + \varphi_1(y)$, and

$$\varphi_1(x)\varphi_1(y) = (x, D(x)) * (y, D(y)) = (xy, xD(y) + yD(x)) = (xy, D(xy)).$$

Hence[2] the map φ_1 is a homomorphism of rings. It is also an k-algebra homomorphism. We then have the k-algebra homomorphism

$$\Phi = (\varphi_1, \varphi_2)\colon A \otimes_k A \longrightarrow A * M.$$

We then restrict our map Φ to the $A \otimes_k A$-module $I \subseteq A \otimes_k A$. If $x = \sum x_i \otimes y_i$ is an element of I then $\Phi(x) = (0, D(x))$, and consequently the restriction map is a map of $A \otimes_k A$-modules $\Phi\colon I \longrightarrow M$. Since $M^2 = 0$, we get an induced A-linear map

$$\varphi\colon \Omega_{A/k} = I/I^2 \longrightarrow M.$$

[2] Here I disagree with [Ma]

That is an element $\varphi \in \mathrm{Hom}_A(\Omega_{A/k}, M)$. When we compose this A-module map φ with the derivation d we get

$$\varphi \circ d(x) = \varphi(1 \otimes x - x \otimes 1) = 1 \cdot D(x) - x \cdot D(1) = D(x).$$

Thus $D = \varphi \circ d$, and the map in question is surjective. □

Lemma 1.7. *If $A = k[x_1, \dots, x_n]$ is the polynomial ring, then*

$$\Omega_{A/k} = \oplus_{i=1}^{n} A d(x_i)$$

is the free A-module with basis $d(x_1), \dots, d(x_n)$.

Proof. Let $x_1, \dots, x_n$ be elements in the ring A, and consider a polynomial expression $f(x_1, \dots, x_n)$ in the elements $x_1, \dots, x_n$. Using the definition of derivations we get that

$$d(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d(x_i).$$

Therefore it follows by Lemma 1.5 that if A is generated by $x_1, \dots, x_n$ as a k-algebra, then $\Omega_{A/k}$ is generated by $d(x_1), \dots, d(x_n)$ as an A-module. In the polynomial ring situation we have the derivations $D_i \in \mathrm{Der}_k(A, A)$ where $D_i(x_j) = \delta_{i,j}$. It then follows that $\mathrm{Hom}_A(\Omega_{A/k}, A)$ is a free module of rank n, from where we get that $\Omega_{A/k}$ had to be free of rank n. □

Theorem 1.8. *If $g \colon A \longrightarrow B$ is a k-algebra homomorphism, then we have the exact sequence of B-modules*

$$\Omega_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega_{B/k} \xrightarrow{\beta} \Omega_{B/A} \longrightarrow 0,$$

where $\alpha(d(x) \otimes y) = y \cdot d(g(x))$, and $\beta(d(y)) = d(y)$.

Proof. One proves this by looking at the dual sequence, for arbitrary M. We then get the sequence

$$0 \longrightarrow \mathrm{Der}_A(B, M) \longrightarrow \mathrm{Der}_k(B, N) \longrightarrow \mathrm{Hom}_B(\Omega_{A/k} \otimes_A B, N).$$

We identify $\mathrm{Hom}_B(\Omega_{A/k} \otimes_A B, N) = \mathrm{Holm}_A(\Omega_{A/k}, N) = \mathrm{Der}_k(A, N)$. The exactness of the dual sequence is the readily checked, and then the exactness of the sequence follows. □

Theorem 1.9. *If $g \colon A \longrightarrow B$ is a surjective k-algebra homomorphism, with kernel $\mathfrak{m}$, then we have the exact sequence of B-modules*

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega_{B/k} \longrightarrow 0,$$

where $\delta(x) = d(x) \otimes 1$.

Proof. This is also proved easily by looking at the dual sequence, for arbitrary B-module M. □

Example 1.10. Let $B = k[x,y]/(x^2+y^2)$, and set $A = k[x,y]$. We then have that $\Omega_{B/k}$ is the B-module

$$Bd(x) \oplus Bd(y)/(2xd(x) + 2yd(y)).$$

Lemma 1.11. *Let $S \subseteq A$ be a multiplicatively closed subset. Then we have an canonical identification*

$$(\Omega_{A/k} \otimes 1, d_A \otimes 1) = (\Omega_{S^{-1}A/k}, d_{S^{-1}A}).$$

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Proof. It suffices to show that the pair $(\Omega_{A/k} \otimes 1, d_A \otimes 1)$ satisfies the same universal properties as $(\Omega_{S^{-1}A/k}, d_{S^{-1}A})$. For any $S^{-1}A$-module M, we have the identification

$$\mathrm{Hom}_{S^{-1}A}(\Omega_{A/k} \otimes S^{-1}A, M) = \mathrm{Hom}_A(\Omega_{A/k}, M).$$

Then the result follows. □

1.12. **Sheaves of Kähler differentials.** Let $f \colon X \longrightarrow Y$ be a separated map[3] of schemes. Then $\Delta \colon X \longrightarrow X \times_Y X$ is a closed immersion, and we have that $\Delta(X)$ is a closed subscheme of $X \times_Y X$. Let $\mathscr{I}$ be the quasi-coherent ideal sheaf defining $\Delta(X)$. Then $\mathscr{I}/\mathscr{I}^2$ is a quasi-coherent sheaf of $\mathscr{O}_{\Delta(X)}$-modules (check this). We define the quasi-coherent sheaf $\Omega_{X/Y}$ on X, as the sheaf

$$\Omega_{X/Y} := \Delta^*(\mathscr{I}/\mathscr{I}^2).$$

If X and Y are Noetherian, and $f \colon X \longrightarrow Y$ we have that $\Omega_{X/Y}$ is a coherent sheaf.

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[3]Separatedness is not needed, but simplifies a bit.