Exercises, Lecture 5, October 4

Let k be a field.

(1a) Show the following homogeneous version of prime avoidance. Let $x_1, x_2, \dots, x_m$ be a finite number of points in $\mathbb{P}^n_k$ ($n \geq 1$). Show that there is a hypersurface that does not contain any of the given points. If k is infinite, show that there is a hyperplane with the same property.

(1b) Show that if $Z \hookrightarrow \mathbb{P}^n_k$ is a closed subscheme of dimension d, then there is a hypersurface $H \hookrightarrow \mathbb{P}^n_k$ such that $Z \cap H$ has dimension $d - 1$.

(1c) Assume that k is infinite. Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$, show that there is $f \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ such that multiplication by $f\colon \mathcal{F}(-1) \to \mathcal{F}$ is injective (use associated points) and conclude (using your argument in (1b)) that $\dim(\operatorname{Supp}(\mathcal{F}) \cap H) < \dim(\operatorname{Supp}(\mathcal{F}))$ where H is the hyperplane defined by f.

(2) Find the Hilbert polynomial for the dth Veronese embedding $j\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ of $\mathbb{P}^1$. Hint: use that $H^i(\mathbb{P}^d, (j_*\mathcal{O}_{\mathbb{P}^1})(m)) = H^i(\mathbb{P}^1, j^*(\mathcal{O}_{\mathbb{P}^d}(m)))$ for all i.

Exercises (1a) and (1b) are variants of 12.3.B (c) and (d). Exercise (1c) is 20.5.A and 20.5.B. Exercise (2) is 20.5.D/E.