1. Exercise, Lecture 1, September 6

I suspect that even making the steps listed below, comprehensive, can be hard.

Plucker embedding. Let E be a quasi-coherent sheaf on a scheme S, and let G_E^n be the Grassmann scheme parameterizing locally free rank n quotients of E. Furthermore, let P denote the Grassmann scheme parameterizing locally free rank 1 quotients of $\Lambda^n E$.

Step a. Establish that we have a natural morphism of schemes

$$\pi\colon G_E^n \longrightarrow P.$$

1.1. We are going to prove (or disprove), that the map π is a closed immersion. We show this by showing that the map π is a closed immersion of functors, that is for any scheme $T \longrightarrow S$, and any map $T \longrightarrow P$ the fiber product

$$G_E^n \times_P T$$

is representable by a closed subscheme T_0 of T.

Step b. Explain that we may assume that T is affine scheme, and that we may assume that $S = T$, in order to prove that π is a closed immersion. Use properties of the fiber product and that $G_E^n \times_S T = G_{f^*E}^n$, where $f\colon T \longrightarrow S$ is the structure map.

1.2. Recall that the Grassmann $G_E^n \longrightarrow S$ over an affine scheme $S = \mathrm{Spec}(A)$ can be covered by standard open affine sets: Let M be the A-module that corresponds to the quasi-coherent sheaf E on $S = \mathrm{Spec}(A)$. Fix an A-module map

$$s\colon \oplus^n A = M_1 \longrightarrow M.$$

Then the standard open set G_E^s is represented by the scheme

$$S_A(M/M_1 \otimes_A M_1),$$

and parameterizes quotients of M that composed with s gives an isomorphism. And $S_A(N)$ means the symmetric tensor algebra of the A-module N.

Step c. Show that the standard open set $P^{\wedge s}$ coming from the map $\wedge^n s\colon A \longrightarrow \wedge^n M$ form a covering of $P = G^1_{\wedge^n E}$, and show that

$$\pi^{-1}(P^{\wedge s}) = G_E^s.$$

1.3. To show that we have a closed immersion, we need only to see that the restricted map

$$\pi\colon G_E^s \longrightarrow P^{\wedge s}$$

is a closed immersion. To do so we need to establish some facts.

Step d. Assume that we have a short exact sequence of A-modules, that split

$$0 \longrightarrow M_1 \longrightarrow M = M_1 \oplus F \longrightarrow F \longrightarrow 0.$$

Show that we have that

$$\wedge^r M = \oplus_{i=0}^r \wedge^{r-i} M_1 \otimes_A \wedge^i F.$$

In particular we have that $\wedge^r M = \wedge^r M_1 \oplus F_1$, for some module F_1. Finally, you show that

$$\mathrm{Hom}_{A-\mathrm{mod}}(F_1, \wedge^r M_1) = \oplus_{i=1}^r \mathrm{Hom}_{A-\mathrm{mod}}(\wedge^i F, \wedge^i M_1).$$

Here you use that

$$\mathrm{Hom}_{A-\mathrm{mod}}(\wedge^{r-i} M_1, \wedge^r M_1) = \wedge^i M_1,$$

and this isomorphism is canonical, and most probably the one place where we use that the module $\wedge^i M_1$ is locally free (of finite rank).

Step e. Identify the standard open set G_E^s with $\mathrm{Hom}_A(F, M_1)$, and the standard open set $P^{\wedge^s}$ with $\oplus_{i=1}^n \mathrm{Hom}_{A-\mathrm{mod}}(\wedge^i F, \wedge^i M_1)$. The map π is then

$$\pi\colon\ \mathrm{Hom}_{A\text{-mod}}(F, M_1) \longrightarrow \oplus_{i=1}^n \mathrm{Hom}_{A\text{-mod}}(\wedge^i F, \wedge^i M_1)$$

that sends a map φ to

$$\pi(\varphi) = (\wedge\varphi, \wedge^2\varphi, \dots, \wedge^n\varphi).$$

Deduce from this that the map π is a closed immersion.