1. Lecture 12, April 24

Let X be a Noetherian, integral, separated scheme which is regular in codimension 1. We refer to such schemes as schemes satisfying (*). It is a fact (see references in [Ha]) that examples of such schemes are non-singular or normal schemes. Being regular in codimension 1 means that for any point $x \in X$ such that the stalk $\mathscr{O}_{X,x}$ is of dimension 1, then the ring is regular. This is equivalent with the local ring being a DVR.

Definition 1.1. Let X be a scheme satisfying (*). A *prime divisor* Y on X is a closed, integral, subscheme of codimension 1. The free abelian group generated by the prime divisors in X is denoted by $\mathrm{Div}(X)$, and an element $D \in \mathrm{Div}(X)$ is called a *Weil divisor*.

Example 1.2. Points on curves, surfaces on three folds.

Let X be a scheme satisfying (*), and let $K = K(X)$ be its function field. If Y is a prime divisor, then the local ring $\mathscr{O}_{X,y}$, where y is the generic point of Y, is a DVR with fraction field K. We have a discrete valuation

$$v_Y \colon K^* \longrightarrow \mathbf{Z},$$

where $v_Y(xy) = v_Y(x) + v_Y(y)$, and where $v_Y(x+y) \geq \min(v_Y(x), v_Y(y))$. The valuation is group homomorphism.

For any $f \in K^*$ we say that f has a zero along Y if $v_Y(f) > 0$, and that f has a pole along Y if $v_Y(f) < 0$.

Lemma 1.3. *Let $f \in K^*$. Then $v_Y(f) = 0$ for all but a finite number of prime divisors Y in X.*

Proof. As X is quasi-compact, it suffices to show the statement for $X = \mathrm{Spec}(A)$ an affine scheme. We have that $f = g/h$ and that $v_Y(f) = v_Y(g) - v_Y(h)$, hence it suffices to show the statement for non-zero $f \in A$. We have $v_Y(f) = 0$ if the corresponding generic point $y \in X \setminus V(f)$. Thus we need only to look at point on $\mathrm{Spec}(A/f)$. Since $\mathrm{Spec}(A/f)$ is a proper subset, meaning not equal to $X = \mathrm{Spec}(A)$, the only possible codimension 1 points of X in $\mathrm{Spec}(A/f)$ are the minimial primes. There are only finite many of these. $\square$

Definition 1.4. Let $f \in K^*$, then we define the divisor

$$(f) = \sum v_Y(f) \cdot Y \qquad \in \mathrm{Div}(X),$$

where the sum runs over all prime divisors in X. The divisor (f) is called a *principal divisor*.

Note that we have a group homomorphism $(K^*, \cdot) \longrightarrow (\mathrm{Div}(X), +)$.

Definition 1.5. Two divisors D_1 and D_2 are *linearly equivalent*, denoted with $D_1 \simeq D_2$ if their difference $D_1 - D_2$ is a principal divisor.

Linear equivalence is an equivalence relation, and we define the divisor class group

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\simeq .$$

Proposition 1.6. *Let A be a Noetherian domain. Then A is a UFD if and only if A is normal, and* $\mathrm{Cl}(\mathrm{Spec}(A)) = 0$.

Remark 1.7. Note that if A is normal, then $\mathrm{Spec}(A)$ will satisfy (*), and in particular we can talk about its class group.

Proof. It is a fact that a UFD is integrally closed = normal. We have furthermore that A is UFD if and only if each prime ideal of height 1 is principal. So, if Y is a prime divisor in $\mathrm{Spec}(A)$, then the corresponding prime ideal $p \subseteq A$ is generated by one element $f \in A$. We have that $v_Y(f) = 1 \cdot Y$, and then it follows that $(f) = Y$, and thus $\mathrm{Cl}(\mathrm{Spec}(A)) = 0$.

Conversely, assume that $\mathrm{Cl}(\mathrm{Spec}(A)) = 0$, and let Y be a prime divisor. Let $P \subseteq A$ be the corresponding prime ideal. By assumption there is $f \in K^*$ such that $(f) = Y$. As $v_Y(f) = 1$ we have that $f \in A_P$, and that f generates PA_P. For any other prime divisor Z in X we have that $v_Z(f) = 0$, which implies that $f \in A_Q$, where $Q \subset A$ is the prime ideal corresponding to Z. As A is integrally closed domain we have (see references in [Ha]) that

$$A = \cap_{htP=1} A_P,$$

and it follows that $f \in A$, not only in the fraction field K. As $A \cap PA_P = P$, we have that $f \in P$. We need to show that f generates P. Let $x \in P$. Then $v_Y(x) \geq 1$ and $v_Z(x) \geq 0$. It follows that for all prime divisors Y in $\mathrm{Spec}(A)$ we have that $v_Y(x/f) \geq 0$. Thus $x/f \in A$ so $x \in fA$, proving our claim. We then have that A is a UFD. □

Example 1.8. We have that the class group of the affine n space over a field, $\mathbf{A}^n_k$, is zero.

Let $X = \mathbf{P}^n_k$ be the projective n-space over a field. If Y is a prime divisor we have that Y is given by the ideal generated by an irreducible polynomial $g \in k[x_0, \ldots, x_n]$ of some degree d. We let $\deg(Y_i)$ be the degree of the homomogeneous element defining it. If $D = \sum n_i Y_i$ is a Weil divisor, we define the degree of D as $\deg(D) = \sum n_i \deg(Y_i)$.

Proposition 1.9. *Let $H \subset \mathbf{P}^n_k$ denote the hyperplane given by $x_0 = 0$. We have the following*

1. *If $f \in K^*$, then the degree of the divisor (f) is zero.*
2. *If d is the degre of a divisor D, then $D \simeq dH$.*
3. *The degree map* $\deg\colon \mathrm{Cl}(X) \longrightarrow \mathbf{Z}$ *is an isomorphism.*

Proof. We did not prove this in detail. We noted the following.

If $f \in K^*$, then $f = g/h$, where g and h are homogeneous polynomials of the same degree.

If $g \in k[x_0, \dots, x_n]$ is a homogeneous polynomial of degree d, then we can write $g = g_1^{n_1} \cdots g_r^{n_r}$, where g_i is irreducible of degree d_i. We have that $d = \sum n_i d_i$, and that each g_i corresponds to a prime divisor Y_i of degree d_i. This proves a).

To prove b) we write a given divisor $D = D_1 - D_2$, where D_1 and D_2 are effective divisors. Effective means that the coefficients $E = \sum n_i Y_i$ are all non-negative. It follows that an effective divisor D_1 is associated to a homogeneous g, of degree $\deg(g) = \deg(D_1)$. And similarily that D_2 is associated to the homogeous h. Consider now the function $f = g/hx_0^d$, where $d = d_1 - d_2$. Then $f \in K$, and we have that the associated divisor

$$(f) = D_1 - D_2 - dH.$$

Since (f) is a prinicpal divisor, the result follows.

Last statement follows as $\deg(H) = 1$. □

Proposition 1.10. *Let $Z \subseteq X$ be a closed, proper subset, and let $U = X \setminus Z$.*

(1) *The map* $\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U)$ *taking* $\sum n_i Y_i$ *to* $\sum n_i Y_i \cap U$ *is surjective.*
(2) *If* $\mathrm{codim}(Z, x) \geq 2$, *then the map in 1) is an isomorphism.*
(3) *If Z is irreducible of codimension 1, then we have an exact sequence*

$$\mathbf{Z} \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0,$$

where the first map takes $n \mapsto nZ$.

Proof. See [Ha]. □

Example 1.11. Let C be an irreducible curve in $\mathbf{P}_k^2$. Then C has a degree d, and it follows that

$$\mathrm{Cl}(\mathbf{P}_k^2 \setminus C) = \mathbf{Z}/(d).$$

Example 1.12. Let $X = \mathrm{Spec}(A)$, where $A = k[x,y,z]/(xy - z^2)$. Then A is a Noetherian domain, and also normal. We will determine its class group.

Note that $P = (y, z)$ is a prime ideal since the quotient $A/P = k[x]$. Thus $\mathrm{Spec}(A/P) = Y$ is a prime divisor on X. We have the exact sequence

$$\mathbf{Z} \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0,$$

where $U = X \setminus Y$. The ideal (y) is not prime, since the quotient $A/(y) = k[x,z]/(z^2)$. However the underlying topological space of $\mathrm{Spec}(A/(y))$ equals $|Y|$. In particular we have that

$$U = X \setminus Y = X \setminus \mathrm{Spec}(A/y) = \mathrm{Spec}(A_y).$$

The ring $A_y = k[y, y^{-1}, z]$ is a UFD, hence $\mathrm{Cl}(U) = 0$. In particular we have that $\mathrm{Cl}(X)$ is generated by the prime divisor Y, that is the left most map in the sequence above is surjective.

The map is however not an isomorphism. Consider the function $y \in A$, as an element of K^*. We are interested in the principal divisor associated to that function. We have that the DVR at the generic point of Y is

$$k[x,y,z]_{(y,z)}/(y - x^{-1}z^2) = k[x,z]_{(z)}.$$

The function $y = x^{-1}z^2$, where x is invertible, and where z is a local generator for the maximal ideal in the DVR. It follows that the valuation of y is $2 \cdot Y$. It follows that $v_Y(y) = 2Y$, and in particular

$$\mathbf{Z}/2\mathbf{Z} \longrightarrow \mathrm{Cl}(X)$$

is surjective. If the kernel was even bigger that would imply that $\mathrm{Cl}(X) = 0$. However, then since A is normal, we would get that A is a UFD, which A is not. Thus, the class group is $\mathbf{Z}/2\mathbf{Z}$.

Department of Mathematics, KTH, Stockholm, Sweden
E-mail address: `skjelnes@kth.se`