1. Lecture 14, May 15

1.1. **Graded algebras.** Let X be a scheme. Let $\mathscr{S}$ be a quasi-coherent sheaf of $\mathscr{O}_X$-algebras, and assume that $\mathscr{S} = \oplus_{d\geq 0}\mathscr{S}_d$ is a graded algebra. We assume that $\mathscr{S}_0 = \mathscr{O}_X$ and that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as a $\mathscr{S}_0$-algebra.

Example 1.2. If $\mathscr{I} \subseteq \mathscr{O}_X$ is a quasi-coherent ideal sheaf, then the Rees algebra $\mathscr{S} = \oplus_{d\geq 0}\mathscr{I}^d$ is a typical example we will consider.

Example 1.3. If $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_X$-module, the the symmetric algebra $\mathscr{S}(\mathscr{E})$ is another typical example of graded $\mathscr{O}_X$-algebras.

If $\mathscr{S}$ is a graded $\mathscr{O}_X$-algebra as above, we define a scheme

$$\pi\colon \operatorname{Proj}(\mathscr{S}) \longrightarrow X$$

over X. For any open affine $U \subseteq X$, the scheme $\operatorname{Proj}(\mathscr{S})_{\pi^{-1}(U)}$ is the Proj of the graded algebra $\Gamma(U, \mathscr{S}_{|U})$. Locally we then also have the invertible sheaves $\mathscr{O}(1)$ defined, and these glue to form an invertible sheaf $\mathscr{O}(1)$ on $\operatorname{Proj}(\mathscr{S})$.

Example 1.4. $X = \operatorname{Spec}(A)$, and $\tilde{\mathscr{E}}$, where E is a free A-module of rank $n+1$. Then the symmetric algebra $S(\tilde{\mathscr{E}}) = S(E)$ is isomorphic to the polynomial ring $A[X_0, \ldots, x_n]$ in $n+1$-variables over A. Then $\operatorname{Proj}(S(E)) = \mathbf{P}^n_A$.

Example 1.5. Let $I = (x, y) \subseteq A = k[x, y]$ the maximal ideal corresponding to the origo in the plane. Then $\oplus_{d\geq 0} I^d = A[T, U]/(Ty - Ux)$, so $\operatorname{Proj}(\oplus I^d)$ is a closed subscheme of the projective line over A. We have that $D_+(T)$ and $D_(U)$ are both isomorphic to the affine plane, and we have that the fiber over $A/I = k$ is the projective line $\mathbf{P}^1_k$.

Proposition 1.6. *Let X be a Noeterian scheme, and $\mathscr{S}$ a graded $\mathscr{O}_X$-algebra where $\mathscr{S}_1$ a coherent $\mathscr{O}_X$-module. Then $\pi\colon \operatorname{Proj}(\mathscr{S}) \longrightarrow X$ is a proper morphism.*

Proof. Properness is a local property, and locally we have proven this statement. □

Proposition 1.7. *Let X be a Noetherian scheme, and $\mathscr{E}$ a coherent, locally free of rank n module. Then $\mathbf{P}(\mathscr{E}) = \operatorname{Proj}(\mathscr{S}(\mathscr{E}))$ has the following universal defining property. Let $g\colon Y \longrightarrow X$ be a scheme. A morphism from Y to $\mathbf{P}(\mathscr{E})$, compatible with the morphism to X, is equivalent with an invertible sheaf $\mathscr{L}$ on Y, and a surjection $g^*\mathscr{E} \longrightarrow \mathscr{L}$.*

Proof. We have proved this locally. For a proof see [Ha] Proposition 7.12. □

Definition 1.8. Inverse image sheaf. Let $f\colon X \longrightarrow Y$ be a morphism of schemes, and let $\mathscr{I}$ be a quasi-coherent sheaf of ideals on Y. We let $f^{-1}\mathscr{I}\mathscr{O}_X$ denote the ideal sheaf on X, given as the image of the natural map of quasi-coherent sheaves $f^*\mathscr{I} \longrightarrow f^*\mathscr{O}_Y = \mathscr{O}_X$.

Proposition 1.9. *Let $\mathscr{I}$ be a (quasi-) coherent idealsheaf on a Noetherian scheme X. Let $\pi\colon \tilde{X} \longrightarrow X$ be the blow-up of X along $\mathscr{I}$.*

1. *The inverse image sheaf $\pi^{-1}(\mathscr{I}\mathscr{O}_{\tilde{X}}$ is invertible.*
2. *Let $U = X \setminus Z$, where Z is the closed subscheme defined by $\mathscr{I}$. Then π restricted to $\pi^{-1}(U)$ is an isomorphism.*

Proof. We proved this as in [Ha], Proposition 7.13. □

Proposition 1.10. *Universal property of blow-up. Let $f\colon Z \longrightarrow X$ be a morphism of schemes. Assume that $f^{-1}\mathscr{I}\mathscr{O}_X$ is invertible, for a coherent ideal sheaf $\mathscr{I}$ on a Noetherian X. Then there exist a unique factorization of f via $Z \longrightarrow \tilde{X}$, where $\tilde{X}$ is the blow-up of X along $\mathscr{I}$.*

Theorem 1.11. *Let X be a quasi-projective variety (integral, separated scheme of finite type over an algebraically closed field k that can be realized as a subscheme of a projective n-space over k). If Z is a variety, and $f\colon Z \longrightarrow X$ is a birational (isomorphism on a dense open subset) map, then $f\colon Z \longrightarrow X$ is isomorphic to the blow-up map for some coherent ideal sheaf $\mathscr{I}$ in X.*

Theorem 1.12. *Let $X = \mathbf{P}^n_A$, the projective n-space over a $Y =$ $\mathrm{Spec}(A)$. We have an exact sequence*

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \mathscr{O}_X(-1)^{n+1} \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

Proof. We proved this as in [Ha], Theorem 8.17. □

Theorem 1.13. *Let X be an irreducible, separated scheme of finite type over an algebraically closed field k. Then $\Omega_{X/k}$ is locally free of rank $n = \dim(X)$ if and only if X is non-singular.*

The tangent sheaf of a nonsingular scheme X is defined as the dual of $\Omega_{X/k}$. If $n = \dim(X)$, then its canonical sheaf is $\omega_X = \wedge^n \mathscr{O}_{X/k}$.

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