1. Lecture 13, May 08

1.1. **Cartier divisors.** Let X be a scheme, for any open *affine* $U = \operatorname{Spec}(A)$, we let $K(U)$ denote the total fraction ring of A. Then $K(-)$ form a presheaf on the basis for the topology on X, and we let $\mathscr{K}_X$ denote its associated sheaf. See the article by Kleiman [Kl][1].

We let $\mathscr{K}_X^*$ denote the subsheaf of $\mathscr{K}_X$ consisting of invertible elements. We have that $\mathscr{O}_X^*$ is a subsheaf of $\mathscr{K}_X^*$. We will in the sequel consider these sheaves as sheaves of abelian groups, that is with respect to their multiplicative structures.

Definition 1.2. A *Cartier divisor* on X is a global section of the sheaf quotient $\mathscr{K}_X^*/\mathscr{O}_X^*$.

Recall that the sheaf quotient is the sheaf associated to the presheaf quotient. A Cartier divisor D is given by an open cover $\{U_i\}$ of X, and elements $f_i \in \Gamma(U_i, \mathscr{K}_X^*)$, such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathscr{K}_X^*)$ for all i, j. By possibly shrinking the open U_i, we can assume that the f_i are elements of the presheaf $K(-)$.

From the exact sequence of sheaves

$$0 \longrightarrow \mathscr{O}_X^* \longrightarrow \mathscr{K}_X^* \longrightarrow \mathscr{K}_X^*/\mathscr{O}_X^* \longrightarrow 0,$$

we get the induced map of global sections $\Gamma(X, \mathscr{K}_X^*) \longrightarrow \Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*)$. The image of this map are called *principal* Cartier divisors, and the cokernel is denoted by $\operatorname{CaCl}(X)$. Two Cartier divisors D and E are said to be linearly equivalent if they have the same image in $\operatorname{CaCl}(X)$.

Proposition 1.3. *Let X be an integral, separated, Noetherian scheme that is locally factorial (each stalk $\mathscr{O}_{X,x}$ is a UFD). Then we have an isomorphism of groups*

$$\operatorname{Div}(X) = \Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*).$$

Moreover, the principal Weil divisors correspond under this isomorphism to principal Cartier divisors. Hence we also have an isomorphism of groups $\operatorname{Cl}(X) = \operatorname{CaCl}(X)$.

Proof. As in [Ha], Proposition 6.11. □

1.4. **Invertible sheaves.**

Proposition 1.5. *Let X be a scheme, and let $\mathscr{L}$ and $\mathscr{M}$ be invertible sheaves on X. Then we have*

(1) *The sheaf $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{M}$ is invertible.*
(2) *The sheaf $\mathscr{H}om(\mathscr{L}, \mathscr{O}_X)$ is invertible.*
(3) *We have an isomorphism $\mathscr{H}om(\mathscr{L}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{L} = \mathscr{O}_X$.*

Proof. Exercise/Clear. □

[1] Steve Kleiman "Misconceptions about K_X"

We have that the set of equivalence classes of invertible sheaves on a scheme X form an Abelian group under tensor product. This group is called the Picard group $\mathrm{Pic}(X)$ of X. The identity element is $\mathscr{O}_X$, and the dual of an invertible sheaf $\mathscr{L}$ is its inverse. We write $\mathscr{L}^{-1} := \mathscr{H}om(\mathscr{L}, \mathscr{O}_X)$.

Let D be a Cartier divisor on X, given by $\{U_i, f_i\}$. Then we form the $\mathscr{O}_X$-submodule $\mathscr{L}(D)$ of $\mathscr{K}_X^*$, that is generated, locally over U_i by f_i^{-1}. One checks that this is indeed well-defined, and independent of the given presentation.[2]

Proposition 1.6. *Let X be a scheme. We have the following*

(1) *The sheaf $\mathscr{L}(D)$ is invertible, and we obtain a 1-1 correspondance between Cartier divisors and invertible $\mathscr{O}_X$-subsheaves of $\mathscr{K}_X^*$.*
(2) *We have that $\mathscr{L}(D - E) = \mathscr{L}(D) \otimes_{\mathscr{O}_X} \mathscr{L}(E)^{-1}$.*
(3) *Two Cartier divisors D and E are linearly equivalent, if and only if $\mathscr{L}(D)$ is isomorphic (abstractly) to $\mathscr{L}(E)$.*

Proof. As in [Ha], Proposition 6.13. □

Corollary 1.7. *We have an injective group homomorphism*

$$\mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X).$$

Proposition 1.8. *If X is integral then* $\mathrm{CaCl}(X) = \mathrm{Pic}(X)$.

Proof. See [Ha], Proposition 6.15. □

1.8.1. *The result holds when X is projective over a field, but not in general.*

Corollary 1.9. *If X is Noetherian, integral, separated, and locally factorial, then* $\mathrm{Cl}(X) = \mathrm{Pic}(X)$.

Corollary 1.10. *Let $X = \mathbf{P}_k^n$, the projective n-space over a field k. Then every invertible sheaf on X is isomorphic to $\mathscr{O}(n)$, for some integer n.*

1.11. **Effective Cartier divisors.** A Cartier divisor D on X, is called *effective* if we can find an cover U_i, such that the Cartier divisor (U_i, f_i) is represented with $f_i \in \Gamma(U_i, \mathscr{O}_{U_i})$, for all i.

Note that if D is effective, and U_i all affine, the local generators $f_i \in A_i = \Gamma(U_i, \mathscr{O}_{U_i})$ are regular elements; that is f_i is not a zero divisor.

Let D be an effective Cartier divisor. For two different open, affine, U_i and U_j we have that f_i/f_j is invertible on the intersection, and it follows that the local generators $f_i \in A_i$ generate an idealsheaf $\mathscr{I}_D \subseteq$

[2] To me it would make more sense to take the submodule generated by f_i, and not its inverse. I would appreciate if someone could explain why not to do so.

$\mathscr{O}_X$. We get a correspondance between effective Cartier divisors on X, and locally principal subschemes where the local generators are regular.

By our earlier conventions we have that if D is an effective Cartier divisor then corresponding ideal sheaf $\mathscr{I}_D$ equals $\mathscr{L}(-D)$.

Example 1.12. Consider $X = \mathbf{P}^2_k$, the projective plane. A global section, non-zero, $f \in \Gamma(X, \mathscr{O}(n))$ with $n > 0$, determines a hypersurface $V(f) \subseteq X$. It also determines an effective divisor $D = V(f)$ by the standard cover $\{D_+(x_i)\}$ of X, and with local generators $f_i = f/x_i^n$.

Two different, non-zero, global sections f and g of $\mathscr{O}(n)$ give different effective Cartier divisors, but their difference is a principal divisor (f/g).

1.13. **Properties of projective n-space.** . Let $X = \mathbf{P}^n_A$, be the projective n-space over $\operatorname{Spec}(A)$. We have that $X = \operatorname{Proj}(S)$, where S is the graded polynomial ring $A[x_0, \dots, x_n]$, where the variables all have degree one. The sheaf $\mathscr{O}_X(1)$ is invertible, and generated by global sections $x_0, \dots, x_n$. Thus the map

$$\oplus_{i=0}^n \mathscr{O}_X e_i \longrightarrow \mathscr{O}_X(1)$$

sending $e_i \mapsto x_i$, is surjective. For any morphism $f\colon Y \longrightarrow X$, we have that $f^*\mathscr{O}_X(1) = \mathscr{L}$ is invertible, and that the global sections $f^*(x_i) = s_i$, generate $\mathscr{L}$.

Theorem 1.14. *Let Y be a scheme. If $\mathscr{L}$ is an invertible sheaf on Y, and $s_0, \dots, s_n$ are global sections that generate $\mathscr{L}$, then there is a unique morphism $f\colon Y \longrightarrow \mathbf{P}^n_A$ such that $f^*\mathscr{O}_X(1) = \mathscr{L}$, and $f^*(x_i) = s_i$, for $i = 0, \dots, n$.*

Proof. We did not have time to do this proof properly, but in essence we followed [Ha], Theorem 7.1. To construct a morphims form Y, we construct morhpisms locally $f_i\colon U_i \longrightarrow D_+(x_i)$. The open cover U_i of Y is given as

$$U_i = \{y \in Y \mid (s_i)_y \notin \mathfrak{m}_y\mathscr{L}\}.$$

We observe that as the section s_i is invertible on U_i, it follows that $\mathscr{L}_{|U_i} = s_i\mathscr{O}_{X|U_i}$. In particular any section s_j restricted to U_i can be written as $s_j = s_{i,j}s_i$, with $s_{i,j} \in \mathscr{O}_{U_i}$. To give a morhism $f_i\colon U_i \longrightarrow D_+(x_i)$ is to give an ring homomorphism

$$A[y_0, \dots, \hat{y}_i, \dots, y_n] \longrightarrow \Gamma(U_i, \mathscr{O}_{U_i}) = \Gamma(U_i, \mathscr{L}),$$

where $y_j = x_j/x_i$. The A-algebra homomorphism we define sends y_j to $s_{i,j}$. One checks that these maps have the right properties. □

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