1. Lecture 11, April 17

Definition 1.1. Let $X = \mathrm{Proj}(S)$, and $\mathscr{F}$ a sheaf of $\mathscr{O}_X$-modules. Then the graded group

$$\Gamma_*(\mathscr{F}) = \oplus_{n\in\mathbf{Z}}\Gamma(X, \mathscr{F}(n)),$$

is called the *associated* S-module. The S-module structure is as follows. Any homogeneous $x \in S_d$ determines a global section $x \in \Gamma(X, \mathscr{O}_X(d))$. The isomorphism $\mathscr{F}(n) \otimes_{\mathscr{O}_X} \mathscr{O}_X(d) = \mathscr{F}(n+d)$ determines a map

$$\Gamma(X, \mathscr{O}_X(d)) \otimes_{\Gamma(X,\mathscr{O}_X)} \Gamma(X, \mathscr{F}(n)) \longrightarrow \Gamma(X, \mathscr{F}(n+d)),$$

which also describes the S-module structure $x \otimes s \mapsto xs$.

Lemma 1.2. *Let X be a quasi-compact scheme, $\mathscr{L}$ an invertible sheaf, and $\mathscr{F}$ a quasi-coherent sheaf. Assume that $f \in \Gamma(X, \mathscr{L})$ is a global section, and let $U_f \subseteq X$ be the open set $\{x \in X \mid f_x \text{ not in } \mathfrak{m}_x\mathscr{L}_x\}$.*

(1) *If $s \in \Gamma(X, \mathscr{F})$ is a global section such that the restriction1 $s_{|U_f} = 0$, then there exists an integer n such that the global section $f^n \otimes s \in \Gamma(X, \mathscr{L}^{\otimes n} \otimes \mathscr{F})$ is the zero section.*

(2) *Assume that the intersection of any two open affines in X, is quasi-compact, and let $t \in \Gamma(U_f, \mathscr{F})$. Then there exists a integer n such that $f^n \otimes t \in \Gamma(U_f, \mathscr{L}^{\otimes n} \otimes \mathscr{F})$ extends to a global section of $\mathscr{L}^{\otimes n} \otimes \mathscr{F}$.*

Proposition 1.3. *Let S be a graded ring, finitely generated by S_1 as a S_0-algebra. Let $\mathscr{F}$ be a quasi-coherent sheaf on $X = \mathrm{Proj}(S)$. Then there is a natural isomorphism*

$$\beta \colon \widetilde{\Gamma_*(\mathscr{F})} \longrightarrow \mathscr{F}.$$

Proof. We define the map of sheaves β by giving a natural map over basic opens $D_+(f) \subseteq X$. By definition, the sections of $\widetilde{\Gamma_*(\mathscr{F})}$ are elements of the form $s = m \otimes a/f^n = am \otimes 1/f^n$, where $\deg(a) + \deg(m) = n\deg(f)$, and where $am \in \Gamma(X, \mathscr{F}(d))$ is a global section with $d = n\deg(f)$. The element $1/f^n$ is a section of $\mathscr{O}_X(-d)$ over the basic open $D_+(f)$. By Proposition **??** (ii) we have that $\mathscr{F}(d) \otimes_{\mathscr{O}_X} \mathscr{F}(-d) = \mathscr{F}$. In particular this identity holds locally over $D_+(f)$. The two sections am and $1/f^n$ naturally give a section $\widetilde{am \otimes 1/f^n}$ of $\mathscr{F}$ over $D_+(f)$. This determines the map of sheaves $\beta \colon \widetilde{\Gamma_*(\mathscr{F})} \longrightarrow \mathscr{F}$.

We need to show that β is an isomorphism, and this we establish locally. Let $U = D_+(f)$ be a basic open. We consider $f \in \Gamma(X, \mathscr{O}_X(d))$, a global section of $\mathscr{O}_X(d)$ with $d = \deg(f)$. With the notation of Lemma 1.2 we have that $U_f = D_+(f)$. Furthermore, we have $\mathscr{F}(U) = M$, for some $S_{(f)}$-module M. Let $s \in \widetilde{\Gamma_*(\mathscr{F})}(U)$ be an element that is mapped to zero by β. Then s is a sum of elements of the form $m \otimes 1/f^p$, where $m \in \Gamma(X, \mathscr{F}(n))$ is a global section, and where $n = p\deg(f)$.

By construction $\beta(s) = m/f^p = 0$, as a section of $\Gamma(U, \mathscr{F}) = M$. By Lemma 1.2 we have that the global section $f^n \otimes s \in \Gamma(X, \mathscr{F}(d))$, where $d = \deg(s) + n\deg(f)$, for some n, is the zero section. But then we have that

$$s = m \otimes \frac{1}{f^p} = f^n m \otimes \frac{1}{f^{p+n}} = 0,$$

and we have shown injectivity.

To show surjectivity, let $m \in M = \mathscr{F}(U)$. By Lemma 1.2 we know that $m \otimes f^n \in \mathscr{F}_{|U}(d)$ comes from a global sections $s' \in \Gamma(X, \mathscr{F}(d))$, for some integer n.

□

1.4. It is however important to note that two different graded S-modules M and N can give the same sheaf $\tilde{M} = \tilde{N}$.

Definition 1.5. Let $f\colon X \longrightarrow \operatorname{Spec}(A)$ be a morphism of schemes, and let $\mathscr{L}$ be an invertible sheaf on X (that is quasi-coherent $\mathscr{O}_X$-module that is locally free of rank 1). The sheaf $\mathscr{L}$ is *very ample* (relative to f) if there exists an immersion

$$i\colon X \longrightarrow \mathbf{P}^n_A,$$

to some projective n-space over A, such that $i^*\mathscr{O}(1) = \mathscr{L}$. A morphism of schemes $i\colon X \longrightarrow Z$ is an immersion if it is an open immersion into a closed subscheme. Thus X is an open subscheme of W, where W is closed in Z.

Example 1.6. We have seen () that a closed subscheme $i\colon X \subseteq \mathbf{P}^n_A$ is given by a homogeneous ideal $I \subseteq A[x_0, \dots, x_n]$. So, $X = \operatorname{Proj}(A[x_0, \dots, x_n]/I$. We then have the sheaf $\mathscr{O}_X(1)$ on X, and we have the sheaf $\mathscr{O}(1)$ on $\mathbf{P}^n_A$. Then we also have that

$$\mathscr{O}_X(1) = i^*\mathscr{O}(1).$$

So, $\mathscr{O}_X(1)$ is very ample.

Definition 1.7. A sheaf $\mathscr{F}$ on a scheme X is *generated* by global sections if there exists global sections $\{s_i \in \Gamma(X, \mathscr{F})\}_{i \in \mathscr{I}}$ such that, for all points $x \in X$, the images of the sections s_i in the stalk $\mathscr{F}_x$, generate the stalk as an $\mathscr{O}_{X,x}$-module.

1.8. Note a global section $s \in \Gamma(X, \mathscr{F})$ of a sheaf $\mathscr{F}$ is equivalent with a map of sheaves $\mathscr{O}_X \longrightarrow \mathscr{F}$. The identity element is a global section, and sending 1 to s determines both a map and a global section of $\mathscr{F}$. Thus, a collection of global sections is equivalent with a map

$$\bigoplus_{i \in \mathscr{I}} \mathscr{O}_X \longrightarrow \mathscr{F}.$$

That the sheaf $\mathscr{F}$ is generated by gloabal sections, means that there exists a map as above, which is surjective as a map of sheaves.

Example 1.9. Let $X = \mathbf{P}^n_A$. From Propositon **??** we have that the sheaf $\mathscr{O}_X(-1)$ has no global sections. Since the sheaf $\mathscr{O}_X(-1)$ is non-zero, it follows that it can not be generated by global sections. The Proposition **??** gives that the A-module generated by $x_0, \ldots, x_n$, the linear forms of the polynomial ring $A[x_0, \ldots, x_n]$ are the global sections of $\mathscr{O}_X(1)$. We know furthermore, that locally, when restricted to $U_i = D_(x_i)$ we have that the sheaf $\mathscr{O}_X(1)_{U_i}$ is free of rank 1, and a basis for the sheaf is $x_{i|U_i}$. In other words, the map of sheaves

$$\oplus_{i=0}^n \mathscr{O}_X \longrightarrow \mathscr{O}_X(1)$$

is surjective.

Theorem 1.10 (Serre)**.** *Let $X \subseteq \mathbf{P}^n_A$ be a closed subscheme, with A a Noetherian ring. Let $\mathscr{O}_X(1) = i^*\mathscr{O}(1)$. For any coherent sheaf on X, there exists an integer n_0 such that for any $n \geq n_0$ the sheaf $\mathscr{F}(n)$ is generated by a finite set of global sections.*

Proof. Let $U_i = D_+(x_i)$ form the opens that gives us the standard open covering of X, with $i = 0, \ldots, n$. By assumption $\mathscr{F}$ is coherent, so $\mathscr{F}_{|U_i} = \tilde{M}_i$, for some finitely generated $A_i = \Gamma(U_i, \mathscr{O}_X)$-module M_i. Let $s_{i,1}, \ldots, s_{i,p_i}$ be elements of M_i that generate it as an A_i-module. Then there exist some N such that $x_i^N s_{i,j}$ is a global section of $\Gamma(X, \mathscr{F}(N))$. The integer N can be chosen such that it works for all $i = 0, \ldots, n$ as well. But then we have our map of sheaves

$$\oplus \mathscr{O}_X \longrightarrow \mathscr{F}(N)$$

that locally, over U_i willl hit the elements $x_i^N s_{i,j}$ with $j = 1, \ldots, p$. □

Corollary 1.11. *Let $X \subseteq \mathbf{P}^n_A$ as above. Any coherent sheaf $\mathscr{F}$ on X can be written as a quotient*

$$\oplus_{i=1}^p \mathscr{O}_X(n_i) \longrightarrow \mathscr{F},$$

for some integers $n_1, \ldots, n_p$. That is, $\mathscr{F}$ is a quotient of a sum of invertible sheaves.

Theorem 1.12. *Let k be a field, and let A be a finitely generated k-algebra. Let $X \subseteq \mathbf{P}^n_A$ be a projective scheme, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then $\Gamma(X, \mathscr{F})$ is a finitely generated A-module.*

Proof. See book of Hartshorne. □

Corollary 1.13. *Let $X \subseteq \mathbf{P}^n_Y$ be a closed subscheme, where Y is a scheme of finite type over a field. Let $f \colon X \longrightarrow Y$ be the projection map, and let $\mathscr{F}$ be a coherent sheaf on X. Then $f_*\mathscr{F}$ is coherent.*

Proof. Can assume $Y = \mathrm{Spec}(A)$. Then $f_*\mathscr{F}$ is the sheafification of the A-module $\Gamma(Y, f_*\mathscr{F}) = \Gamma(X, \mathscr{F})$, which is finitely generated by the theorem. □

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