1. Lecture 10, April 03

1.1. **Graded modules.** Let S be a graded ring, and let M be a graded S-module. That means that $\deg(s \cdot m) = \deg(s) + \deg(m)$ for all $s \in S$ and all $m \in M$. On $X = \mathrm{Proj}(S)$ we get the sheaf $\tilde{M}$ by defining

$$\tilde{M}_{|D_+(f)} = (M \otimes_S S_f)_{(0)} := M_{(f)},$$

for any homogeneous $f \in S$. The usual argument (...) shows that this determines a sheaf on X. By construction the sheaf $\tilde{M}$ is a quasi-coherent sheaf of $\mathscr{O}_X$-modules.

1.2. **Shifting.** For any integer $n \in \mathbf{Z}$ we define a new graded module $M[n]$ by shifting the degrees in M by the integer n: If $M = \oplus_{d \in \mathbf{Z}} M_d$ is the decomposition of M into its graded components, then

$$M[n] = \bigoplus_{d \in \mathbf{Z}} M_{d+n}.$$

That means that degree d elements in $M[n]$ are degree $d+n$ elements in M, thus $M[n]_d = M_{n+d}$.

Definition 1.3. Let $X = \mathrm{Proj}(S)$, and let n be an integer. Then we let

$$\mathscr{O}_X(n) = \widetilde{S[n]}.$$

And, for any $\mathscr{O}_X$-module $\mathscr{F}$ we set $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$.

Example 1.4. Let $S = k[x,y]$. Then $\mathscr{O}_X(1)_{|D+(x)}$ is given by the $k[\frac{y}{x}]$-module $yk[\frac{y}{x}]$.

Proposition 1.5. *Let S be a graded ring that is generated by S_1 as a S_0-algebra. Let $X = \mathrm{Proj}(S)$. Then we have the following*

1. *The sheaf $\mathscr{O}_X(n)$ is invertible.*
2. *For any graded S-module M and N we have that $\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} = \widetilde{M \otimes_S N}$. In particular we have that*
$$\widetilde{M} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) = \widetilde{M[n]}$$
and we have
$$\mathscr{O}_X(n) \otimes_{\mathscr{O}_X} \mathscr{O}_X(m) = \mathscr{O}_X(m+n).$$
3. *Let T be a graded ring, also generated in degree 1 as a T_0-algebra. Let $\varphi\colon S \longrightarrow T$ be a grading preserving homomorphism of rings. Then there is an induced map of schemes $f\colon U \longrightarrow X$, for some open $U \subseteq Y = \mathrm{Proj}(T)$, and we have that*
$$f^*\mathscr{O}_X(n) = \mathscr{O}_Y(n)_{|U} \quad \textit{and} \quad f_*(\mathscr{O}_Y(n)_{|U} = (f_*\mathscr{O}_U)(n).$$

Proof. Let $f \in S$ be homogeneous. By definition we have that $\mathscr{O}_X(n)_{D_+(f)}$ is $\tilde{M}$, where

$$M = S[n]_{(f)} = \{\frac{g}{f^m} \in S_f \mid \deg(g) = m \cdot \deg(f) + n\}.$$

We need to show that M is free of rank 1 as an A-module, where $A = S_{(f)}$. The S_f-module map $S_f \longrightarrow S_f$ sending $x \mapsto f^n x$ is an isomorphism. When we restrict this map to the subring $S_{(f)} = A$ we get an A-module isomorphism identifying A with its image. If $\deg(f) = 1$ then the image is $S[n]_{(f)}$. This proves the first assertion.

To prove Assertion 2, we let $M = \oplus_{d_1 \in \mathbf{Z}} M_{d_1}$ and $N = \oplus_{d_2 \in \mathbf{Z}} N_{d_2}$. Then we have the decomposition

$$M \otimes_S N = \bigoplus\nolimits_{d_1+d_2=d} M_{d_1} \otimes_S N_{d_2}$$

as a graded S-module. This means that $m \otimes n$ in $M \otimes_A N$ is homogeneous of degree d, iif $m \in M$ and $n \in N$ are homogenous, and $d = \deg(m) + \deg(n)$. We have that

$$M \otimes_S N \otimes_S S_f = M \otimes_S S_f \otimes_{S_f} N \otimes_S S_f = M_f \otimes_{S_f} N_f. \tag{1.5.1}$$

If f is homogenous, then the degree zero part is

$$(M \otimes_S N) \otimes_S S_f)_{(0)} = \{m \otimes n \otimes \frac{s}{f^p} \mid \deg(m) + \deg(n) + \deg(s) = p \deg(f)\}.$$

Note that we can write $m \otimes n \otimes (s/f^p) = sm \otimes n \otimes 1/f^p$. If $\deg(f) = 1$ then we can identify, under 1.5.1,

$$m \otimes n \otimes \frac{1}{f^p} = \frac{m}{f^{q_1}} \otimes \frac{n}{f^{q_2}},$$

where $q_1 = \deg m$ and $q_2 = \deg n$. This gives that

$$(M \otimes_S N) \otimes_S S_f)_{(0)} = M_{(f)} \otimes_{S_{(f)}} N_{(f)}$$

is an isomorphism. Note that in the right expression we are taking the tensor product over $S_{(f)}$ not S_f. This proves 2. The last statement one checks over affine opens, covering the open set $U \subseteq Y$. □

Lemma 1.6. *Let $\varphi \colon \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves on a scheme $X = \operatorname{Proj}(S)$, where S is a graded ring generated by S_1 as an S_0-algebra. Let n be an integer. Then we have that*

1. *The map φ is injective if and only if $\mathscr{F}(n) \longrightarrow \mathscr{G}(n)$ is injective.*
2. *The map φ is surjective if and only if $\mathscr{F}(n) \longrightarrow \mathscr{G}(n)$ is surjective.*

Proof. Both statements follows as $\mathscr{O}_X(n)$ is an invertible sheaf. Or, if you like $\mathscr{O}_X(n) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-n) = \mathscr{O}_X$. □

Definition 1.7. Let $X = \operatorname{Proj}(S)$, and $\mathscr{F}$ a sheaf of $\mathscr{O}_X$-modules. Then the graded group

$$\Gamma_*(\mathscr{F}) = \oplus_{n \in \mathbf{Z}} \Gamma(X, \mathscr{F}(n)),$$

is called the *associated* S-module. The S-module structure is as follows. Any homogeneous $x \in S_d$ determines a global section $x \in$

$\Gamma(X, \mathscr{O}_X(d))$. The isomorphism $\mathscr{F}(n) \otimes_{\mathscr{O}_X} \mathscr{O}_X(d) = \mathscr{F}(n+d)$ determines a map

$$\Gamma(X, \mathscr{O}_X(d)) \otimes_{\Gamma(X,\mathscr{O}_X)} \Gamma(X, \mathscr{F}(n)) \longrightarrow \Gamma(X, \mathscr{F}(n+d)),$$

which also describes the S-module structure $x \otimes s \mapsto xs$.

Proposition 1.8. *Let $S = A[x_1, \dots, x_r]$ be the polynomial ring in $r \geq 2$ variables, and let the variables all have degree 1. Then $\Gamma_* \mathscr{O}_X = S$.*

Proof. If $s \in \Gamma(X, \mathscr{O}_X(n))$ then the restriction of $s_{|D_+(x_i)}$ is a degree n element of S_{x_i}. If we simply consider all n simultaneously, we realize that $\Gamma_* \mathscr{O}_X$ is the global sections of the structure sheaf $\mathscr{O}_U$ on $U = \cup D_+(x_i)$. We have that $U = \mathbf{A}^r \setminus V(x_r, \dots, x_r)$. The global sections of U we have computed for $r = 2$, and the general situation is computed similarily. Note that the variables x_i are not zero-devisors, so the localizations maps are injective. We can therefore embed everything into $S_{x_1 \cdots x_r}$. Anyhow, we get that for $r \geq 2$ the global sections are the same as for $\mathbf{A}^r$, that is $S = A[x_1, \dots, x_r]$. □

Lemma 1.9. *Let X be a quasi-compact scheme, $\mathscr{L}$ an invertible sheaf, and $\mathscr{F}$ a quasi-coherent sheaf. Assume that $f \in \Gamma(X, \mathscr{L})$ is a global section, and let $U_f \subseteq X$ be the open set $\{x \in X \mid f_x \text{ not in } \mathfrak{m}_x \mathscr{L}_x\}$.*

1. *If $s \in \Gamma(X, \mathscr{F})$ is a global section such that the restriction1 $s_{|U_f} = 0$, then there exists an integer n such that the global section $f^n \otimes s \in \Gamma(X, \mathscr{L}^{\otimes n} \otimes \mathscr{F})$ is the zero section.*
2. *Assume that the intersection of any two open affines in X, is quasi-compact, and let $t \in \Gamma(U_f, \mathscr{F})$. Then there exists a integer n such that $f^n \otimes t \in \Gamma(U_f, \mathscr{L}^{\otimes n} \otimes \mathscr{F})$ extends to a global section of $\mathscr{L}^{\otimes n} \otimes \mathscr{F}$.*

Proof. We can find a finite, affine covering $\{U_i\}_{i=1}^r$ where $\mathscr{L}$ trivializes. Let $\psi_i \colon \mathscr{L}_{U_i} \longrightarrow \mathscr{O}_{X|U_i} = \tilde{A}_i$ be a trivialization, for each $i = 1, \dots, r$. Then the global section f restrict to give an element $g_i = \psi_i(f_{U_i}) \in A_i$. By construction we have that $U_f \cap U_i = \operatorname{Spec}(A_{i,g_i})$. The quasi-coherent module $\mathscr{F}$ is such that $\mathscr{F}_{|U_i} = \tilde{M}_i$, for some A_i-module M_i. The global section $s \in \Gamma(X, \mathscr{F})$ gives by restriction an element $s_i \in M_i$. By assumption $s_{|U_f} = 0$, so s_i is zero in the localized module M_{i,g_i}. But that means that $g_i^{n_i} s_i = 0$ in M_i, for some $n_i \geq 0$. Let n be the maximum of $\{n_1, \dots, n_r\}$. Then we have that $g^n s_i = 0$, for all i, and this translates to $f_{|U_i} \otimes \cdots \otimes f_{|U_i} \otimes s_i = 0$ in $\mathscr{L}_{|U_i}^{\otimes n} \otimes \mathscr{F}_{|U_i}$, for all $i = 1, \dots, n$. But, this means that $f^n \otimes s = 0$ as an element of $\Gamma(X, \mathscr{L}^{\otimes n} \otimes \mathscr{F})$, proving the first assertion.

Let now $t \in \Gamma(U_f, \mathscr{F})$ be a section. We use the covering as above. We have $U_f \cap U_i = \operatorname{Spec}(A_{i,g_i})$, and we get that $t_{|U_f \cap U_i}$ is given by an element of the localized module M_{i,g_i}. Then there is an element $t_i \in M_i$ such that $g_i^{n_i} t_i = t_{|U_f \cap U_i}$. Let n be one integer that works for all $i = 1, \dots, n$. We then have that the elements $f_{|U}^n \otimes t_i \in \Gamma(U_i, \mathscr{L}^{\otimes n} \otimes \mathscr{F})$, for each

$i = 1, \ldots, r$. Consider now the restriction of two of these $f^n \otimes t_i$ and $f^n \otimes t_j$ to the intersection $X_{i,j} = U_i \cap U_j$. Their difference

$$s = s_{i,j} = f^n \otimes t_i - f^n \otimes t_j$$

is a global section of $\Gamma(X_{i,j}, \mathscr{G})$, where $\mathscr{G}$ is the quasi-coherent sheaf $(\mathscr{L}^{\otimes n} \otimes \mathscr{F})_{|X_{i,j}}$. The restriction of the section s to the open

$$U_f \cap X_{i,j} = \{x \in X_{i,j} \mid f_x \text{ not in } \mathfrak{m}_x \mathscr{L}_x\},$$

where we consider f as a global section of $f \in \Gamma(X_{i,j}, \mathscr{L}_{X_{i,j}})$, is zero. By the first statement proven above, there exists an integer $m_{i,j}$ such that $f^{m_{i,j}} s = 0$ as a global section of $\mathscr{L}^{\otimes m_{i,j}}_{|X_{i,j}} \otimes \mathscr{L}^{\otimes n}_{|X_{i,j}} \otimes \mathscr{F}_{|X_{i,j}}$. That means that $f^{m_{i,j}+n} t_i = f^{m_{i,j}+n} t_j$. Let m be the maximum of $\{m_{i,j}\}$ and we are done since the local sections $f^{m+n} t_i$ agree on intersections, hence there is a global section that restricts to these local sections. $\square$

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