1. Exercises for Lecture 11, November 20

Exercise I. Let F be a covariant functor from the category of B-modules to sets. What does it mean that F is representable?

Exercise II. Let $A \longrightarrow B$ be a homomorphism of rings, and let $I \subseteq B \bigotimes_A B$ be the kernel of the multiplication map $B \bigotimes_A B \longrightarrow B$. Set $\Omega_{B/A} = I/I^2$. We have an exact sequence of $C = B \bigotimes_A B/I^2$-modules

$$0 \longrightarrow \Omega_{B/A} \longrightarrow C \longrightarrow B \longrightarrow 0,$$

and in particular we have that the ideal $\Omega_{B/A}$ is such that $\Omega_{B/A}^2 = 0$. We have the induced coprojection maps $\mu_i \colon B \longrightarrow C$, lifting the identity map on B. Show that $d_B = \mu_1 - \mu_2$ is an element of $\mathrm{Der}_A(B, \Omega_{B/A})$.

Exercise III. For a B-module M, we have the B-algebra $B[M]/M^2$. Show that for any derivation $d \in \mathrm{Der}_A(B, M)$ we have the A-algebra homomorphism

$$\varphi \colon B \bigotimes_A B \longrightarrow B * M$$

sending pure tensors $(x \otimes y) \mapsto (xy, xdy + ydx)$, and then We get an induced B-module map $f \colon \Omega_{B/A} \longrightarrow M$.

Exercise IV. Show that $(\Omega_{B/A}, d_B)$ represents the functor $F = \mathrm{Der}_A(B, -)$.

Exercise V. Let $X \longrightarrow Y$ be a separated morphism of schemes. Then the diagonal map $\Delta \colon X \longrightarrow X \times_Y X$ is a closed immersion, given by the ideal sheaf I. Let $\Omega_{X/Y}$ be the quasi-coherent sheaf $\Delta^*(I/I^2)$ on X. Let $Z \longrightarrow Y$ be a morphism, and let $p \colon X \times_Y Z \longrightarrow X$ be the projection map. Show that

$$\Omega_{X \times_Y Z/Z} = p^* \Omega_{X/Y}.$$