1. Exercises for Lecture 10, November 13

Exercise I. Let $D \longrightarrow E \longrightarrow F \longrightarrow 0$ be a complex of A-modules. If the sequence is an exact sequence, then we have that the induced sequence

$$0 \longrightarrow \mathrm{Hom}_A(F, X) \longrightarrow \mathrm{Hom}_A(E, X) \longrightarrow \mathrm{Hom}_A(D, X)$$

is exact, for all A-modules X. Is the converse statement true?

Exercise II. Let $A \longrightarrow B \longrightarrow C$ be homomorphisms of rings, and consider the induced sequence of C-modules

$$C \bigotimes_B \Omega_{B/A} \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0.$$

(1) Show directly that the sequence above is exact.
(2) Use the universal defining properties of the Kähler differentials to show that the sequence is exact.
(3) Give a non-trivial example where the leftmost map in the relative contangent sequence, is injective.

Exercise III. Let $A \longrightarrow B$ be a homomorphism of rings, and let $C = B/I$.

(1) Show that we have an exact sequence of C-modules

$$I/I^2 \longrightarrow C \bigotimes_B \Omega_{B/A} \longrightarrow \Omega_{C/A} \longrightarrow 0.$$

(2) Consider the surface (families of surfaces) in $B = A[x, y, z]$, determined by the ideal $I = (x^3y^2 - z^2)$. Show that in the conormal exact sequence, the leftmost map is injective.
(3) Can you generalize the surface example?