Development and Investigation of Reactivity Measurement Methods in Subcritical Cores

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Abstract

Subcriticality measurements during core loading and in future accelerator driven systems have a clear safety relevance. In this thesis two subcriticality methods are treated: the Feynman-alpha and the source modulation method.

The Feynman-alpha method is a technique to determine the reactivity from the relative variance of the detector counts during a measurement period. The period length is varied to get the full time dependence of the variance-to-mean. The corresponding theoretical formula was known only with stationary sources. In this thesis, due to its relevance for novel reactivity measurement methods, the Feynman-alpha formulae for pulsed sources for both the stochastic and the deterministic cases are treated. Formulae neglecting as well as including the delayed neutrons are derived. The formulae neglecting delayed neutrons are experimentally verified with quite good agreement.

The second reactivity measurement technique investigated in this thesis is the so-called source modulation technique. The theory of the method was elaborated on the assumption of point kinetics, but in practice the method will be applied by using the signal from a single local neutron detector. Applicability of the method therefore assumes point kinetic behaviour of the core. Hence, first the conditions of the point kinetic behaviour of subcritical cores was investigated. After that the performance of the source modulation technique in the general case as well as and in the limit of exact point kinetic behaviour was examined. We obtained the unexpected result that the method has a finite, non-negligible error even in the limit of point kinetic behaviour, and a substantial error in the operation range of future accelerator driven subcritical reactors (ADS). In practice therefore the method needs to be calibrated by some other method for on-line applications.

Keywords: reactivity measurement, subcritical cores, core loading, Accelerator Driven Systems (ADS), Feynman-alpha formulae, source modulation method, point kinetics
This licentiate thesis consists of an introductory text and the following four appended research papers, henceforth referred to as **Paper I**, **Paper II**, **Paper III**, and **Paper IV**.

I. J. Wright and I. Pázsit, *Derivation and analysis of the Feynman-alpha formula for deterministically pulsed sources*, CTH-RF-179, 2004, Chalmers University of Technology, Göteborg, Sweden


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Contents

1 Introduction 1

2 Reactivity Measurement Methods 3
  2.1 Feynman-alpha method ............................................. 3
  2.2 Rossi-alpha method .................................................. 4
  2.3 Source modulation method ......................................... 5
  2.4 Break frequency noise analysis method .......................... 5
  2.5 Californium-252 method .............................................. 6

3 Pulsed Feynman-alpha Formulae 9
  3.1 Feynman-alpha formula neglecting delayed neutrons ............. 9
    3.1.1 Deterministic pulsing ........................................ 11
    3.1.2 Stochastic pulsing ........................................... 14
  3.2 Feynman-alpha formula including delayed neutrons .............. 16
    3.2.1 Deterministic pulsing ....................................... 18
    3.2.2 Stochastic pulsing ........................................... 20

4 Source Modulation Method 21
  4.1 Kinetics of subcritical cores ..................................... 21
    4.1.1 Point reactor approximation .................................. 23
    4.1.2 Kinetics of subcritical cores with a source ............... 23
  4.2 Analysis of the performance of the source modulation method 25
  4.3 Possible methods for improving the performance ................ 28

5 Concluding remarks 31

Bibliography 33

Abbreviations 37

Paper I-IV 39
Introduction

Reactivity measurements are important both in power reactors as well as in future source-driven subcritical systems. Recently, the topic has received new actuality due to a few incidents such as in Tokaimura in 1999 [1–3] or at Dampierre in 2001 [4]. In Tokaimura a criticality accident occurred at a uranium conversion facility, when several times more than the specified limit of a solution of enriched uranium had been added in a precipitation tank until the critical mass was achieved, resulting in a self-sustaining nuclear fission chain reaction [3]. Unfortunately, three workers suffered acute radiation syndrome and several workers as well as members of the public received radiation doses. The incident in France happened during a routine refuelling of the Dampierre-4 PWR, where one fuel assembly was skipped by mistake when the work shift changed. When the error was noticed near the end of the refuelling sequence, the core configuration was quite different from that intended due to the fact that 113 fuel assemblies were incorrectly positioned [4]. Under unfavourable circumstances this could have resulted in prompt criticality.

The two examples above illustrate the renewed actuality of subcriticality measurements. In particular, there are two major areas of interest; core loading in power reactors and future Accelerator Driven Systems (ADS). Today there are no satisfactory methods for reactivity measurements and monitoring during core loading. In general, the available systems use a Start-up and Intermediate Monitors (SIRM) system that causes a reactor shut down or scram if the amplitude of the neutron flux or the increase rate of the flux becomes larger than empirical limits. If the core configuration is unfavourable, it can lead to prompt criticality. Under such circumstances the SIRM system is not enough and it is necessary to measure criticality during core loading to avoid a configuration that leads to an unexpected event. As is seen in the case at Dampierre it is possible to load a large number of fuel elements incorrectly. However, for a BWR only 30 misplaced fuel elements can lead to prompt criticality [5].

Recently, ADS has received increased interest, since it is a promising system for nuclear waste transmutation. An ADS is a non-self-sustaining subcritical reactor driven by a strong external neutron source. Some experimental programs are being performed to develop such systems [6–8]. As the experimental programs also underline, on-line monitoring of the subcritical reactivity is one of the central operational and safety issues of a future ADS. The reason is that the reactivity variations under no circumstances should lead neither to criticality nor delayed criticality. In the EU project MUSE, a large number of reactivity measurement methods applicable in zero or low power systems, such as the Feynman- and the Rossi-alpha methods, and pulse decay methods, including the area ratio method of Sjöstrand [9] were investigated. However, it is obvious that at the higher operating level of a power producing ADS, zero noise methods need to be replaced by other ones, just as in an operating critical reactor inverse kinetic methods are used.
for reactivity measurements instead of the Feynman-alpha or pulse decay methods.

There are several similarities between the subcriticality measurements during core loading and in ADS. To measure reactivity during core loading one needs an external source analogous to the driving source in ADS. In addition, since both systems are subcritical, their neutron kinetic characteristics are similar. Therefore, similar reactivity measurement techniques can be used for both cases.

The outline of this thesis is as follows: Chapter 2 is a continuation of the introduction with an overview of the common subcriticality measurement methods. In Chapter 3 a derivation of the Feynman-alpha formulae for pulsed sources is given. We treat the stochastic as well as the deterministic pulsing and even include delayed neutrons. Further, in Chapter 4 the so-called source modulation technique for reactivity measurement in subcritical cores is investigated. In Chapter 5, finally, concluding remarks are made.

The thesis also contains four appended papers. Paper I, Paper II, and Paper III deal with the pulsed Feynman-alpha formulae. Paper IV is an investigation of the source modulation method including a revision of the conditions of point kinetic behaviour in subcritical cores. Paper III also includes a derivation of pulsed Rossi-alpha formulae, omitted in this thesis.
Reactivity Measurement Methods

There is a number of methods for conventional subcriticality measurements. Here, a short description of the Feynman-alpha, the Rossi-alpha, the source modulation, the break frequency noise analysis, and the Cf-252 methods are given. Both the Feynman- and Rossi-alpha methods are based on neutron fluctuations and the measurement of the second moment of the statistics of the detector counts. The break frequency and the Cf-252 methods are based on measurements of auto-power and/or cross-power spectral densities. However, the basis of all these methods is that the statistical properties of the count sequence depend on the dynamic properties of the multiplying system. The methods that use a single detector are based on the assumption of point kinetic behaviour of the system.

2.1 Feynman-alpha method

The Feynman-alpha method, also called variance-to-mean method, is based on the measurement of the relative variance of the detected neutron counts as a function of the measurement time $t$ [10, 11]:

$$\frac{\sigma_z^2(t)}{\langle Z(t) \rangle} = 1 + \varepsilon \sum_{i=0}^{6} A_i \left( 1 - \frac{e^{-\alpha_i t}}{\alpha_i t} \right),$$

(2.1)

where $Z(t)$ is the detector counts, $\langle Z(t) \rangle$ its expectation value, and $\sigma_z^2(t)$ is the variance of $Z$. In practice, one uses an external neutron source and repeatedly detects the number of detector counts during a time interval $t$. The measurement interval is varied and for each interval the mean and the variance are determined.

Assume both an infinite reactor and detector, and a Poisson source, i.e. the probability to emit one neutron in $dt$ being $Sdt$. Then, using the master equation formalism for the probability distributions the Feynman-alpha formula is obtained [12] as

$$\frac{\sigma_z^2(t)}{\langle Z(t) \rangle} = 1 + \varepsilon \sum_{i=0}^{6} A_i \left( 1 - \frac{e^{-\alpha_i t}}{\alpha_i t} \right),$$

(2.2)

where $\varepsilon$ is the detection efficiency.

For the prompt part of the sum, i.e. $i = 0$, which includes the reactivity and is thus the most important part, the following relationships are given:

$$\alpha_0 \approx \frac{\beta - \rho}{\Lambda}, \quad \rho = \frac{k - 1}{k}, \quad \alpha_i \ll \alpha_0; \quad i \geq 1$$

(2.3)

and

$$A_0 = \frac{\langle \nu \nu - 1 \rangle}{\bar{\nu}^2} \frac{1}{(\beta - \rho)^2} = \frac{D_\nu}{(\beta - \rho)^2},$$

(2.4)
where $\alpha_0$ is the prompt neutron decay constant, $\beta$ is the delayed neutron fraction, $\rho$ the subcritical reactivity, and $\Lambda$ the prompt neutron generation time. Further, $\bar{\nu}$ is average total number of neutrons per fission, $\langle \nu(\nu - 1) \rangle$ is the second-order factorial moment of total number of neutrons generated per fission, and $D_{\nu}$ is the fission Diven factor, defined as

$$D_{\nu} = \frac{\nu(\nu - 1)}{\bar{\nu}^2}. \quad (2.5)$$

Obviously, the interesting information is contained in the deviation of the relative variance from unity, i.e. in the Feynman $Y$-function

$$Y(t) = \frac{\sigma_z^2(t)}{\langle Z(t) \rangle} - 1 = \frac{\mu_z(t)}{\langle Z(t) \rangle}, \quad (2.6)$$

where

$$\mu_z(t) = \sigma_z^2(t) - \langle Z(t) \rangle = \langle Z(t)(Z - 1) \rangle - \langle Z(t) \rangle^2 \quad (2.7)$$

is called the modified variance. The reason is that if all neutrons were statistically independent, as those emitted by a radioactive source, they would have Poisson statistics and the relative variance in eq. (2.2) would equal unity. In other words, we would not gain any information in the system. In a multiplying medium, however, each neutron induces a chain of correlated neutrons. Positive correlations lead to a variance higher than Poisson and it is the part that is larger than unity that contains the useful system information. In a subcritical reactor each individual chain will die out exponentially with the time constants $\alpha_i$, hence the relative variance will saturate.

The parameter $\alpha_0$ is obtained by curve fitting to experimental data and then the reactivity can be calculated using eqs. (2.3), which does not require knowledge of the detector efficiency or the source strength.

### 2.2 Rossi-alpha method

The Rossi-alpha method (Orndoff’s type), also called the correlation or covariance method, is based on the measurement of the covariance function of the detector counts in infinitesimal time intervals $dt$ around two different times $t$ and $t + \tau$ [13]. Assuming a Poisson source, i.e. the probability to emit one neutron in $dt$ being $Sdt$, the Rossi formula is often written as

$$P(\tau)d\tau = C_{zz}(\tau) = \frac{\langle Z(t + \tau)Z(t) \rangle - \langle Z(t + \tau) \rangle \langle Z(t) \rangle}{\langle Z(t) \rangle}, \quad (2.8)$$

where $C_{zz}(\tau)$ is the stationary value of the covariance function, $Z(t)$ and $Z(t + \tau)$ are the detector counts during $dt$ around $t$ and $t + \tau$, respectively, and $\langle Z(t) \rangle$ and $\langle Z(t + \tau) \rangle$ are the first moments of the stationary detector counts in $(t, dt)$ and $(t + \tau, dt)$, respectively. Because of the stationarity, the quantities depend not on $t$, but on $\tau$.

Using the master equations, an analytical expression for the covariance can be derived:

$$P(\tau)d\tau = \varepsilon d\tau \sum_{i=0}^{6} C_ie^{-\alpha_i\tau}, \quad (2.9)$$
where the \( C_i \) are similar to \( A_i \) and the other parameters are the same as in eq. (2.6). If all the neutrons were independent, the covariance would be zero. Since the neutrons generated by fission are correlated, the covariance becomes larger than zero. Similarly to the Feynman formula, the neutron chains will die out with the same time constants. Again, the parameter \( \alpha_0 \) and the reactivity is obtained by curve fitting to experimental data.

### 2.3 Source modulation method

The source modulation method is proposed by Carta and D’Angelo [14] for subcriticality monitoring in ADS. However, the method could be used during core loading as well. The idea of source modulation method is that by varying the amplitude of the external source neutron fluctuations are induced. Using the point kinetic equations, for such fluctuations the following relationship can be derived at plateau frequency

\[
\frac{\delta P(\omega)}{P_0} = \frac{\rho(\$)}{\rho(\$) - 1} \frac{\delta q(\omega)}{Q_0},
\]

where \( \delta P(\omega) \) and \( P_0 \) are the fluctuations and the mean value of the amplitude factor of the flux, respectively, \( \delta q(\omega) \) and \( Q_0 \) are the fluctuations and the mean value of the source, and \( \rho(\$) \) is the reactivity in dollars, which is defined as \( \rho(\$) = \rho / \beta \), where \( \rho \) is the reactivity and \( \beta \) is the delayed neutron fraction. The equation above is suggested to be used to a calibration-free determination of the reactivity.

This method is further investigated in Chapter 4.

### 2.4 Break frequency noise analysis method

The break frequency noise analysis method is based on the measurement of the auto power spectral density (APSD) from one detector or the cross-power spectral density (CPSD) from at least two detectors [15]. Plotting the APSD or CPSD as a function of frequency and least square fitting to the subcritical zero power transfer function, the fundamental mode break frequency, \( f_b \), is obtained. Figure 2.1 shows a typical transfer function for a subcritical system. The break frequency which equals \( (\beta - \rho) / \Lambda \) is indicated in the figure, where \( \Lambda \) is the neutron generation time, \( \beta \) is the delayed neutron fraction and \( \rho \) is the reactivity.

After curve fitting to experimental data, the effective multiplication factor, \( k_{eff} \) can be calculated from

\[
\frac{f_b}{f_{bdc}} = \frac{1 - k_{eff}}{\beta k_{eff}} + 1,
\]

where \( f_{bdc} \) is the break frequency at delayed criticality. Since the change from the delayed critical state to the current subcritical state changes the neutron lifetime and the effective delayed neutron fraction, eq. (2.11) needs to be corrected for the specific subcritical level. The break frequency at delayed criticality is obtained from a reference measurement.
2.5 Californium-252 method

The Californium-252 method, first suggested by Mihalczo [15,16] and hence also called Mihalczo method, is based on measurement of the spectral densities from two or three detectors. Here, the three detector technique is briefly described. One of the detectors, $D_1$, is an ionisation chamber including a Cf-252 source, and hence acting as a fission counter, while the two others, $D_2$ and $D_3$, are ordinary detectors. The Cf-252 detector is used as a random source of neutrons, while simultaneously detecting the fission products of the spontaneous fissions of Cf-252, thus providing a means to measure the emission time. The two other detectors are two regular neutron detectors. The detector with the source is usually placed centrally in the core. It is important that the two neutron detectors are at a sufficient distance from the Cf-252 detector, so that instead of directly detecting source neutrons, they detect neutrons from the fission chains generated by source neutrons. In addition, the two regular detectors have to be placed in a sufficient distance from each other so they detect neutrons from two different fission chains. One of the advantages of this method compared to the Feynman- or the Rossi-alpha methods is that without disturbing the fission chains by neutron detection, the initiators of the chains are detected with the Cf-252 detector.

The reactivity is obtained measuring the auto power spectral density (APSD), $G_{11}$, for the Cf-252 detector and the cross power spectral densities (CPSD), $G_{12}$, $G_{13}$, and $G_{23}$ for all the three detectors. Then, one obtains the effective multiplication factor, $k_{eff}$, by calculating the ratio of spectral densities as

$$
\frac{G_{12}G_{13}}{G_{11}G_{23}} = \frac{1 - k_{eff} \nu_c I_c}{k_{eff} \nu I_{RX}} \frac{1}{\nu I_{RX}},
$$

(2.12)

where the subscript $c$ denotes quantities that originate from the californium source and symbols without subscripts denote quantities that originate from the reactor, that is $\nu_c$.
and $\nu$ are the average number of neutrons per one fission in $^{252}\text{Cf}$ source and in $^{235}\text{U}$ in the core, respectively. $I_c$ is the importance of the source neutrons from $^{252}\text{Cf}$, $I$ is the core average importance of neutrons in the fission chain, $X$ is the Diven factor defined in eq. (2.5), and $R$ is a factor that corrects for deviations from point kinetics. In the equation above any inherent sources are neglected.

Since the method is efficient and applicable to measure even deep subcriticalities, $k_{eff} < 0.8$, it is suggested to be used in future ADS. Also, the method seems to be promising for reactivity measurements during core loading.

In the following two chapters some advances on the application of the Feynman-alpha method with a pulsed source are described and a detailed analysis of the source modulation method is given.
In this chapter an effective method for calculating the pulsed Feynman-alpha formula is described. In section 3.1 only prompt neutrons are included in the formula. The delayed neutrons are included in section 3.2. Detailed derivations are found in Paper II and Paper III.

One objective of the analysis was that although the pulsed Feynman method is not a strong candidate for measuring reactivity during core loading, however the mathematical treatment used will also be useful for the treatment of the Californium-252 method. In addition, the pulsed Feynman-alpha measurements have a clear relevance for future ADS.

### 3.1 Feynman-alpha formula neglecting delayed neutrons

For Feynman-alpha experiments with a pulsed source, two alternative pulsing methods have been developed, i.e. the deterministic and the stochastic one [17]. The deterministic method is performed in such a way that the neutron counting is synchronized with the pulsing of the neutron source. The synchronization is achieved by starting the neutron counting at the beginning of one pulse. For the stochastic method, such synchronization is not established.

The Feynman $Y$-formula for a pulsed source can be derived from a master equation. We use the so-called backward master equation, which means that a core with a stationary source is modelled by assuming that the source is switched on at $t = 0$, and the measurement is made when $t \to \infty$. The derivation is made in two steps. First, the probability of the number of neutrons in the system and the number of detector counts within a time period $(t - T, t)$ is determined as induced by one source neutron. Then, the probability for the same quantities for the case with an external source present is calculated from the earlier derived quantities.

The probability generating function for the probability, $P$, of finding $N$ neutrons at $t$ and $Z$ counts in the time interval $(t - T, t)$, due to one neutron starting the process at $t = 0$, can be derived directly from a mixed-type backward master equation for the probability generating function

$$G(x, z, t) \equiv \sum_{N=0}^{\infty} \sum_{Z=0}^{\infty} x^N z^Z P(N, Z, t).$$

(3.1)
The master equation used is

\[
\frac{\partial G(x, z, t)}{\partial t} = - (\lambda_c + \lambda_f + \lambda_d) G(x, z, t) + (\lambda_c + \lambda_d) \\
+ \lambda_f \sum_{n=0}^{\infty} p(n) G^n(x, z, t) \\
+ \lambda_d (z - 1) \Delta(T; t),
\]

where \(\lambda_c\) is the probability that one neutron is captured per unit time, \(\lambda_f\) the probability that one neutron undergoes a fission reaction per unit time, \(\lambda_d\) the probability that one neutron is detected per unit time by a neutron detector, and \(p(n)\) the probability that \(n\) prompt neutrons are emitted in one fission reaction. The function \(\Delta(T; t)\) in the last term of eq. (3.33) is defined as

\[
\Delta(T; t) = \begin{cases} 
1; & 0 < t \leq T \\
0; & T < t
\end{cases}
\]

and means that the detector registers the number of neutrons when \(\Delta(T; t)\) equals unity.

To connect the distribution, \(P\), of the single particle induced observables to the distribution \(\tilde{P}\) of the source induced observables, the so-called Bartlett formula for the generating functions is used:

\[
\tilde{G}(x, z, t) = \exp \left[ \int_0^t S(t') \{ G(x, z, t - t') - 1 \} \, dt' \right].
\]

The above formula was derived using another master equation and it has been known for a long time. However, the name is misleading since the formula was derived by Sevast’yanov [18]. Observe that the measurement time \(T\) is not denoted above and that it is the asymptotic form, when \(t \to \infty\), that is considered. However, with a non-stationary source like a pulsed one, the upper limit of the integration cannot directly be extended to infinity. Instead it is done when the exact form of the source is defined.

For stochastic pulsing we need a generalization of the Bartlett formula, as described in [19]. This is made by the introduction of a random variable \(\xi \in [0, T_0)\):

\[
\tilde{G}(x, z, t) = \int_0^{T_0} \tilde{G}(x, z, t | \xi) p(\xi) \, d\xi \equiv \left\langle \tilde{G}(x, z, t | \xi) \right\rangle,
\]

where

\[
\tilde{G}(x, z, t | \xi) = \exp \left[ \int_0^t S(t' | \xi) \{ G(x, z, t - t') - 1 \} \, dt' \right],
\]

\(S(t' | \xi)\) is the “randomization” of the pulse, and the \(\langle \cdots \rangle\) symbol denotes for the expectation value w.r.t. \(\xi\).

The source is analytically expressed as a periodic sum:

\[
S(t | \xi) = S_0 \sum_{n=0}^{\infty} f(t - nT_0 - \xi),
\]
where \( f(t) \) is the pulse shape, defined as:

\[
f(t) = \begin{cases} 
\geq 0; & 0 \leq t < T_0, \\
0; & \text{otherwise},
\end{cases}
\]  

(3.8)

which is assumed to be normalized and the intensity is described by a factor \( S_0 \).

For deriving the Feynman-alpha formula the asymptotic (i.e. \( t \to \infty \)) first and second factorial moments of the source induced distributions, i.e. \( \tilde{N}(t | \xi), \tilde{Z}(t, T | \xi), \) \( \tilde{M}_Z(t, T | \xi) \) and/or \( \langle \tilde{\mu}_Z(t, T | \xi) \rangle \), are needed. Integrating the corresponding factorial moments of the single particle induced distributions, i.e. \( \langle N(t) \rangle, \langle Z(t) \rangle \) and using eq.(3.5), equations for the source induced quantities are obtained. The factorial moments are obtained as first or second order derivatives of the generating function w.r.t. \( z \) with \( x = z = 1 \). For example the first, single particle induced moment is:

\[
\langle N(t) \rangle = \frac{\partial G(x, z, t)}{\partial x} \bigg|_{x=z=1} \equiv N(t) = e^{-\alpha t},
\]  

(3.9)

where \( \alpha \) is the prompt neutron decay constant, given by \(-\rho/\Lambda\). Here \( \rho \) is the subcritical reactivity (to be determined in the measurement) and \( \Lambda \) the prompt neutron generation time. Expressions for all other factorial moments used are found in \( \text{Paper II} \).

All equations for the source-induced moments include integrals that need to be evaluated. Using Laplace transform of \( \tilde{N}(t | \xi) \), one can obtain a Fourier series expansion for \( \tilde{N}(t | \xi) \), after inverse Laplace transform. The solution can be given for arbitrary pulse shapes, since the pulse shape only affects certain terms of the Fourier expansion. This is due to the fact that the pulse shape only affects the numerical values of the residues, but not the residue structure. By integrating termwise and letting \( t \to \infty \), \( \tilde{Z}(t | \xi) \) is obtained.

When the source induced distributions, \( \tilde{N}(t | \xi), \tilde{Z}(t, T | \xi), \) and \( \langle \tilde{\mu}_Z(t, T | \xi) \rangle \), are calculated, the expectation values of the quantities w.r.t. \( \xi \) need to be taken.

### 3.1.1 Deterministic pulsing

The treatment of the pulsing methods means taking the expectation values of the expressions for the first and second moments of the detector count w.r.t. the random variable \( \xi \). For the deterministic pulsing the probability distribution of \( \xi \) is given as

\[
p(\xi) = \delta(\xi),
\]  

(3.10)

with

\[
t = kT_0 + T; \quad k = 0, 1, \cdots.
\]  

(3.11)

From (3.5) and (3.10) it is seen that the resulting deterministic formulae are obtained by substituting \( \xi = 0 \) into the relevant general formulae, eqs. (3.6) and (3.7). The resulting mean and the modified variance are given by (for simplicity, the expectation value sign, indicating the averaging w.r.t. \( \xi \), will be omitted):

\[
\tilde{Z}(T) = S_0\lambda \left[ a_0T + \sum_{n=1}^{\infty} \frac{a_n \{ 1 - \cos (\omega_n T) \} + b_n \sin (\omega_n T)}{\omega_n} \right],
\]  

(3.12)
\[
\tilde{\mu}_Z (T) = S_0 \lambda_0^2 \lambda_f \langle \nu (\nu - 1) \rangle \times \left[ a_0 T \left( 1 - \frac{1 - e^{-\alpha T}}{\alpha T} \right) + \sum_{n=1}^{\infty} \left\{ a_n A_n (T) + b_n B_n (T) \right\} \right] + \left( 1 - e^{-\alpha T} \right)^2 \sum_{n=1}^{\infty} \frac{a_n (-\omega_n) + b_n (2\alpha)}{\omega_n^2 + (2\alpha)^2}.
\] (3.13)

In the equations above the following notation is used:

\[\omega_n \equiv \frac{2n\pi}{T_0}; \quad n = 1, 2, \ldots, \] (3.14)

\[a_0 = \frac{f (s = 0)}{\alpha T_0}, \] (3.15)

\[a_n = \frac{2 \omega_n \Re \{ f (s = i\omega_n) \} - \alpha \Im \{ f (s = i\omega_n) \}}{\omega_n^2 + \alpha^2}, \] (3.16)

\[b_n = \frac{2 \alpha \Re \{ f (s = i\omega_n) \} + \omega_n \Im \{ f (s = i\omega_n) \}}{\omega_n^2 + \alpha^2}, \] (3.17)

\[A_n (T) \equiv p_n (0, T) - 2p_n (\alpha, T) + p_n (2\alpha, T), \] (3.18)

\[p_n (\alpha, T) \equiv e^{-\alpha T} \int_0^T e^{\alpha t} \sin (\omega_n t) \, dt \]

\[= \frac{\alpha \sin (\omega_n T) + \omega_n \{ e^{-\alpha T} - \cos (\omega_n T) \}}{\omega_n^2 + \alpha^2}, \] (3.19)

\[B_n (T) \equiv q_n (0, T) - 2q_n (\alpha, T) + q_n (2\alpha, T), \] (3.20)

\[q_n (\alpha, T) \equiv e^{-\alpha T} \int_0^T e^{\alpha t} \cos (\omega_n t) \, dt \]

\[= \frac{\omega_n \sin (\omega_n T) - \alpha \{ e^{-\alpha T} - \cos (\omega_n T) \}}{\omega_n^2 + \alpha^2}. \] (3.21)

In eq. 3.15 the \(f (s = 0)\) is the Laplace transform of the pulse shape function in eq. (3.8). It is, however, not a full Laplace transform, but defined as

\[f (s) \equiv \int_0^{T_0} f (t) e^{-st} \, dt, \] (3.22)
The deterministic pulsed Feynman $Y(T)$ function has similarities with the case with continuous sources. First, since the Feynman $Y(T)$ function is given as $Y(T) = \tilde{\mu}_Z(T)/\tilde{Z}(T)$, the source strength $S_0$ cancels out from the asymptotic formula. Second, both $\tilde{Z}(T)$ and $\tilde{\mu}_Z(T)$ are linearly dependent on the measurement time $T$, hence the $Y(T)$ curve goes into saturation. In addition, the formulae above also show that the oscillating deviations from the continuous Feynman-curve asymptotically approach zero with increasing time gate $T$. It is also worth mentioning that the relative weight of the oscillations depends on the pulse angular frequency and pulse width as follows; for high pulse repetition frequency, the oscillations are small, furthermore, with increasing pulse width $W$ the relative magnitude of the oscillations decreases. For short pulses with low pulse frequency, the deviations from the smooth $Y(T)$ become quite significant.

Equations (3.15)–(3.17) are valid for any general pulse form $f(t)$, and the different pulse shapes affect only $a_0$, $a_n$, and $b_n$, which is the major advantage of the solution method. The particular cases of square and Gaussian pulses can be then obtained by substituting appropriate expressions for $a_0$, $a_n$, and $b_n$, into (3.12)–(3.13), see PAPER II. For the case with square pulses the pulse form is described by

$$f(t) = H(t) - H(t - W),$$  \hspace{1cm} (3.23)

and the sequence of square pulses by

$$S(t) = S_0 \sum_{n=0}^{\infty} \{H(t - nT_0) - H(t - nT_0 - W)\},$$  \hspace{1cm} (3.24)

where $H$ is Heaviside’s step function, $T_0$ the pulse period, and $W$ the pulse width. In that case, eq.(3.22) becomes

$$f(s) = \frac{1 - e^{-sW}}{s}. \hspace{1cm} (3.25)$$

Inserting this into (3.15)–(3.17) will yield the following results for $a_0$, $a_n$ and $b_n$:

$$a_0 = \frac{W}{\alpha T_0}, \hspace{1cm} (3.26)$$

$$a_n = \frac{\omega_n \sin(\omega_n W) + \alpha \{1 - \cos(\omega_n W)\}}{n\pi (\omega_n^2 + \alpha^2)}, \hspace{1cm} (3.27)$$

and

$$b_n = \frac{\alpha \sin(\omega_n W) - \omega_n \{1 - \cos(\omega_n W)\}}{n\pi (\omega_n^2 + \alpha^2)}. \hspace{1cm} (3.28)$$

The method has been used for calculation of the $\alpha$-value from pulsed Feynman-alpha measurements at the KUCA reactor (Kyoto University Critical Assembly) [20]. The measurements were made with various subcriticality levels, by various pulse repetition frequencies (or periods), and pulse widths. In Fig. 3.1.1 the measurement results for four subcritical reactivities, $0.65$&, $1.30$&, $2.07$&, and $2.72$& are shown, for the case of $T_0 = 20$ ms. The experimentally determined $Y$ values are shown as symbols together with the experimental errors, and the fitted curves are shown as continuous lines. The
reference alpha values and the ones obtained from the curve fitting are shown in all four sub-figures.

It is seen that for the deep subcriticalities, the fitted values agree with the reference values quite well. Correspondingly, the fitted $Y(T)$ curves agree well, but note completely, with the measurements. When the system is near criticality, the alpha values and the curves obtained by curve fitting deviate significantly from the reference alpha value and the measured curve. The reason for this deviation is not yet understood.

Figure 3.1: Measured and fitted results for the deterministic pulsing.

### 3.1.2 Stochastic pulsing

For the stochastic pulsing the probability distribution of $\xi$ is:

$$p(\xi) = \frac{1}{T_0^3},$$

(3.29)

which means that we need to integrate over the pulse period, $T_0$, w.r.t. $\xi$ to obtain the expectation values w.r.t. $\xi$. However, it turns out that the oscillating parts in the
expected value of the detector counts disappears, resulting in an particularly simple smooth function for the case of stochastic pulsing, i.e.

$$\tilde{Z}(T) = S_0 \lambda d a_0 T.$$  \hspace{1cm} (3.30)

Therefore it is practical to give directly the $Y(T)$ function. It will have the relatively simple form

$$Y(T) = \frac{\lambda d \lambda t (\nu (\nu - 1))}{\alpha^2} \left( 1 - \frac{1 - e^{-\alpha T}}{\alpha T} \right) + \frac{2 S_0 \lambda d}{a_0 T} \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{\omega_n^2} \sin^2 \left( \frac{\omega_n}{2} T \right),$$  \hspace{1cm} (3.31)

where $a_0$, $a_n$, and $b_n$ are given by eqs. (3.15)–(3.17).

One significant difference between the stochastic and the deterministic case is the relative weight of the oscillating part to the smooth part. Most notably, for the stochastic case the oscillating part is linear in the source strength, i.e. the source strength does not disappear from the relative variance. This is a clear consequence of the “randomization” of the pulse, which leads to a qualitatively different dispersion of the source neutrons, as remarked in [17]. The properties of the dispersion affect the statistics of the neutron chain and that of the detector counts. Another difference is that the Diven factor of fission is absent in the oscillating part in eq. (3.31), which means that the oscillating part is controlled by the source statistics instead of the statistics of the multiplication of the fission chain. In addition, with a strong source, the relative oscillations become large.

The relative weight of the oscillating part in eq. (3.31) will increase much faster with increasing subcriticalities than in the case of deterministic pulsing. The reason is that the smooth part of is proportional to $1/\alpha$, whereas the oscillating part, through the factor $1/a_0$, is proportional to $\alpha$. Again, this seems to depend on the fact that the oscillating part is influenced mostly by the source properties, whose significance increases in deep subcritical systems.

Similar to the deterministic case, the different pulse shapes affect only $a_0$, $a_n$, and $b_n$. The particular cases of square and Gaussian pulses are obtained by substituting appropriate expressions for $a_0$, $a_n$, and $b_n$, into (3.31), as shown in PAPER II. For the specific case of square pulses, the formula has already been calculated by Ceder and Pázsit [21], although the solution method was less general and explicitly utilised the properties of the square pulse.

Also the stochastic method was tested for alpha parameter calculation from the same measurements at the KUCA [20]. In Fig. 3.1.2 the measurement results for four subcritical reactivities, 0.65$, 1.30$, 2.07$, and 2.72$, are shown, for the case of $T_0 = 20$ ms. The experimentally determined $Y$ values are shown as symbols together with the experimental errors, and the fitted curves are shown as continuous lines. The reference alpha values and the ones obtained from the curve fitting are shown in all four sub-figures. For all subcriticalities, both the fitted alpha values and the curves agree with the reference alpha values and the measured curves rather well. Based on these results it is recommended to prefer the stochastic pulsed method over the deterministic one in practical applications.
3.2 Feynman-alpha formula including delayed neutrons

Inclusion of one group of delayed neutrons means that the probability distributions and
the master equations have to be modified. The generating function becomes:

\[ G(x, y, z; T; t) = \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z=0}^{\infty} x^N y^C z^Z P(N, C, Z; T; t) , \]

(3.32)

where \( P(N, C, Z; T; t) \) is the probability of finding \( N \) neutrons and \( C \) delayed neutron
precursors at time \( t \) in the system and detecting \( Z \) counts in the time interval \([t - T, t)\),
due to one initial neutron injected at time \( t = 0 \). The master equation for the generating
function is
\[
\frac{\partial G(x, y, z, T; t)}{\partial t} = -(\lambda_c + \lambda_f + \lambda_d) G(x, y, z, T; t) + (\lambda_c + \lambda_d)
\]
\[
+ \lambda_f \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p(n, m) G^n(x, y, z, T; t)
\]
\[
\times \left\{ \lambda \int_0^t dt' G(x, y, z, T; t') e^{-\lambda(t-t')} + ye^{-\lambda t} \right\}^m
\]
\[
+ \lambda_d (z - 1) \Delta(T; t),
\]
where \(\lambda_c\) is the probability that one neutron is captured per unit time, \(\lambda_f\) the probability that one neutron undergoes a fission reaction per unit time, \(\lambda_d\) the probability that one neutron is detected per unit time by a neutron detector, \(\lambda\) the delayed neutron decay constant, and \(p(n, m)\) the probability that \(n\) prompt neutrons and \(m\) delayed neutron precursors are emitted in one fission reaction [22]. The function \(\Delta(T; t)\) in the last term of eq. (3.33) is defined as in eq. (3.3).

The generalized so-called Bartlett formula, that connects the single-particle induced probability generating function with the source induced ones is unaffected by the inclusion of delayed neutrons, because the source does not emit delayed neutrons. As earlier the Bartlett formula is given by
\[
\tilde{G}(x, y, z, T; t) = \int_0^{T_0} d\xi p(\xi) \tilde{G}(x, y, z, T; t | \xi),
\]
with
\[
\tilde{G}(x, y, z, T; t | \xi) = \exp \left[ \int_0^t dt' S(t' | \xi) \{ G(x, y, z, T; t - t') - 1 \} \right].
\]
where \(S(t)\) is the intensity of the external neutron source, defined as in eq. (3.7) and the generating function \(\tilde{G}(x, y, z, T; t)\) is defined as
\[
\tilde{G}(x, y, z, T; t) \equiv \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z=0}^{\infty} x^N y^C z^Z \tilde{P}(N, C, Z, T; t),
\]
where \(\tilde{P}(N, C, Z, T; t)\) is the probability of having \(N\) neutrons and \(C\) delayed neutron precursors in the system at time \(t\) and \(Z\) counts in the time interval \([t - T, t]\), due to an extraneous neutron source switched on at time \(t = 0\).

As for the case without delayed neutrons, single-particle induced moments are calculated by differentiating \(G(x, y, z, T; t)\) w.r.t. \(z\) and then substituting \(x = y = z = 1\). However, when delayed neutrons are included, the single particle neutron number is now obtained on the form
\[
N(t) \equiv \sum_{j=p,d} A_j e^{-\alpha_j t},
\]
where the prompt and delayed time parameters are given by
\[
\alpha_p \equiv \frac{(\alpha + \lambda) \Lambda + \sqrt{(\alpha + \lambda)^2 \Lambda^2 + 4\alpha \Lambda \rho}}{2\Lambda}
\]
and
\[ \alpha_d \equiv \frac{(\alpha + \lambda) \Lambda - \sqrt{(\alpha + \lambda)^2 \Lambda^2 + 4\Lambda \lambda \rho}}{2\Lambda}, \]
respectively. Furthermore, one has
\[ A_j \equiv \frac{\alpha_j - \lambda}{\alpha_j - \alpha_j'}; \quad j, j' = p, d; \quad j' \neq j. \]

Observe that the expected neutron number contains two terms instead of only one as when delayed neutrons are neglected, see eq. (3.9). Laplace transform of the time-derivative of \( Z(T; t) \) gives the single-particle induced neutron number and detector counts. Then, using the generalization of the Bartlett formula, the source-induced moments are derived using Laplace transform once more. The results for deterministic and stochastic pulsing are found below. Detailed calculations are found in Paper III.

### 3.2.1 Deterministic pulsing

The probability distribution of \( \xi \) is given in eq. (3.10). So, in the case with deterministic pulsing one obtains:

\[
Y(T) = \frac{\lambda_d S_0 A f(s = 0)}{Z(T) (-\rho) T_0} T \sum_{j=p,d} Y_j \left( 1 - \frac{1 - e^{-\alpha_j T}}{\alpha_j T} \right) \\
+ \frac{\lambda_d S_0}{Z(T)} \sum_{n=1}^{\infty} \left\{ a_n A_n(T) + b_n B_n(T) \right\} \\
+ \frac{\lambda_d S_0}{Z(T)} \sum_{n=1}^{\infty} \left\{ \sum_{j=p,d} \sum_{k=p,d} \mathcal{Y}_{j,k}^{(2)} \left( 1 - e^{-\alpha_j T} \right) \left( 1 - e^{-\alpha_k T} \right) \frac{a_n (-\omega_n) + b_n (\alpha_j + \alpha_k)}{(\alpha_j + \alpha_k)^2 + \omega_n^2} \right\},
\]

where \( \omega_n \) is defined in eq. (3.14) as in the case without delayed neutrons, and \( \tilde{Z}(T) \) is defined as
\[
\tilde{Z}(T) = \frac{\lambda_d S_0 A f(s = 0)}{(-\rho) T_0} T + \lambda_d S_0 \sum_{n=1}^{\infty} \mathcal{C}_n(T),
\]
where
\[
\mathcal{C}_n(T) \equiv \lim_{k \to \infty} \mathcal{C}_n(T; k T_0 + T) = \frac{a_n \{ 1 - \cos (\omega_n T) \} + b_n \sin (\omega_n T)}{\omega_n}.
\]

Here
\[
a_n \equiv \frac{2}{T_0} \sum_{j=p,d} A_j \omega_n \Re \left\{ f(s = i \omega_n) \right\} - \alpha_j \Im \left\{ f(s = i \omega_n) \right\} \left( \omega_n^2 + \alpha_j^2 \right) ; \quad n = 1, 2, \ldots ,
\]
and
\[
b_n \equiv \frac{2}{T_0} \sum_{j=p,d} A_j \frac{\omega_n \Re \left\{ f(s = i \omega_n) \right\} + \alpha_j \Im \left\{ f(s = i \omega_n) \right\}}{\omega_n^2 + \alpha_j^2} ; \quad n = 1, 2, \ldots ,
\]
with $A_j$ being the same as in eq. (3.40). Here, both $a_n$ and $b_n$ contain one prompt and one delayed term. Neglecting prompt-delayed and delayed-delayed neutron correlations one obtains:

$$Y_j \equiv \frac{\lambda d \lambda f \langle \nu_p (\nu_p - 1) \rangle}{(\alpha_j + \alpha_{j'}) (\alpha_j - \alpha_{j'})} \left( 1 - \frac{\lambda^2}{\alpha_j^2} \right); \quad j, j' = p, d; \quad j' \neq j,$$

(3.46)

$$A_n (T) \equiv Y^{(0)} p_n (0, T) - 2 \sum_{j=p,d} Y^{(1)}_j p_n (\alpha_j, T) + \sum_{j=p,d} \sum_{k=p,d} Y^{(2)}_{j,k} p_n (\alpha_j + \alpha_k, T),$$

(3.47)

$$B_n (T) \equiv Y^{(0)} q_n (0, T) - 2 \sum_{j=p,d} Y^{(1)}_j q_n (\alpha_j, T) + \sum_{j=p,d} \sum_{k=p,d} Y^{(2)}_{j,k} q_n (\alpha_j + \alpha_k, T),$$

(3.48)

$$Y^{(0)} \equiv \frac{\lambda d \lambda f \langle \nu_p (\nu_p - 1) \rangle}{\alpha_p \alpha_d} \left( \frac{\lambda^2}{\alpha_p \alpha_d} \right),$$

(3.49)

$$Y^{(1)}_j \equiv \frac{\lambda d \lambda f \langle \nu_p (\nu_p - 1) \rangle}{\alpha_{j'} (\alpha_j - \alpha_{j'})} \left( \frac{\lambda}{\alpha_j} - \frac{\lambda^2}{\alpha_j^2} \right); \quad j, j' = p, d; \quad j' \neq j,$$

(3.50)

$$Y^{(2)}_{j,k} \equiv \frac{\lambda d \lambda f \langle \nu_p (\nu_p - 1) \rangle}{(\alpha_j - \alpha_{j'}) (\alpha_k - \alpha_{k'})} \left[ 1 - \left( \frac{\lambda}{\alpha_j} + \frac{\lambda}{\alpha_k} \right) + \frac{\lambda^2}{\alpha_j \alpha_k} \right]; \quad j, j' = p, d; \quad j' \neq j; \quad k, k' = p, d; \quad k' \neq k,$$

(3.51)

$$p_n (\alpha, T) \equiv \frac{\alpha \sin (\omega_n T) + \omega_n \left\{ e^{-\alpha T} - \cos (\omega_n T) \right\}}{\alpha^2 + \omega_n^2},$$

(3.52)

and

$$q_n (\alpha, T) \equiv \frac{\omega_n \sin (\omega_n T) - \alpha \left\{ e^{-\alpha T} - \cos (\omega_n T) \right\}}{\alpha^2 + \omega_n^2}.$$

(3.53)

As in the cases with only prompt neutrons, the pulse form affects only $a_n$ and $b_n$. Equations (3.44) and (3.45) above are valid for any pulse shape $f(t)$. The particular cases of square and Gaussian pulses are obtained by substituting appropriate expressions for $a_n$ and $b_n$ instead of the general ones above, into (3.41). More details are found in Paper III.

The deterministic pulsed Feynman $Y$-formula in eq. (3.41) shows a clear similarity to the case with only prompt neutrons. Also, the generalization to inclusion of delayed neutrons is easily seen. Even the traditional Feynman $Y$-function corresponding to a static source and including one group of delayed neutrons is recognized in the first term in eq. (3.41).
3.2.2 Stochastic pulsing

The probability distribution of $\xi$ is defined in eq. (3.29). So, in the case with stochastic pulsing one obtains:

$$Y(T) = \sum_{j=p,d} Y_j \left( 1 - \frac{1 - e^{-\alpha_j T}}{\alpha_j T} \right) + \frac{2\lambda_d S_0 (-\rho) T_0}{\Lambda f(s = 0) T} \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{\omega_n^2} \sin^2 \left( \frac{\omega_n T}{2} \right), \quad (3.54)$$

where $Y_j$, $a_n$, and $b_n$ is given in eqs. (3.46), (3.44) and (3.45).

As earlier, the pulse form affects only $a_n$ and $b_n$. Equations (3.44) and (3.45) in section 3.2.1 are valid for any pulse form $f(t)$. The particular cases of square and Gaussian pulses are obtained by substituting appropriate expressions for $a_n$ and $b_n$ instead of the general ones above, into (3.54). Detailed calculations are found in Paper III.

As in the deterministic case, the derived Feynman $Y$-formula in eq. (3.54) shows a clear similarity to the case with only prompt neutrons. Also, the generalization to inclusion of delayed neutrons is easily seen. Even the traditional Feynman $Y$-function corresponding to a static source and including one group of delayed neutrons is recognized in the first term in eq. (3.54).
4
Source Modulation Method

In the previous chapter we described the Feynman-alpha method for subcritical measurements. In the present chapter we continue with the source modulation method. Since the method is based on the point reactor approximation, in section 4.1 the kinetics of subcritical cores with a source is briefly revisited. Then, in section 4.2 we continue with a short derivation of the proposed formula, analyse its performance, and briefly describe a possible improvement of the method. Detailed derivations are found in Paper IV.

4.1 Kinetics of subcritical cores

In this section, the same one-dimensional bare reactor model with one group of delayed neutrons as in [23] is used, but with slight change of the notation for better clarity for the case of the subcritical kinetics. There is a need to distinguish between three different quantities; the static (and now subcritical) equilibrium state, the time-dependent quantities, and also the fundamental flux or adjoint, with corresponding buckling, which belongs to a hypothetical critical system. Therefore, we denote the static core quantities, belonging to the subcritical case, by a subscript \( \rho \); the fundamental mode eigenfunction with a subscript zero, and the time dependent quantities have no subscript. The source, which is unaffected by the reactivity of the system, is excepted from the above convention and has a static value denoted by a zero subscript and a time-dependent deviation.

In a one-dimensional homogeneous one-group diffusion model the static equation with a steady source reads as

\[
D \nabla^2 \phi_{\rho}(x) + (\nu \Sigma_f - \Sigma_a) \phi_{\rho}(x) + S_0(x) = 0.
\] (4.1)

The usual diffusion theory boundary condition is given as

\[
\phi_{\rho}(x_B) = 0,
\] (4.2)

where \( x_B \) is any of the two boundaries of the system. The other symbols have their usual meaning. The system is assumed to be subcritical, i.e. \( k_{\text{eff}} < 1 \), and the subcriticality can be calculated from the eigenvalue equation for the critical flux.

Solving equation (4.1) with Green’s function technique gives

\[
\phi_{\rho}(x) = \int G(x, x') S_0(x') \, dx',
\] (4.3)
where
\[ G(x, x') = \begin{cases} 
\frac{\sin[B_\rho(a - x')] \sin[B_\rho(a + x)]}{DB_\rho \sin(2B_\rho a)} & x \leq x' \\
\frac{\sin[B_\rho(a + x')] \sin[B_\rho(a - x)]}{DB_\rho \sin(2B_\rho a)} & x > x' 
\end{cases} \] \tag{4.4}

with \( B_\rho \) being defined as
\[ B^2_\rho = \frac{\nu \Sigma_f - \Sigma_a}{D}. \] \tag{4.5}

\( G(x, x') \) in eq. (4.4) is the solution to the corresponding static Green’s function equation
\[ \nabla^2 G(x, x') + B_\rho G(x, x') + \frac{1}{D} \delta(x - x') = 0. \] \tag{4.6}

Now, assume that the fluctuations of the flux are induced by the temporal and spatial variations of the external source. We use the one-dimensional space-time dependent diffusion equations:
\[ \frac{1}{v} \frac{\partial \phi_\rho(x, t)}{\partial t} = D \nabla^2 \phi_\rho(x, t) + [(1 - \beta)\nu \Sigma_f - \Sigma_a] \phi_\rho(x, t) + \lambda C(x, t) + S(x, t) \] \[ \frac{\partial C(t)}{\partial t} = \beta \nu \Sigma_f \phi_\rho(x, t) - \lambda C(x, t) \] \tag{4.7}

with the boundary conditions
\[ \phi_\rho(x_B, t) = C(x_B, t) = 0. \] \tag{4.8}

Inserting the time-dependent quantities split up into stationary values and fluctuations in eq. (4.7), subtracting the static equations and eliminating the fluctuations of delayed neutrons by a temporal Fourier-transform, for the neutron noise in frequency domain following equation is obtained:
\[ \nabla^2 \delta \phi(x, \omega) + B^2_\rho(\omega) \delta \phi(x, \omega) + \frac{1}{D} \delta S(x, \omega) = 0, \] \tag{4.9}

where
\[ B^2_\rho(\omega) = B^2_\rho \left(1 - \frac{1}{\rho_\infty G_0(\omega)}\right). \] \tag{4.10}

\( G_0(\omega) \) is the usual zero transfer function, given by
\[ G_0(\omega) = \frac{1}{i\omega \left[\Lambda + \frac{\beta}{i\omega + \lambda}\right]}, \] \tag{4.11}

which is similar to the critical zero reactor transfer function, but with material parameters of the actual subcritical system. As in the critical case, \( G_0(\omega) \) diverges for vanishing frequencies.

Equation (4.9) can also be solved using the corresponding Green’s function equation. The solution of eq. (4.9) is given as
\[ \delta \phi(x, \omega) = \int G(x, x', \omega) \delta S(x', \omega) \, dx', \] \tag{4.12}
where $G(x, x', \omega)$ is the dynamic Green’s function

$$ G(x, x', \omega) = \begin{cases} 
\frac{\sin[B(\omega)(a - x')] \sin[B(\omega)(a + x)]}{DB(\omega) \sin(2B(\omega)a)} & x \leq x' \\
\frac{\sin[B(\omega)(a + x')] \sin[B(\omega)(a - x)]}{DB(\omega) \sin(2B(\omega)a)} & x > x'.
\end{cases} \quad (4.13) $$

### 4.1.1 Point reactor approximation

In general, the reactor kinetic approximations are based on a factorisation of the space-time dependent flux into an amplitude factor and a shape function. In the one-dimensional case we have:

$$ \phi(x, t) = P(t) \psi(x, t), \quad (4.14) $$

where $P(t)$ is as usual the amplitude function and $\psi(x, t)$ the shape function. For the subcritical system we use the same definition of the point reactor approximation as is used by Pázsit and Arzhanov [23], and in Paper IV, i.e.

$$ \phi(x, t) = P(t) \phi_p(x), \quad \text{or} \quad \psi(x, t) = \phi_p(x) \quad \forall \ t \quad (4.15) $$

where $\phi_p(x)$ is the static subcritical flux.

Linearising by splitting up all the time-dependent quantities into mean values and fluctuations and then inserting the linearised quantities in eq. (4.14) and neglecting the second order terms gives

$$ \delta \phi(x, t) = \phi(x) \delta P(t) + \delta \psi(x, t). \quad (4.16) $$

Using the point kinetic equations for the fluctuations, the solution in frequency domain is

$$ \delta P(\omega) = \Lambda G_P(\omega) \delta q(\omega), \quad (4.17) $$

and the point kinetic approximation gives

$$ \delta \phi_{pk}(x, \omega) = \Lambda G_P(\omega) \delta q(\omega) \phi_p(x), \quad (4.18) $$

where $G_P(\omega)$ is the zero-transfer function of the subcritical system:

$$ G_P(\omega) = \frac{1}{i \omega \left[ \Lambda + \frac{\beta}{i \omega + \lambda} \right] - \rho}. \quad (4.19) $$

The point kinetic solution (4.18) does not diverge for $\omega \to 0$, as it is expected, due to the appearance of the reactivity in the denominator of (4.19).

### 4.1.2 Kinetics of subcritical cores with a source

For the source modulation method it is interesting to investigate the performance of the point reactor approximation for a variable strength source. In general, the validity of the point kinetic approximation depends on the frequency of the perturbation and
the system size. For source driven subcritical cores, as was pointed out earlier in [23],
the spatial and temporal variation of the source fluctuation also affect the validity. In
particular, to get the point kinetic behaviour to dominate for low frequencies or small
system sizes, it is necessary that the time and space dependence of the source fluctuations
is factorised into an arbitrary time function and a spatial dependence which is identical
with that of the static flux. This condition is fulfilled with the source modulation
technique.

In Paper IV, both small and large systems were considered and the system sizes
were chosen so that they would count as small and large if they were critical. We
used a one-dimensional bare homogeneous system with boundaries at \( x = \pm a \), which is
described by the material parameters \( D, \Sigma_a, \nu \Sigma_f, \beta, \lambda \). The two investigated sizes are
\( a = 30 \text{ cm} \) and \( a = 150 \text{ cm} \). The corresponding reactivity values for the critical systems
are \( \rho_\infty(\$) = 3.401 \) and \( \rho_\infty(\$) = 0.136 \), respectively. The cross sections data were taken,
with modifications, from Garis et al. [24]. The desired subcriticality was obtained by
changing \( k_{eff} \) via the macroscopic fission cross section. Since this procedure will not
change the ratio of the leakage rate and the absorption rate significantly, these systems
can still be regarded as small and large, respectively, in the subcritical state.

For a time-dependent source the source fluctuations are described by
\[
S(x,t) = S(t) \delta(x - x_p) = [S_0 + \delta S(t)] \delta(x - x_p),
\]
(4.20)
where \( x_p \) is the position of the beam impact point. In this case the exact dynamic
transfer function for the neutron noise is
\[
\frac{\delta \phi(x,\omega)}{\delta S(\omega)} = G(x,x_p,\omega),
\]
(4.21)
where \( \delta S(\omega) \) is the Fourier transform of \( \delta S(t) \). In this case, using eq. (4.18) the point
kinetic transfer function becomes
\[
\frac{\delta \phi_{pk}(x,\omega)}{\delta S(\omega)} = \Lambda v G_\rho(\omega) \phi^1_0(x_p) \phi_\rho(x),
\]
(4.22)
where the adjoint critical flux is defined as
\[
\phi^1_0(x) = C \cos(B_0 x),
\]
(4.23)
where \( C \) is an arbitrary factor, that is determined from a normalisation condition, and
\[
B_0^2 \equiv \frac{\nu \Sigma_f / k_{eff} - \Sigma_a}{D} = \left( \frac{\pi}{2a} \right)^2.
\]
The point kinetic behaviour occurs when \( B_0^2(\omega) \approx B_0^2 \). This means that \( | \rho_\infty G_0(\omega) | \)
has to be much larger than 1, which is obviously the case when \( \omega \rightarrow 0 \), because then
\( G_0(\omega) \rightarrow \infty \). Hence, the point reactor approximation works well independent of system
size, for sufficiently low frequencies. For subcritical systems there are however other pos-
sibilities to achieve point kinetic behaviour. Briefly, for a small system, e.g. a research
reactor, the approximation works relatively well at plateau frequencies for all subcriti-
calities. In addition, for relatively large subcritical systems at plateau frequencies and
4.2 Analysis of the performance of the source modulation method

The source modulation method is based on the linearised point kinetic equations and their frequency domain solution for the amplitude factor fluctuation, $\delta P(\omega)$. Using the plateau frequency approximation, i.e. $\lambda \ll \omega \ll (\beta - \rho)/\Lambda$, and introducing the relative fluctuations normalized by static values, $\tilde{\delta} P(\omega)$ and $\tilde{\delta} q(\omega)$, dividing the dynamic equation (4.17) with the static one, one obtains

$$\tilde{\delta} P(\omega) = \frac{\rho(\$)}{\rho(\$) - 1} \tilde{\delta} q(\omega),$$  (4.24)

Figure 4.1: The noise induced by a fluctuating strength beam at various frequencies and $\rho(\$) = -3.00 and for a large reactor. The solid line and the dashed line denotes the exact solution and the point kinetic approximation, respectively.

deep subcriticalities, where $|\rho(\$)| \gg 1$, the system achieves point kinetic behaviour. This is shown in Fig. 4.1. In the figure, the exact and point kinetic transfer functions for a large reactor and for varying frequencies are shown. More detailed calculations and figures for a small reactor are found in Paper IV.

Summarizing, in contrast to critical systems where the only two factors determining the kinetic behaviour of the system is the frequency of the perturbation and the size of the system, in subcritical cores the static subcritical reactivity of the system also influences the kinetic behaviour. In the next section these conditions will be used in the analysis of the source modulation method for reactivity measurements.

4.2 Analysis of the performance of the source modulation method

The source modulation method is based on the linearised point kinetic equations and their frequency domain solution for the amplitude factor fluctuation, $\delta P(\omega)$. Using the plateau frequency approximation, i.e. $\lambda \ll \omega \ll (\beta - \rho)/\Lambda$, and introducing the relative fluctuations normalized by static values, $\tilde{\delta} P(\omega)$ and $\tilde{\delta} q(\omega)$, dividing the dynamic equation (4.17) with the static one, one obtains

$$\tilde{\delta} P(\omega) = \frac{\rho(\$)}{\rho(\$) - 1} \tilde{\delta} q(\omega),$$  (4.24)
which is the formula suggested by Carta and D’Angelo [14], for reactivity measurements. However, in reality only $\delta \phi(x, \omega)$ and $\phi_p(x)$ and not $\delta P(\omega)$ and $P_0$ can be measured. To obtain an estimation of reactivity, $\tilde{\delta}P(\omega)$ needs to be replaced with the normalised space-dependent flux $\tilde{\delta} \phi(x, \omega) = \delta \phi(x, \omega)/\phi_p(x)$ at some position $x$. The reactivity is obtained as

$$\rho_{est}(x, \omega) = \frac{\tilde{\delta} \phi(x, \omega)}{\delta \phi(x, \omega) - \delta S(\omega)}.$$  (4.25)

Using the exact solution for the neutron noise in the frequency domain and the static flux for the model in Section 4.1, the accuracy of the estimation in eq. (4.25) can be investigated. This leads to

$$\rho_{est}(x, \omega) = G(x, x_p, \omega) - G(x, x_p)\tilde{\delta} \rho / (\tilde{\delta} \phi(x, \omega) - \delta S(\omega)).$$  (4.26)

where $G(x, x_p, \omega)$ and $G(x, x_p)$ are defined in eqs. (4.13) and (4.4), respectively.

The formula in eq. (4.26) can be generalised to all frequencies. Instead of using the plateau frequency approximation the exact subcritical transfer function defined in eq. (4.19) can be used and then eq. (4.24) becomes

$$\tilde{\delta} P(\omega) = \frac{\rho}{\rho - G_0^{-1}(\omega)} \tilde{\delta} q(\omega).$$  (4.27)

For the estimated reactivity one obtains

$$\rho_{est}(x) = \frac{G_0^{-1}(\omega)G(x, x_p, \omega)}{G(x, x_p, \omega) - G(x, x_p)},$$  (4.28)

where $G_0(\omega)$ is defined in eq. (4.11).

It is expected that the method will work well when the system behaviour is point kinetic, which means that a measurement of the flux fluctuations in one point gives a proper approximation of the fluctuations of the point kinetic amplitude factor. Our investigation, based on quantitative analysis of eq. (4.28), shows however that the accuracy of the method depends strongly on the detector position even when the system behaviour is point kinetic. This is illustrated in Fig. 4.2, which shows the ratio of the estimated reactivity and the real reactivity for a large reactor with $a = 150$ cm, for $\omega = 20$ rad/s, which is a typical plateau frequency. Subcritical reactivities corresponding to $k_{eff}$ equal to 0.9996, 0.99, 0.95, and 0.90, were chosen, or in dollars $-0.061, -1.55, -8.10, -17.9$. The corresponding $\rho_{\infty}$ values are 0.0075, -1.42, -7.95, and -17.09, respectively. This system behaves point kinetically at plateau frequencies for $k_{eff} \leq 0.99$, as is shown in Fig. 4.1 for $k_{eff} = 0.98$. Figure 4.2 illustrates that the estimated reactivity value shows large deviations from the reference value for all four subcriticalities. The conclusion is that for a large system the performance of the method is surprisingly poor at plateau frequency.

For a small reactor the performance is reasonably good in slightly subcritical cases. However, it is interesting to note that with subcriticality levels that are more likely to be used in planned power ADS, the method breaks down even in a small system at plateau frequencies. Since a large system is the more interesting case from the practical...
4.2 Analysis of the performance of the source modulation method

Figure 4.2: The ratio of estimated and real reactivity for $\omega = 20 \text{ rad/s}$ and different subcriticalities, for a large reactor with the source at the center of the core.

point of view, figures for the small reactor is omitted here and later. Nevertheless, they are found in PAPER IV.

Since one could think that the reason for the large deviations is the large system size and the deviation from point kinetics even when this deviation is small, the performance of the method was investigated in the point kinetic limit, i.e. when $|\rho_\infty| \rightarrow \infty$ or $\omega \rightarrow 0$. Here we describe briefly the latter case using the generalized formula, i.e. eq. (4.28). The other case, i.e. $|\rho_\infty| \rightarrow \infty$ leads to a similar result. Detailed calculations are found in PAPER IV.

By defining $\varepsilon$ as

$$\varepsilon = \frac{1}{2\rho_\infty G_0(\omega)}; \quad |\varepsilon| \ll 1,$$

a Taylor expansion gives for $\rho_{asy}$, for $x_p = 0$ for the leading term

$$\rho_{asy}(x) \approx \frac{G_0^{-1}(\omega) 2\rho_\infty G_0(\omega)}{1 - B_\rho [a \tan(B_\rho a) - (a + x) \cot(B_\rho(a + x))] = f(x)\rho_\infty,} \quad (4.29)$$

where $f(x)$ is given as

$$f(x) = \begin{cases} 
2 & x \leq x_p \\
\frac{1 - B_\rho [a \tan(B_\rho a) - (a + x) \cot(B_\rho(a + x))]}{2} & x > x_p.
\end{cases} \quad (4.30)$$

From the above for the ratio of the estimated reactivity to the true reactivity the following relationship is obtained:

$$\frac{\rho_{asy}(x)}{\rho} = f(x)\frac{\rho_\infty}{\rho} \quad (4.31)$$
Consequently, the estimated reactivity has a finite error even when the system asymptotically approaches point kinetic behaviour and regardless of how the limit is reached, i.e. by $|\rho| \to \infty$ or by $\omega \to 0$.

Figure 4.2 shows the asymptotic formula, eqs. (4.29)–(4.30), and the generalized formula, eq. (4.28), for $\omega = 0.001$ rad/s for a large system. The solid line and the line with symbols denote the estimated ratio and the asymptotic ratio, respectively. The two formulae agree well with each other. Except for the case of a very slight subcriticality, the error of the estimation is non-negligible. For the reactivity level planned for a future ADS, i.e. $k_{eff} \approx 0.95$, the deviation between the estimated and true reactivity is quite large, even at this very low frequency.

![Figure 4.3](image.png)

Figure 4.3: The ratio of estimated and real reactivity and the ratio of asymptotic and real reactivity for $\omega = 0.001$ rad/s and different subcriticalities, for a large reactor with the source at the center of the core. The solid line and the line with x denotes the estimated ratio and the asymptotic ratio, respectively.

### 4.3 Possible methods for improving the performance

The investigations in the previous Section showed that using a single neutron detector, unless it is by chance placed at a position where the theoretical error of the method is zero or very small, will give an error which will not disappear even in the limit of exact point kinetic behaviour of the system. However, the original formula was derived for the amplitude factor of the factorised neutron noise, i.e. $\delta P(\omega)$ and not for the space-dependent noise $\delta \phi(x, \omega)$. The amplitude factor $\delta P(\omega)$ can be recovered from $\delta \phi(x, \omega)$ as

$$\delta P(\omega) = \int \delta \phi(x, \omega) \phi_0(x) dx. \quad (4.32)$$

In other words, instead of using the local signals, a weighted integral, yielding the amplitude factor, could be used in the applications of the method. Using the correct
$\delta P(\omega)$ instead of the local flux fluctuations, means that there is no need for the system to behave in a point kinetic way. Hence, the method can also be applied at plateau frequencies in a large system at arbitrary subcriticality level.

By using several detectors distributed relatively evenly at different positions in the core, and trying to approximate the integral as a sum, $\delta P(\omega)$ from (4.32) can be approximated. A similar approach was used in other noise problems earlier, see [25]. The equation (4.28) can be approximated as

$$\rho_{\text{est}}(\omega) = \frac{D(B_0^2 - B_p^2) \sum_i G(x_i, x_p, \omega) \cos(B_0 x_i) dx_i}{D(B_0^2 - B_p^2) \sum_i G(x_i, x_p, \omega) \cos(B_0 x_i) dx_i - \cos(B_0 x_p)} \quad (4.33)$$

Obviously, the quality of this approximation depends on the number of the terms used in the sum (i.e. number of detectors in the measurement), and on the relative positioning of the detectors. It is obvious that detectors both close to and far away from the source are needed to be able to eliminate the space dependent term $\delta \psi(x, \omega)$ in eq. (4.16).

The applicability of this procedure was investigated quantitatively. Briefly, it is found that the number of detectors needed is most likely larger than the number of detectors available for the purpose of reactivity monitoring in practice. Hence, the method of summing up the detector signal fluctuations does not give the expected measure of improvement of the method. In a real case the situation may be even more disadvantageous, since in contrast to the investigations here that used a one-dimensional model, the dynamics is three-dimensional and the space-dependent effects are even more significant. Hence, the source modulation method can most likely only be used in a relative way, after having been calibrated by other methods.
In this thesis two subcriticality measurement methods were investigated, the pulsed Feynman-alpha method and the source modulation method. For the pulsed Feynman-alpha formula an efficient Fourier series solution method was developed, which can be used for any pulse shape. During the investigation of the source modulation method the conditions of point kinetic behaviour in subcritical cores were revised. The investigation led to the unexpected conclusion that in practical applications of the method difficulties turn up when it is implemented by using single detector signals to approximate the point kinetic amplitude factor. Hence, the source modulation method can most likely only be used in a relative way, after having been calibrated by other methods.

The results regarding the Feynman technique have been confirmed experimentally in measurements made at the KUCA reactor of the Kyoto University Research Reactor Institute. The results were also used in the interpretation of pulsed Feynman-alpha measurements within the EU project MUSE, and are likely to be used in future ADS projects. There is no experimental application of the source modulation technique yet, but the conclusions have a clear relevance on the usefulness of the method, and have thus to be taken into account in the planning of future experiments.

In the continuation of this work, the experience gained in the work so far will be applied to investigate the performance of the Cf-252 method. A formalism, similar to that used in the study of the Feynman-alpha method will be used, but the aspects of space-dependence, as well as the significance of point kinetic behaviour will also be investigated. Another method that might have practical relevance is the so-called break frequency method, whose applicability will also be investigated.
Bibliography


Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>ADS</td>
<td>Accelerator Driven Systems</td>
</tr>
<tr>
<td>APSD</td>
<td>Auto-Power Spectral Density</td>
</tr>
<tr>
<td>BWR</td>
<td>Boiling Water Reactor</td>
</tr>
<tr>
<td>CPSD</td>
<td>Cross-Power Spectral Density</td>
</tr>
<tr>
<td>KUCA</td>
<td>Kyoto University Critical Assembly</td>
</tr>
<tr>
<td>MUSE</td>
<td>MUltiplication avec une Source Externe</td>
</tr>
<tr>
<td>PWR</td>
<td>Pressurised Water Reactor</td>
</tr>
<tr>
<td>SIRM</td>
<td>Start-up and Intermediate Monitors</td>
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Paper I
Derivation and analysis of the Feynman-alpha formula for deterministically pulsed sources

J. Wright and I. Pázsit
Derivation and analysis of the Feynman-alpha formula for deterministically pulsed sources

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Abstract

The purpose of this report is to give a detailed description of the calculation of the Feynman-alpha formula with deterministically pulsed sources. In contrast to previous calculations [1], Laplace transform and complex function methods are used to arrive at a compact solution in form of a Fourier series-like expansion. The advantage of this method is that it is capable to treat various pulse shapes. In particular, in addition to square- and Dirac-delta pulses, a more realistic Gauss-shaped pulse is also considered here. The final solution of the modified variance-to-mean, that is the Feynman $Y(t)$-function, can be quantitatively evaluated fast and with little computational effort.

The analytical solutions obtained are then analysed quantitatively. The behaviour of the number of neutrons in the system is investigated in detail, together with the transient that follows the switching on of the source. An analysis of the behaviour of the Feynman $Y(t)$-function was made with respect to the pulse width and repetition frequency. Lastly, the possibility of using the formulae for the extraction of the parameter alpha from a simulated measurement is also investigated.
1. Introduction

The theory of the Feynman-alpha method with pulsed sources became interesting recently in connection with the future accelerator-driven systems (ADS). An ADS is a subcritical reactor driven by a strong external neutron source. The source needs to be an accelerator-based one, in order to achieve sufficient intensity. The most frequently proposed candidate for such a source is based on the spallation reaction. Due to technical reasons, such accelerators will most likely be operated in a pulsed mode. This is also the case with the international MUSE project, which is a 5th FrameWork program of the EU, for the experimental verification of some basic ADS principles. The MUSE experiments are being performed on a fast research reactor called MASURCA, driven by a pulsed neutron generator called GENEPI [2]. The calculations described in this report were partly made for the theoretical support of evaluation of the MUSE experiments.

There are different ways of describing a pulse of neutrons from a neutron generator. One possibility is to assume a finite pulse width, usually with a square shape, and assume that during the pulse all neutrons arrive at random with Poisson statistics. In between the pulses there are no neutrons emitted. This method was used in Refs [1], [3]-[5], and this is also the approach that we shall pursue in this report, although here both square and Gaussian-shaped pulses will be considered. The other possibility is to assume that the pulses have the width of a Dirac delta function in time, but at each pulse a random number of neutrons is injected into the system. This approach was used in Refs [6]-[8]. In certain special cases the results from the two methods are quantitatively equivalent, as it will be touched upon in the report.

In either cases, i.e. either finite width or infinitely short pulses, there are still two possibilities to perform or evaluate measurements. These are generally termed as deterministic and stochastic pulsing, respectively. The deterministic pulsing means that the start of the measurement, i.e. neutron detection or pulse counting in the neutron detector, is synchronised with the source pulse train. The usual assumption is that the neutron detection always starts at the start of a pulse from the neutron generator. The stochastic pulsing means, as the name indicates, that the neutron detection and the source pulsing are not synchronised, so that the start of the neutron measurement is randomly distributed with a uniform probability over a time period of the source pulse.

For the case of finite sources, which is also our subject here, both the deterministic ([1], [5]) and the stochastic ([4], [5]) pulsing was treated earlier. However, in these earlier works the case of the deterministic pulsing was treated with a method such that the corresponding temporal differential equations were solved piece-wise for each consecutive pulse and the periods in between the pulses [1]. Although the method did yield quantitatively correct results with very short computation times, it was at the same time clumsy to use for different values of the input parameters, and did not appear to be suitable for parameter unfolding by curve fitting.

Another, more significant drawback was that the solution method was based on the piecewise constant behaviour of the source. Extension of the method to more complicated (and hence realistic) pulse shapes would incur a substantial complication.

In the case of stochastic pulsing, another approach was used. It was noticed that several expressions for the first and second moments of the neutron number and the number of detections were, or could be cast into, the form of convolutions. One could make use of the fact that the Laplace transform of such convolutions could be written as the product of the individual
Laplace transform, and the original convolution could be obtained in the time domain through a Laplace inversion by complex function methods. It was therefore suggested that also the case of deterministic pulsing also be treated this way, and even the feasibility was demonstrated [5]. However, in order to arrive to tractable compact results, a significantly larger effort was needed than in the case of stochastic pulsing. This was only achieved relatively recently.

It was immediately realised, that this method could be extended to other pulse shapes, and in particular to a Gauss-shaped pulse, which is actually the case of the pulses of the neutron generator GENEPI in the MUSE experiments. In fact, the singularity structure (number and position of the poles) of the function to be inverse transformed is determined only by the periodicity of the source pulse train, and it is only the values of the residues at these poles that are dependent on the actual pulse form. This means that it is only a relatively small part of the calculations that needs to be re-done when calculating the Feynman-alpha function for various pulse shapes. This is demonstrated in the report by calculating the $Y$-function for both square pulses and Gaussian pulses.

The purpose of this report is to describe the flow of these calculations in a detail that is larger than what is customary in a journal publication. Also, based on the compact form results, a qualitative and quantitative analysis of the Feynman-alpha formula is made and reported, what regards the dependence of the characteristics of the $Y(t)$-curve for various combinations of the pulse width and repetition frequency. Finally, the possibility of extracting (unfolding) the parameter alpha from some simulated measurements, generated by the formulæ derived and adding some extra noise, is investigated. These show that the formula is suitable for the determination of subcritical reactivity from Feynman-alpha measurements with deterministic pulsing, given that the pulse period is short enough in comparison to $1/\alpha$.

2. General theory

The principles and the basic theory of the Feynman-alpha method are described in several publications, including the pre-decessors of the present work (e.g. in [1], [3] and [5]). The method is based on the measurement of the detector counts $\tilde{Z}(T)$ during a measurement time period $T$ in a stationary system driven with a source S. Both the source emission, the diffusion and multiplication of the neutrons in the medium, as well as the detection process, are subject to random fluctuations. The probability of emission of a source neutron during time $t$ is given as $Sdt$. The condition of stationarity requires that the measurement is made a long time after that the source was switched on, such that all transients after the switch-on have decayed. With a repeated measurement of the random variable $\tilde{Z}$, its mean $\langle \tilde{Z} \rangle \equiv \tilde{Z}$ and variance $\sigma^2_Z(T)$ can be determined. With a repetition of the measurement for various measurement time lengths, the dependence of the relative variance, or the variance-to-mean $\sigma^2_Z(T)/\tilde{Z}$ (Feynman-alpha function) can be determined. In practice, it is more customary to use the deviation of the relative variance from unity, which is called the Feynman $Y$-function:

$$Y(T) \equiv \frac{\sigma^2_Z(T)}{\tilde{Z}(T)} - 1 = \frac{\tilde{\mu}_Z(T)}{\tilde{Z}(T)}$$  \hfill (1)

where

$$\tilde{\mu}_Z(T) \equiv \sigma^2_Z(T) - \tilde{Z}(T)$$  \hfill (2)
is called the modified variance. The advantage of introducing the modified variance is the convenience that one can derive equations for it directly.

The mean and the variance of the detector counts can be calculated from a master equation, i.e., from a probability balance equation. Usually, it is more advantageous to use the so-called backward master equation, which means that one has to operate on initial variables [9]. This excludes the possibility to switch on the source in \(-\infty\), rather it is assumed that the source is switched on at \(t = 0\) and the measurement is made when \(t \to \infty\). For this reason one needs to go in two steps; first the probability of the number of neutrons in the system and detector counts during the time period \((t - T, t)\) is determined, and then from this the probability of the same quantities is calculated for the case of an external source.

Because we shall be dealing with a time-dependent source, which constitutes a complication compared to the previous cases, for the time being we shall neglect the existence of the delayed neutrons in this work. They will be treated in a coming publication.

The quantities that will appear in the derivation are thus as follows.

\[ P(N, Z, t) \tag{3} \]

is the probability of finding \(N\) neutrons at \(t\) and \(Z\) counts in the time interval \((t - T, t)\), due to one neutron starting the process at \(t = 0\). One also introduces the probability generating function of \(P\) as

\[ G(x, z, t) \equiv \sum_N \sum_Z x^N z^Z P(N, Z, t) \tag{4} \]

If a master equation for the generating function \(G\) is obtained, equations for the various moments can be obtained by differentiating \(G\) w.r.t. \(x\) or \(z\). Such a master equation was derived a long time back in the theory of neutron fluctuations in nuclear reactors. For later reference, we only quote the two first-moment quantities that will be used in later parts of the paper in the calculations. The expected number of neutrons at time \(t\) is given by

\[ N(t) = e^{-\alpha t} \tag{5} \]

where \(\alpha\) is the prompt neutron time constant, given by \(\alpha = -\rho / \Lambda\). Here \(\rho\) is the subcritical reactivity (to be determined in the measurement) and \(\Lambda\) the prompt neutron generation time. Both \(\rho\) and \(\Lambda\) can be expressed by nuclear physics parameters such as cross sections and neutron speed, and are known in a calculation.

The second quantity we shall need is the expected number of counts, which is given as

\[ Z(t, T) = \lambda_d \int_0^t N(t - t') \Delta(t', T) dt' \tag{6} \]

where

\[ \Delta(t, T) = \begin{cases} 1; & 0 \leq t \leq T \\ 0; & t > T. \end{cases} \tag{7} \]

From (5)-(7) it readily follows that
In a similar manner, 

\[
\tilde{P}(N, Z, t|t_0)
\]  

(9)

denotes the probability of finding \(N\) neutrons at \(t\) and \(Z\) counts in the time interval \((0, t)\), due to a source being switched on at \(t = t_0\). As it is usual in the literature, all distributions and moments, corresponding to the source-induced case, will be denoted by a tilde. The generating function of \(\tilde{P}\) is defined as

\[
\tilde{G}(x, z|t_0) = \sum_N \sum_Z x^N z^Z \tilde{P}(N, Z, t|t_0)
\]  

(10)

As it will be seen in the next section, one can derive a direct relationship between \(G\) and \(\tilde{G}\) such that the latter is given as an exponential integral over the former. Calculation of the moments of the source-induced distribution requires in general the calculation of multiple nested integrals over certain functions of the various moments of the single-particle induced distribution.

### 3. Calculation of the Variance-to-mean for Deterministically Pulsed Sources

As mentioned earlier, the novelty of the Feynman-alpha method with a pulsed source consists of the time dependence of the source. Some characteristic properties follow directly from the fact that the source consists of a train of pulses, independently of the form of the pulses. These will be first investigated here. For the sake of concreteness, whenever explicit formulae are necessary, square pulses will be assumed. The case of Gaussian pulses will be treated in Subsection 3.5.

#### 3.1. The source and its Laplace transform for square pulses

The time-dependent neutron source is represented by a sequence of square functions:

\[
S(t) = S_0 \sum_0^\infty [H(t - nT_0) - H(t - nT_0 - W)]
\]  

(11)

where \(H\) is Heaviside’s step function, \(T_0\) the pulse period, and \(W\) the pulse width. In Fig. 1 the function is plotted.

The Laplace transform of the source is given by:

\[
S(s) = \int_0^\infty e^{-st}S(t)dt = \sum_0^\infty (e^{-sT_0})^n S_0 \int_0^W e^{-st}dt = \frac{S_0(1 - e^{-sW})}{s(1 - e^{-sT_0})}; \quad Re\{s\} > 0.
\]  

(12)

Eqn (12) shows that the singularities of \(S(s)\) are determined by the zeros of the denominator,
which arise from the periodicity of the pulse train.

3.2. Calculation of the source induced neutron number (square pulses)

We know from earlier calculations that the Bartlett-formula, i.e the relationship between the generating functions of the source induced and single-particle induced distributions, for the case of a time-dependent source read as

$$\tilde{G}(x, z, t) = \exp \left\{ \int_0^t S(t')G[(x, z, t - t') - 1]dt' \right\}. \quad (13)$$

From eq. (13) we obtain that the source induced neutron number, $\tilde{N}(t)$ is given by

$$\tilde{N}(t) = \left. \frac{\partial \tilde{G}(x, z, t)}{\partial x} \right|_{x = z = 1} = \int_0^t S(t')N(t - t')dt' \quad (14)$$

and the source induced detector count, $\tilde{Z}(t, T)$ reads as

$$\tilde{Z}(t, T) = \left. \frac{\partial \tilde{G}(x, z, t)}{\partial z} \right|_{x = z = 1} = \int_0^t S(t')Z(t - t', T)dt'. \quad (15)$$

Further, the source induced modified variance, defined in (2), derives from the above as

$$\tilde{\mu}_Z(t, T) = \int_0^t S(t')M_Z(t - t', T)dt' \quad (16)$$

Here

$$M_Z(t, T) = \int_0^t Q_Z(t', T)N(t - t')dt' \quad (17)$$

Fig. 1. The source function $S(t)$, numerical values are found in Table 1 on page 18.
with
\[ Q_Z(t, T) = \lambda_f \langle \nu(\nu - 1) \rangle Z^2(t, T). \]  

(18)

and where \( Z(t, T) \) is the single-particle induced detector count, given in eq. (8). Hence, using eq. (14) and eq. (15) we can write eq. (16) as
\[ \tilde{\mu}_Z(t, T) = \int_0^t Q_Z(t', T) \tilde{N}(t-t') dt'. \]  

(19)

The Laplace transform of eq. (14) is
\[ \tilde{N}(s) = S(s)N(s) \]  

(20)

where
\[ N(s) = \frac{1}{s + \alpha} \]  

(21)

since
\[ \tilde{N}(t) = e^{-\alpha t} \]  

(22)

where \( \alpha = \frac{\rho}{\Lambda} \) and \( \rho = \frac{k-1}{k} \) as usual.

Hence
\[ \tilde{N}(s) = \frac{(1-e^{-sW})}{s(1-e^{-s\tilde{T}_b})(s+\alpha)}. \]  

(23)

From eq. (23) we can obtain \( \tilde{N}(t) \) with inverse Laplace transform. Let
\[ \tilde{N}(s) = (1-e^{-sW})f(s) \]  

(24)

where
\[ f(s) = \frac{1}{s(1-e^{-s\tilde{T}_b})(s+\alpha)} \]  

(25)

then
\[ \tilde{N}(t) = f(t) - H(t-W)f(t-W) = \begin{cases} f(t) & t \leq W \\ f(t) - f(t-W) & t > W \end{cases}. \]  

(26)

For the inversion of eq. (23), we note that its singularities are defined by three different types of poles, each corresponding to a different type of behaviour in the time domain after inversion.

1. A pole at \( s = 0 \); the corresponding residue gives the asymptotic mean value of the oscillating function \( \tilde{N}(t) \);
2. a pole at \( s = -\alpha \), which describes the transient after switching on the source at \( t = 0 \);
3. an infinite number of complex conjugate roots on the imaginary axis, yielding harmonic functions in the time domain, representing a Fourier series expansion of the oscillating part of \( \tilde{N}(t) \).
As mentioned earlier, the positions of these poles are independent of the form of the pulse shape, because they are given by the zeros of the function \(1 - e^{-sT_0}\), which latter was given rise by the summation of the geometric series in (12) expressing the periodicity of the pulse. It is only the value of the residues which is affected by the pulse shape. In other words, the result for the asymptotic value of \(\tilde{N}_a(t)\) can always be written in the form of eq. (35) below, only the values of the \(a_n\) and \(b_n\) will be different.

Let us write now the inverse Laplace transform of eq. (23) in terms of the inverse of the function \(f(s)\), introduced in (25), in a sum corresponding to the three types of poles above. That is, let us write \(f(t) = c(t) + e(t) + g(t)\) and calculate each term separately. Then the theorem of residues gives:

\[
c(t) \equiv \text{Res}_{s=0} \left( \frac{e^{st}}{s(1-e^{-sT_0})(s+\alpha)} \right) = \frac{2(t\alpha - 1) + \alpha T_0}{2\alpha^2 T_0}
\]  

\[
e(t) \equiv \text{Res}_{s=-\alpha} \left( \frac{-e^{st}}{s(1-e^{-sT_0})(s+\alpha)} \right) = -\frac{e^{-\alpha t}}{\alpha(1-e^{-\alpha T_0})}
\]  

and

\[
g(t) \equiv \sum_{n} \text{Res}_{s=s_n} \left( \frac{e^{st}}{s(1-e^{-sT_0})(s+\alpha)} \right) = T_0 \sum_{n=1}^{\infty} \frac{\sin \left( \frac{2n\pi t}{T_0} \right) \alpha T_0 - \cos \left( \frac{2n\pi t}{T_0} \right) 2n\pi}{\pi n [ (2n\pi)^2 + (\alpha T_0)^2 ]}.
\]  

Hence, also in view of eq. (26), one has

\[
\tilde{N}(t) = c(t) + e(t) + g(t) - (c(t - W) + e(t - W) + g(t - W))H(t - W).
\]  

A plot of \(\tilde{N}(t)\) is shown in Fig. 2. It is obvious that after a number of pulses the initial transient decays and the system converges to an asymptotic state.

![Fig. 2. \(\tilde{N}(t)\) with its asymptotic behaviour indicated as a dotted line. Numerical values are found in Table 1 on page 18.](image-url)
When $t \to \infty$, $\tilde{N}(t)$ is simplified to:

$$\tilde{N}_a(t) = \frac{W}{T_0\alpha} + g(t) - g(t - W).$$

(31)

Using trigonometric identities and introducing $\omega_n \equiv \frac{2\pi n}{T_0}$, we get:

$$ggw(t) \equiv g(t) - g(t - W) = \frac{1}{\pi} \sum_{n=1}^{\infty} a_n \sin(\omega_n t) + b_n \cos(\omega_n t)$$

(32)

where

$$a_n = \frac{[1 - \cos(\omega_n W)]\alpha + \omega_n \sin(\omega_n W)}{n(\omega_n^2 + \alpha^2)}$$

(33)

and

$$b_n = \frac{\alpha \sin(\omega_n W) - \omega_n [1 - \cos(\omega_n W)]}{n(\omega_n^2 + \alpha^2)}.$$  

(34)

Thus, the asymptotic source induced neutron number, $\tilde{N}_a(t)$, is:

$$\tilde{N}_a(t) = \frac{W}{T_0\alpha} + \frac{1}{\pi} \sum_{n=1}^{\infty} a_n \sin(\omega_n t) + b_n \cos(\omega_n t).$$

(35)

This function is shown in Fig. 3.

Fig. 3. $\tilde{N}_a(t)$ and the constant term in $\tilde{N}_a(t)$ i.e. $\tilde{N}_a(t) - ggw(t)$. Numerical values are found in Table 1 on page 18.

As will be seen in the next two subsections, in all subsequent calculations of the source-induced asymptotic detector count $\tilde{Z}_a(t)$ and the Feynman $Y(t)$-function, the solution given for $\tilde{N}_a(t)$ above is used in various nested integrals in combination with functions that do not
depend on source properties. The functional dependence of \( \tilde{N}_a(t) \) is given by the trigonometric functions that do not depend on the pulse shape. Hence in the resulting expressions using a different pulse shape means just changing the parameters \( a_n \) and \( b_n \) and the first term on the r.h.s. of (35).

### 3.3. Calculation of the source induced detector count for arbitrary pulses

The single-particle induced detector count, \( Z(t, T) \) is given, as usual (c.f. eqn (8)):

\[
Z(t, T) = \lambda_d \int_0^t \Delta(t', T) N(t - t') dt' = \begin{cases} 
\frac{\lambda_d}{\alpha} (1 - e^{-\alpha t}); & 0 \leq t \leq T \\
\frac{\lambda_d}{\alpha} e^{-\alpha t} (e^{\alpha T} - 1); & t > T.
\end{cases}
\quad (36)
\]

where \( \Delta(t, T) \) was defined in (7)

From eq. (15) the source induced detector count, \( \tilde{Z}(t, T) \), is given by:

\[
\tilde{Z}(t, T) = \lambda_d \int_0^t \Delta(t', T) \tilde{N}(t - t') dt'.
\quad (37)
\]

The easiest way to calculate the integral is not with the same method as was used for \( \tilde{N}(t) \), i.e. writing the Laplace transform of \( \tilde{Z}(t, T) \) as:

\[
\tilde{Z}(s, T) = \lambda_d \Delta(s, T) \tilde{N}(s) = \frac{\lambda_d}{s} \left( 1 - e^{-sT} \right) \frac{1}{s(1 - e^{-s\tau_0})s + \alpha}.
\quad (38)
\]

The reason is, that for eq. (38) it is rather complicated to use the same trick as for eq. (24) due to the factor \( (1 - e^{-sT})(1 - e^{-s\tau}) \). Further, it is easier to let \( t \to \infty \) in eq. (37) than in eq. (38). But, since we have a simple expression for \( \tilde{N}_a(t) \), i.e. eq. (35), we are able to obtain the asymptotic source induced detector count by calculating the integral in eq. (37) when \( t \to \infty \). Thus,

\[
\tilde{Z}_a(T) = \lim_{t \to \infty} \lambda_d \int_0^t \Delta(t', T) \tilde{N}_a(t - t') dt'.
\quad (39)
\]

As a result of deterministic pulsing we have \( t = KT_0 + T \) and it is obvious that letting \( t \to \infty \) equals to letting \( K \to \infty \). With that, we have:

\[
\tilde{Z}_a(T) = \lambda_d \int_0^{KT_0 + T} \Delta(t', T) \tilde{N}_a(t - t') dt' = \lambda_d \int_{KT_0}^{KT_0 + T} \tilde{N}_a(t') dt' = \lambda_d \tilde{N}_a(t').
\quad (40)
\]

The last step above results from the periodic character of \( \tilde{N}_a(t) \). The above gives, with eq. (35):
\[
\tilde{Z}_a(T) = \lambda_d \left\{ \frac{WT}{T_0 \alpha} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{\omega_n} \left\{ a_n (1 - \cos(\omega_n T)) + b_n \sin(\omega_n T) \right\} \right\}.
\]  
(41)

Fig. 4 shows both the asymptotic source induced detector count, \(\tilde{Z}_a(t)\) and only the linear term, \(\tilde{Z}_l\), of \(\tilde{Z}_a(t)\). The oscillating part, \(\tilde{Z}_{osc}\), is plotted in Fig. 5. All curves are normalized with a factor \(1/\lambda_d\).

**Fig. 4.** \(\tilde{Z}_a(t)\) and the linear term \(\tilde{Z}_l\). The solid line denotes \(\tilde{Z}_a(t)\) and the dashed line \(\tilde{Z}_l\). Numerical values are found in Table 1 on page 18.

**Fig. 5.** The oscillating part \(\tilde{Z}_{osc}\), other numerical values are found in Table 1 on page 18.
3.4. Calculation of the modified variance for arbitrary sources

The asymptotic modified variance is obtained from eq. (19) using the asymptotic source induced neutron number, $\tilde{N}_a(t)$:

$$
\tilde{\mu}_2(t, T) = \int_0^t Q_2(t', T) \tilde{N}_a(t - t') dt'
$$

where $Q_2(t, T)$ is defined in eq. (18) with $Z(t, T)$ given by eq. (36).

Let $C = \lambda_d \langle \nu (\nu - 1) \rangle$, and for $t < T$ we define with eq. (36):

$$
Q_1(t) = C \left( \frac{\lambda_d}{\alpha} \right)^2 (1 - e^{-\alpha t})^2
$$

and for $t > T$:

$$
Q_2(t) = C \left( \frac{\lambda_d}{\alpha} \right)^2 e^{-2\alpha T} (e^{\alpha T} - 1)^2.
$$

Correspondingly, the expression for the modified variance, eq. (42) will be broken up into two parts, i.e. $\tilde{\mu}_2(t, T) = \mu_1(T) + \mu_2(T)$, with $\mu_1(T)$ and $\mu_2(T)$ corresponding to the integrals over $Q_1(t)$ and $Q_2(t)$, respectively. If we also let $t = KT_0 + T$, we obtain:

$$
\tilde{\mu}_2(KT_0 + T, T) = \int_0^T Q_1(t) \tilde{N}_a(T - t) dt + \int_T^{KT_0 + T} Q_2(t, T) \tilde{N}_a(T - t) dt.
$$

When $t \to \infty$:

$$
\tilde{\mu}_2(KT_0 + T, T) = \int_0^T Q_1(t) \tilde{N}_a(T - t) dt + \int_T^{T} Q_2(T + t, T) \tilde{N}_a(T - t) dt.
$$

Using eq. (31) and eq. (32) the $\mu_1(t)$-integral becomes:

$$
\mu_1(T) = \int_0^T Q_1(t) \left( \frac{W}{T_0 \alpha} + ggw(T - t) \right) dt = \frac{W}{T_0 \alpha} \int_0^T Q_1(t) dt + \int_0^T Q_1(T - t) ggw(t) dt.
$$

Using eq. (43), the result of the first integral is:

$$
\frac{W}{T_0 \alpha} \int_0^T Q_1(t) dt = C \left( \frac{\lambda_d}{\alpha} \right)^2 \frac{W}{T_0 \alpha} \frac{2 T \alpha + 4 e^{-\alpha T} - e^{-2\alpha T} - 3}{2 \alpha}.
$$

The second integral in eq. (47) is trickier to evaluate. We notice that, according to eq. (43), we have:

$$
Q_1(T - t) = C \left( \frac{\lambda_d}{\alpha} \right)^2 (1 - e^{-\alpha(T - t)})^2 = 1 - 2 e^{-\alpha T} e^{\alpha T} + e^{-2\alpha T} e^{2\alpha T}.
$$
is given by eq. (32). It is suitable to evaluate the integral in two parts. First, the one with the sine and second, the one with the cosine functions. Now, let

\[ p_n(\alpha, T) \equiv e^{-\alpha T} \int_0^T e^{\alpha t} \sin(\omega_n t) dt = \frac{\alpha \sin(\omega_n T) + \omega_n (e^{-\alpha T} - \cos(\omega_n T))}{\alpha^2 + \omega_n^2}. \] (50)

Then,

\[ A_n(T) \equiv \int_0^T Q_1(T - t) \sin(\omega_n t) dt = C\left(\frac{\lambda d}{\alpha}\right)^2 \left( p_n(0, T) - 2p_n(\alpha, T) + p_n(2\alpha, T) \right). \] (51)

Analogously, let

\[ q_n(\alpha, T) \equiv e^{-\alpha T} \int_0^T e^{\alpha t} \cos(\omega_n t) dt = \frac{\alpha \sin(\omega_n T) + \omega_n (e^{-\alpha T} - \sin(\omega_n T))}{\alpha^2 + \omega_n^2}. \] (52)

Then,

\[ B_n(T) \equiv \int_0^T Q_1(T - t) \cos(\omega_n t) dt = C\left(\frac{\lambda d}{\alpha}\right)^2 \left( q_n(0, T) - 2q_n(\alpha, T) + q_n(2\alpha, T) \right) \] (53)

and

\[ \int_0^T Q_1(T - t) ggw(t) dt = C\left(\frac{\lambda d}{\alpha}\right)^2 \frac{1}{\pi} \sum_{n=1}^{\infty} \{ a_n A_n(T) + b_n B_n(T) \}. \] (54)

Finally, we have from eq. (48) and eq. (54):

\[ \mu_1(T) = C\left(\frac{\lambda d}{\alpha}\right)^2 \left( \frac{W}{T_0} \frac{2T_0\alpha + 4e^{-\alpha T} - e^{-2\alpha T} - 3}{2\alpha} + \frac{1}{\pi} \sum_{n=1}^{\infty} \{ a_n A_n(T) + b_n B_n(T) \} \right). \] (55)

Using eq. (31) and eq. (32) the \( \mu_2(t) \)-integral in eq. (46) becomes:

\[ \mu_2(T) = \int_0^T Q_2(T + t, T) \left[ \frac{W}{T_0\alpha} + ggw(-t) \right] dt = \]

\[ C\left(\frac{\lambda d}{\alpha}\right)^2 (1 - e^{-\alpha T})^2 \int_0^T e^{-2\alpha t} \left[ \frac{W}{T_0\alpha} + ggw(-t) \right] dt. \] (56)

The first term in the integral above results in:

\[ \int_0^\infty e^{-2\alpha t} dt = \frac{W}{2T_0\alpha^2}. \] (57)

The first part of the \( ggw(t) \)-integral gives:
\[ \int_{0}^{\infty} e^{-2\alpha t} \sin(-\omega_n t) dt = \frac{-\omega_n}{\omega_n^2 + (2\alpha)^2}. \] (58)

And the second:

\[ \int_{0}^{\infty} e^{-2\alpha t} \cos(-\omega_n t) dt = \frac{2\alpha}{\omega_n^2 + (2\alpha)^2}. \] (59)

Then the result to eq. (56) is:

\[ \mu_2(T) = C \left( \frac{\lambda_d}{\alpha} \right)^2 \left( 1 - e^{-\alpha T} \right)^2 \left\{ \frac{W}{2T_0\alpha^2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{-a_n\omega_n + 2\alpha b_n}{\omega_n^2 + (2\alpha)^2} \right\}. \] (60)

The sum of eq. (55) and eq. (60) is the result to eq. (46):

\[ \tilde{\mu}_z(T) = C \left( \frac{\lambda_d}{\alpha} \right)^2 \left( \frac{WT}{2T_0\alpha^2} \left( 1 - \frac{1 - e^{-\alpha T}}{\alpha T} \right) \right) + \frac{1}{\pi} \sum_{n=1}^{\infty} \{ a_n A_n(T) + b_n B_n(T) \} + \left( 1 - e^{-\alpha T} \right) \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{-a_n\omega_n + 2\alpha b_n}{\omega_n^2 + (2\alpha)^2}. \] (61)

Finally, the Feynman Y-function is given by eq. (41) and eq. (61), as usual as:

\[ Y(T) = \frac{\tilde{\mu}_z(T)}{Z_\alpha(T)}. \] (62)

Fig. 6 shows the resulting Feynman Y-function for the case with square pulsed. The function is multiplied with a factor \( \alpha^2/(\lambda_d\lambda_f(v(v-1))), \) The source pulses and the case with no pulsing are included in the figure as well.

\[ \begin{array}{c}
\text{pulsed} \\
\text{steady} \\
\text{source}
\end{array} \]

\[ \begin{array}{c}
0 \quad 0.002 \quad 0.004 \quad 0.006 \quad 0.008 \quad 0.01 \quad 0.012 \\
0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1
\end{array} \]

\[ \text{Fig. 6. Feynman } Y \text{-curve for square pulses. The numerical values used are found in Table 1 on page 18.} \]
3.5. The case with Gaussian pulses

It is possible to use Gaussian pulses instead of the square pulses in eq. (11). Then the time-dependent neutron source is represented by:

\[ S(t) = \sum_{n} S_n(t) = \sum_{n=0}^{\infty} \frac{\tilde{s}}{\sqrt{2\pi\sigma}} \exp \left( \frac{-\left( t - \frac{W}{2} - nT_0 \right)^2}{2\sigma^2} \right) \]  

(63)

where \( \sigma = W/4 \) and \( \tilde{s} = \sqrt{2\pi\sigma} \). The parameter \( \tilde{s} \) here plays the role of the source intensity, i.e. the same as \( S_0 \) for the square pulses in (11). The parameter \( \tilde{s} \) here plays the role of the source intensity, i.e. the same as \( S_0 \) for the square pulses in eq. (11). Its numerical value does not play a role in the derivation of the formulae, since it drops out from the Feynman-alpha function, which is one of the advantages of the variance-to-mean method. Hence its value was chosen such that the maximum value of the pulse function is unity. This choice has no other motivation than easy comparison of the source forms in the plots. Likewise, the choice of \( \sigma = W/4 \) is also arbitrary, and again was made in order that the square and Gaussian pulses be comparable. One representation of such a Gaussian train, with its square pulse companion with the same repetition frequency and corresponding width \( W \), is shown in Fig. 7.

The Laplace transform of \( S_n(t) \) is then

\[ S_n(s) = \int_{0}^{\infty} e^{-st} S_n(t) dt = e^{-nst} \int_{0}^{\infty} S_0(t)e^{-st} dt; \quad Re\{s\} > 0. \]  

(64)

The task is hence to calculate the integral for the term \( n = 0 \). This can be simplified as follows:
As is seen, the lower limit of the integral was extended to minus infinity. The error committed by this step is rather small, given the fast decay of the Gauss function. This may not be so obvious when it is expressed in terms of the very first pulse, which starts close to the origin. However, for the later pulses it becomes a better and better approximation. Since the Feynman-alpha measurement relates to the stationary case, i.e. times long after the switching on the source, the error committed by this approximation is indeed negligible. In return, it leads to a compact analytic form.

Thus, the Laplace transform of the sum in (65) is equal to

\[ S(s) = \frac{\hat{s} e^{-sT_0}}{1 - e^{sT_0} s + \alpha} \]  \hspace{1cm} (66)

According to eq. (20) and eq. (21) we have:

\[ \tilde{N}(s) = \frac{-sW + (\sigma s)^2}{2} e^{-sW} \]  \hspace{1cm} (67)

For the inversion of eq. (67) we note that the singularities are of the same three types as in eq. (23). We write the inverse Laplace transform, in a similar manner as for the square pulse, as

\[ c(t) \equiv \operatorname{Res}_{s=0} \frac{\hat{s} e^{st}}{1 - e^{-sT_0} (s + \alpha)} e^{\frac{-sW + (\sigma s)^2}{2}} = \frac{\hat{s}}{T_0 \alpha} \]  \hspace{1cm} (68)

\[ e(t) \equiv \operatorname{Res}_{s=-\alpha} \frac{\hat{s} e^{st}}{1 - e^{-sT_0} (s + \alpha)} e^{\frac{-sW + (\sigma s)^2}{2}} = \frac{\hat{s} e^{-\alpha t}}{1 - e^{-\alpha T_0}} \]  \hspace{1cm} (69)

and

\[ g(t) \equiv \sum_{n} \operatorname{Res}_{s=s_n} \frac{\hat{s} e^{st}}{1 - e^{-sT_0} (s + \alpha)} e^{\frac{-sW + (\sigma s)^2}{2}} = \frac{1}{\pi} \sum_{n=1}^{\infty} (a_n \sin(\omega_n t) + b_n \cos(\omega_n t)) \]  \hspace{1cm} (70)

where

\[ a_n = \frac{2 \pi \hat{s} e^{-\frac{W}{2}}}{(\alpha T_0)^2 + (2 \pi n)^2} \left\{ \alpha T_0 \sin\left(\frac{\omega_n W}{2}\right) + 2 \pi n \cos\left(\frac{\omega_n W}{2}\right) \right\} \]  \hspace{1cm} (71)
It is obvious that when \( t \to \infty \), \( e(t) \to 0 \) and the asymptotic source induced neutron number is:

\[
\tilde{N}_a(t) = \tilde{s} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left( a_n \sin(\omega_n t) + b_n \cos(\omega_n t) \right)
\]

where \( a_n \) and \( b_n \) are as in eq. (71) and eq. (72), respectively.

Above, \( \tilde{N}_a(t) \) is written in a similar way as in the case with the square pulse, i.e.

\[
\tilde{N}_a(t) = C_s + \frac{1}{\pi} \sum_{n=1}^{\infty} \left( a_n \sin(\omega_n t) + b_n \cos(\omega_n t) \right)
\]

where \( C_s \) is a constant that depends on the pulse form. Thus, we are able to use the same formulas for the source induced detector count, eq. (41), and modified variance, eq. (61) as for the square pulse with minor modifications. So, we have:

\[
\tilde{Z}_a(T) = \frac{\tilde{s} T}{e^{-\alpha T}} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left( a_n (1 - \cos(\omega_n T)) + b_n \sin(\omega_n T) \right)
\]

and

\[
\tilde{Z}_l = \tilde{Z}_{osc}
\]

\[
\tilde{Z}_l(T) = C \left( \frac{\lambda_d}{\alpha} \right)^2 \left( \frac{\tilde{s} T}{e^{-\alpha T}} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left( a_n A_n(T) + b_n B_n(T) \right) \right)
\]

\[
+ \left( 1 - e^{-\alpha T} \right)^2 \frac{1}{\pi} \sum_{n=1}^{\infty} \left( -a_n \omega_n + 2 b_n \right)
\]

where \( A_n \) and \( B_n \) are as in eq. (51) and eq. (53), respectively. The resulting Feynman \( Y \)-function is given by eq. (62), as usual.

Fig. 8 shows the resulting curve for both the Gaussian and the square pulse. In the Figure, the function is multiplied with a factor \( \alpha^2/\left( \lambda_d^{\lambda_f} \langle \nu(v - 1) \rangle \right) \). The source pulses and the case with no pulsing are included in the Figure as well. The Figure shows that the \( Y \)-curve due to the Gaussian source shape is smoother than the one due to the square counterpart. This is because a Gaussian pulse it does not contain sharp edges (discontinuous derivatives) as the square function.
4. Convergence of the Feynman $Y$-function

In the numerical work that follows, in this and the following Section, the following numerical values will be used:

\[
W = 0.0005, \quad T_0 = 0.004, \quad \sigma = W/4, \quad \tilde{s} = \sigma \sqrt{2\pi}
\]

![Figure 8. Feynman $Y$-curve for Gaussian and square pulse. The numerical values included are $W = 0.0005, T_0 = 0.004, \sigma = W/4, \tilde{s} = \sigma \sqrt{2\pi}$](image)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_0$</td>
<td>$2 \cdot 10^{-3}$ s</td>
</tr>
<tr>
<td>$W$</td>
<td>$5 \cdot 10^{-4}$ s</td>
</tr>
<tr>
<td>$S_0$</td>
<td>1 n/s</td>
</tr>
<tr>
<td>$k$</td>
<td>0.95</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>$5 \cdot 10^{-5}$ s</td>
</tr>
</tbody>
</table>

Table 1: Numerical parameters

With the above data, one obtains $\alpha = -\rho / \Lambda = 1052, \ 6 \ s^{-1}$

It is important to know how many terms in the sums in eq. (61) are necessary for adequate results. One part of the asymptotic source induced detector count, $\hat{Z}(t)$, given in eq. (41) is oscillating. However, the sum $\hat{Z}_{osc}$ converges relatively fast, see Fig. 9. In this figure, $\hat{Z}_{osc}$ is plotted for increasing number, $n$, of terms in the sum, for some arbitrary values of time $t$. For $n=5$ we get a relatively good approximation of $\hat{Z}_{osc}$. 
For the two sums in $\tilde{\mu}_z(T)$, in eq. (61) it is suitable to normalize the resulting sums with the asymptotic value i.e. the value which includes $n = 30$ terms in the sum. This makes it easier to evaluate for which $n$ the approximation is good enough. Examples of the non-normalized sums are shown in Fig. 10. In this figure the first sum in $\tilde{\mu}_z(T)$ is plotted.

**Fig. 9.** Convergence of the oscillating part, $\tilde{Z}_{osci}$, the inset magnifies the behaviour near zero.

**Fig. 10.** Convergence of the first sum in $\tilde{\mu}_z(T)$, non-normalized values
The first sum in $\tilde{\mu}_z(T)$, see eq. (61) is plotted in Fig. 11 for increasing number of terms, $n$, in the sum, for a few values of time $t$. The sum is normalized with the asymptotic value.

The second sum in eq. (61) is plotted in Fig. 12 for increasing number of terms, $n$, in the sum, for some arbitrary values of time $t$. The sum is normalized with the asymptotic value.

The figures above indicate that it is the second sum in $\tilde{\mu}_z(T)$ that determines the value of $n$ so the error is small enough. Therefore it is convenient to use $n = 15$.
5. Analysis of the Behaviour

In this section a number of plots of Feynman $Y$-function for a number of $W$- and $T_0$-values are presented. All curves are normalized with a factor $\alpha^2/\langle \lambda_d \lambda_f \rangle (\nu (\nu - 1))$.

The effect of the pulse width is illustrated in Figs 13 and 14 for square pulses. It is seen that the wider the pulse is, the smoother the curves become. This is not surprising, since the continuous source corresponds to the case when the pulse is as wide as the pulsing period, which gives a completely smooth curve. Fig. 14 corresponds to the pulsing with a very narrow pulse width.
pulse. Such results were obtained by other groups in Japan, and are in agreement with our results (Refs [6]-[8]).

For sake of comparison, a Feynman $Y$-curve with Gaussian pulses, with the same repetition frequency as with the square pulses above, is shown in Fig. 15. Despite that it corresponds to much narrower pulses than the case in Fig. 13, for the square pulses, the $Y$-curve is just as smooth as for a wider square pulse. This is because of the smoother character of the Gaussian pulse shape.

The effect of the repetition frequency, with a given prompt neutron time constant, is shown in Figs 16-19. The figures show that for a sufficiently high repetition frequency (Figs 16 and 17), the pulsed curve remains smooth and close to the continuous curve even for narrow pulses. The same conclusion has been drawn earlier in calculations based on delta-function pulses [7]. However, for a repetition frequency which is low compared to the reactor prompt time constant, the deviations between the pulsed and the continuous case are rather large (Figs 18 and 19). This is the case with the majority of the MUSE experiments. The only way of compensating for this would be to use pulses as wide as possible.

![Fig. 15. The resulting Feynman Y-curve for gaussian and square pulse, included numerical values are $W = 0.0005$, $T_0 = 0.002$, $\sigma = W/4$, $\tilde{\sigma} = \sigma \sqrt{2\pi}$]
Fig. 16. Feynman Y-curve for $W=0.0002$ and $T_0=0.001$

Fig. 17. Feynman Y-curve for $W=0.0005$ and $T_0=0.001$
Fig. 18. Feynman Y-curve for $W=0.001$ and $T_0=0.005$

Fig. 19. Feynman Y-curve for $W=0.0005$ and $T_0=0.005$
A similar figure, showing the Y-curve for both Gaussian and square pulses is shown in Fig. 20 below. It illustrates the already mentioned fact that the Y-curve corresponding to Gauss pulse shapes is smoother than its counterpart which is due to square pulses.

Fig. 20. The resulting Feynman Y-curve for Gaussian and square pulse, included numerical values are

\[ W = 0.0005, \ T_0 = 0.004, \ \sigma = 0.000125 \]
6. Determination of the Parameter $\alpha$ from a Simulated Measurement

It is possible to simulate a measurement using the formulas in Section 3. and adding a random noise to it, to simulate the imperfect character of an experiment. Using a MATLAB routine, *lsqcurvefit*, which solves curve-fitting problems in the least-squares sense, it is possible to estimate the value of $\alpha$. Figs 21 and 22 show Feynman $Y$-curves with simulated measurements for two different noise levels. In this study only square pulses are assumed, and the pulse repetition frequency and the pulse width are assumed to be known exactly. In reality this is not true, and a sensitivity analysis of the unfolding method to inaccuracies in those parameters should be also performed, which will be made at a later stage.

The resulting curve for the estimated $\alpha$ is included in the Figures below as well. The true $Y$-curve and the one obtained by the parameter $\alpha$ from the fitting procedure cannot be distinguished in the figures. In all figures, $\alpha_0$ denotes the original value and $\alpha_n$ denotes the estimated value. In Fig. 21 and Fig. 22 the noise level is 4 and 8 percent of the asymptotic value of the original Feynman-$Y$ curve, respectively.

In both cases, the true value $\alpha_0$ was 1053. In the case with lower noise level, the $\alpha_n$ determined from the curve fitting was 1052, while in the second case $\alpha_n$ was 1051. However, the precision of the method depends on the level of the added noise and even for the same noise strength, the particular realisation of the random noise. This is illustrated with the case shown in Fig. 23. The noise level is the same as in Fig. 22, i.e. 8 percent, but the estimated $\alpha$ is 1062.

The few cases shown here support the statement that with the formula for the deterministically pulsed Feynman-alpha measurements, the prompt neutron time constant can be estimated with curve fitting similarly to the traditional case of constant source. Test of the method with real measurement data will be reported in a forthcoming publication.

![Feynman-Y curve for a simulated measurement, the noise level is 4%](image-url)
Fig. 22. Feynman-Y curve with simulated noise, the noise level is 8%.

Fig. 23. Feynman-Y curve with simulated noise, the noise level is 8%.
7. Conclusions

Calculation of the deterministically pulsed Feynman-alpha formula with the method introduced in this report leads to a compact solution which is easy to use in numerical work. One particular advantage, demonstrated in this report, is the ease with which various forms of the pulse shape can be handled. All that is needed is a Fourier-series expansion of the asymptotic form of the source induced neutron number. Once the coefficients of this quantity are obtained, they can be substituted into a general formula for the relative variance, which was derived in the report.

The Feynman formula obtained was investigated quantitatively for various pulse parameters and even shapes. The possibility of using the formula for determining the prompt neutron time constant, and through that the reactivity, was investigated in simulations. It was found that despite the much more complicated structure of the Feynman $Y$-curve as compared to that with constant (time-independent) sources, the prompt alpha parameter can be extracted from a simulated experiment with methods of parameter fitting.

References


Paper II
Calculation of the pulsed Feynman-alpha formulae and their experimental verification

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Abstract

An effective method of calculating the pulsed Feynman-alpha formula for finite width pulses is introduced and applied in this paper. The method is suitable for calculating both the deterministic and the stochastic Feynman-alpha formulae, while also being capable of treating various pulse shapes through very similar steps and partly identical formulae. In the paper both the deterministic and the stochastic cases are treated for square and Gaussian pulses. The solutions show a very good agreement with the results of currently performed experiments by some of the authors at the Kyoto University Critical Assembly (KUCA).

The formulae obtained are also used for a quantitative evaluation of the prompt neutron decay constant from a large number of experiments made at the KUCA for a wide range of parameters such as subcritical reactivity, pulse repetition frequency and pulse width. The suitability of the formulae to determine the prompt neutron decay constant by curve fitting to the measured data was investigated. It was found that, despite the larger deviation from the traditional Feynman Y(T)-curves from the traditional ones with a constant source (i.e., larger ripples superimposed on a smooth curve), the stochastic pulsing method is superior to the deterministic one in that it yields the correct \( \alpha \) value for all subcriticalities. The deterministic method also works fine for most cases, but its application is not so straightforward.

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1. Introduction

The theory of the Feynman-alpha method with pulsed sources became interesting recently in connection with the future accelerator-driven systems (ADS). Current on-going experimental pilot projects with the aim of studying ADS basics use pulsed neutron generators as a source (Soule, 2001, 2002; Kitamura et al., 2004a,b). Even the spallation sources, planned to be used in a future full-scale ADS facility, might be operated in a pulsed mode, for technical reasons. Although there are several competing methods to be used in pulsed source experiments to determine reactivity, most notably the area-ratio method (Sjöstrand, 1956; Soule, 2001, 2002), the Feynman- or Rossi-alpha methods have always been considered as interesting complements. Hence their applicability and performance has been a matter of current interest.

In line with the above, formulae for the Feynman-alpha method with pulsed sources have been elaborated in the past few years. Several methodologies to model the physical situation have been applied, which differ in the definition of the pulse shape (including Dirac-delta descriptions), and also in how the measurement is evaluated. With respect to the latter, it is customary to distinguish between deterministic and stochastic data evaluation techniques (Degweker, 2000; Pázsit and Kuang, 2001; Ceder and Pázsit, 2003). These will also be referred to as the “deterministic (stochastic) pulsing” or “deterministic (stochastic) Feynman method”. They only differ in whether the counting gate opens in a synchronized manner with the pulsing, or randomly. Since the time-to-digital conversion is nowadays made with a high resolution, a measurement consists of the registration of the arrival time of each detector count, such that the neutron pulse trigger is also recorded. Hence a measurement can be evaluated by both the deterministic or the stochastic method.

The first papers on this subject assumed Dirac-delta pulses (Degweker, 2000; Yamane et al., 2000, 2001, 2002; Kitamura et al., 2003, 2004a,b). This method was then extended to describe the production of neutrons in a spallation source with Monte-Carlo simulations of the intranuclear cascade (Muñoz-Cobo et al., 2001). In another line, finite width pulses were considered, first with square pulses, such that the neutrons within one pulse were regarded as being independent (i.e., Poisson-distributed with a time-dependent intensity); however, the case of exponentially decaying pulses was also considered (Degweker, 2003). The deterministic Feynman-alpha formula for square pulses was derived by solving the corresponding equations for each pulse in a piecewise manner (Pázsit and Kuang, 2001). This method was clumsy in the sense that its extension to more complicated (and hence realistic) pulse shapes was not feasible. The stochastic pulsing was then solved with a much more powerful method, using Laplace transform and complex function techniques (Ceder and Pázsit, 2003). This technique was then applied also to the deterministic case (Pázsit et al., 2004), but its potential was not fully utilized, and the first results were only restricted to the reconstruction of the previous quantitative results of Pázsit and Kuang (2001).

In this paper the solution based on the Laplace transform technique is developed fully by evaluating some integrals explicitly. These were left symbolic and were...
evaluated at run-time by Mathematica during each calculation in the predecessor paper (Pázsit et al., 2004) for the deterministic case. Such a solution naturally yielded both a slow and a non-transparent algorithm. With the integrals evaluated explicitly, we were able to give compact and robust solutions.

One particular advantage is the ease with which various pulse shapes can be treated with very little extra effort. A second advantage is the possibility of giving a common treatment for both the deterministic and the stochastic pulsing in a unified frame. Due to the Gaussian form of the pulse shape (in time) of the GENEI neutron source used in the MUSE project, the Feynman-formula was derived with Gaussian pulses, in addition to the square pulses treated already earlier.

That is, altogether four different cases will be treated here: deterministic and stochastic pulsing with square and Gaussian pulses. Illustrative cases of the Feynman $Y(T)$ curves are shown, and the results discussed. One significant difference between the deterministic and stochastic case, irrespective of the pulse shape, is that in the latter, the relative magnitude of the deviations between the pulsed and the traditional case with continuous sources depends on the source strength. No such dependence exists for the deterministic case. This important aspect was overlooked so far, but it is confirmed here not only by formulae, but also by experiments. In particular, this circumstance can explain the difficulties in applying the Feynman method to the MUSE experiments.

The validity and applicability of the formulae was checked by comparisons to recent experiments. Recently, a series of experiments were performed at the Kyoto University Critical Assembly (KUCA) (see Kitamura et al., 2004b). The measurements were then processed by both the deterministic and stochastic techniques, and the corresponding $Y(T)$ curves were evaluated. These show excellent qualitative and quantitative agreement with the theoretical curves. The theoretical expressions were used to evaluate the prompt neutron decay constant from the measurements, and could be compared with the reference values (which were known from the area-ratio method). A very good agreement was found between the reference values and those fitted from the experiments for the case of the stochastic pulsing, for all measurements. With the deterministic method, still good results were obtained, but its application was slightly less straightforward, as it will be described below. The success of the stochastic method here represents a definite change in the assessment of the applicability of the two methods, compared to earlier work. In Pázsit et al. (2004), it was judged that the deterministic method is more promising. This conclusion was based mostly on application of the theory to MUSE measurements, which had poor statistical quality as compared to the measurements reported on in this paper.

The comparison and good agreement with experiments can be interpreted as a confirmation of the basic theoretical approach, even if certain further development is envisaged. These concern accounting for the fact that neutrons in a short pulse cannot be treated as fully independent, like it is made in the current method used. The corresponding development will be given in a subsequent communication.
2. General theory

The principles and the basic theory of the Feynman-alpha method are described in several publications, including the predecessors of the present work (e.g. in Pázsit and Kuang, 2001; Pázsit et al., 2004). The method is based on the measurement of the first two moments of the detector counts $\tilde{Z}(T)$ during a measurement time period $T$ in a stationary system driven with a source $S$. (As usual, the tilde indicates that we deal with a quantity induced by the source, in contrast to the neutron number, detector count, etc. due to one single source particle). The measurement yields the dependence of the relative variance, or the variance-to-mean function $\frac{\sigma^2_{\tilde{Z}}(T)}{\langle \tilde{Z}(T) \rangle}$ on measurement time. In practice, it is more customary to use the deviation of the relative variance from unity, which is called the Feynman $Y$-function:

$$Y(T) = \frac{\sigma^2_{\tilde{Z}}(T)}{\langle \tilde{Z}(T) \rangle} - 1 = \frac{\tilde{\mu}_Z(T)}{\langle \tilde{Z}(T) \rangle},$$

(1)

where

$$\tilde{\mu}_Z(T) \equiv \sigma^2_{\tilde{Z}}(T) - \langle \tilde{Z}(T) \rangle = \langle \tilde{Z}(\tilde{Z} - 1) \rangle - \langle \tilde{Z} \rangle^2.$$  

(2)

is called the modified variance. The purpose of the theory is to serve an explicit formula for the Feynman $Y$-function.

Such a formula can be calculated from a master equation, i.e., from a probability balance equation. Usually, it is more advantageous to use the so-called backward master equation, which means that one has to operate on initial variables. In this formalism, a core with a stationary source is modeled by switching on the source at $t = 0$ rather than at $t = -\infty$, and the measurement is made when $t \to \infty$. For reasons inherent in the properties of the backward master equation approach, one needs to go in two steps; first the probability of the number of neutrons in the system and detector counts during the time period $(t - T, t)$ is determined, and then from this the probability of the same quantities is calculated for the case of an external source. The joint distribution of observables due to a single particle,

$$P(N, Z, t),$$

(3)

or its probability generating function,

$$G(x, z, t) \equiv \sum_{N=0}^{\infty} \sum_{Z=0}^{\infty} x^N z^Z P(N, Z, t),$$

(4)

can be derived directly from a backward equation, and this has been done a long time ago. Here $P$ is the probability of finding $N$ neutrons at $t$ and $Z$ counts in the time interval $(t - T, t)$, due to one neutron starting the process at $t = 0$. As (3) and (4) indicate, we shall neglect the existence of the delayed neutrons in this work. They will be treated in a subsequent publication (Kitamura et al., 2005a).

The next step is to derive an expression that connects the distribution of the same observables as being due to an external source switched on at $t = 0$, and those due to a single particle, (3) or (4). Such a relationship can be derived from another master...
equation, and its solution has also been known for long. The present authors have often referred to the formula as the Bartlett formula, but the formula was known even earlier (Sevast’yanov, 1951), as it was pointed out to us recently (Pál, 2004). At the level of the generating functions, it usually reads as

$$\tilde{G}(x, z, t) = \exp \left[\int_0^t S(t') \{G(x, z, t - t') - 1\} \, dt'\right]. \quad (5)$$

The measurement time $T$ is not denoted in the above notation. The formula needs to be considered in its asymptotic form when $t \to \infty$. In the traditional case of constant source, this can be made in (5) by extending the upper limit of the integration to infinity. However, with a non-stationary (periodic) source, this is not possible directly in (5), rather at the level of the various moments when the concrete form of the source is defined.

Since we also will discuss the case of stochastic pulsing, the relationship (5) needs to be generalized (Pázsit and Kuang, 2001). This is made by the introduction of a random variable $\xi \in [0, T_0)$

$$\tilde{G}(x, z, t) = \int_0^{T_0} \tilde{G}(x, z, t|\xi) p(\xi) \, d\xi \equiv \langle \tilde{G}(x, z, t|\xi) \rangle, \quad (6)$$

where

$$\tilde{G}(x, z, t|\xi) = \exp \left[\int_0^t S(t'|\xi) \{G(x, z, t - t') - 1\} \, dt'\right], \quad (7)$$

and the $\langle \cdot \rangle$ symbol stands for the expectation value w.r.t. $\xi$ in (6). The “randomization” of the source means simply that

$$S(t|\xi) = S(t - \xi). \quad (8)$$

What regards the source, the pulsed character is formulated mathematically that the pulse train consists of a periodic sum of the same pulse shape (“pulse mother function” in recent wavelet terminology) $f(t)$ such that it is non-vanishing only within $[0, T_0)$, i.e.,

$$f(t) = \begin{cases} \geq 0, & 0 \leq t < T_0, \\ 0, & \text{otherwise}. \end{cases} \quad (9)$$

Here $f(t)$ is assumed to be normalized in some sense, whereas the intensity of the source will be described by an intensity factor $S_0$. With this and in view of (9), the source can be written as

$$S(t) = S_0 \sum_{n=0}^{\infty} f(t - nT_0) = S_0 \sum_{n=0}^{\infty} f_n(t), \quad (10)$$

and with (8) one has

$$S(t|\xi) = S_0 \sum_{n=0}^{\infty} f(t - nT_0 - \xi). \quad (11)$$
Various forms of the pulse shape \( f(t) \) will be specified later. For the discussion of the solution technique that follows below, it can be left unspecified. This will exemplify one advantage of the technique used, namely that it can be applied to various pulse shapes with roughly the same effort, and that a substantial part of the steps does not depend on the pulse shape. This means that new pulse shapes can be treated with a rather moderate extra effort, once the problem was fully solved for one particular pulse shape.

The two different pulsing methods, i.e., deterministic and stochastic, can actually be treated in a common framework by specifying the corresponding probability distributions of the random variable \( \xi \). For the stochastic pulsing, one will have

\[
p(\xi) = \frac{1}{T_0},
\]

whereas for the deterministic case it is given as

\[
p(\xi) = \delta(\xi).
\]

From (7) and (13) it is seen that the results of the deterministic case are equivalent with substituting \( \xi = 0 \) into the formulae which contain \( \xi \), whereas those of the stochastic case will require integration over the pulse period (see (12)). This latter can also be achieved relatively easily, because the kernel functions, over which the integrations need to be taken, will be given as a Fourier expansion over the same period. For this reason we will first derive generic results with the dependence on the random variable \( \xi \) retained, from which then the deterministic and stochastic cases can be obtained through the procedures described above.

For the Feynman-alpha formula one needs the first and second factorial moments of the source induced distributions. These are given, from (5)–(7), through integrals of the corresponding first two factorial moments of the single particle induced distributions. Before the treatment of the concrete cases of deterministic and stochastic pulsings with square and Gaussian pulse shapes, we list the relevant formulae. This will facilitate the overview of the flow of the calculations, and demonstrate also the fact how the various pulse shapes can be treated in a common framework.

The relevant formulae for the moments of the single particle induced distributions can be summarized as follows. For the first moments one has

\[
\langle N(t) \rangle = \frac{\partial G(x, z, t)}{\partial x} \bigg|_{x=z=1} \equiv N(t) = e^{-\lambda t},
\]

where \( \lambda \) is the prompt neutron decay constant, given by \(-\rho/\Lambda\). Here \(-\rho\) is the subcritical reactivity (to be determined in the measurement) and \(\Lambda\) the prompt neutron generation time. As indicated in Eq. (14), and in accordance with several previous publications, the notation \( \langle \ldots \rangle \) for the expectation value is omitted here and from now on for the first moment quantities, as long as it does not lead to confusion. Further
\[ \langle Z(t, T) \rangle \equiv Z(t, T) = \lambda_d \int_0^t \Delta(t', T) N(t - t') \, dt' = \begin{cases} \frac{\lambda_d}{\alpha} (1 - e^{-\alpha t}), & 0 \leq t \leq T, \\ \frac{\lambda_d}{\alpha} (e^{\alpha T} - 1) e^{-\alpha t}, & t > T, \end{cases} \]

where

\[ \Delta(t, T) = \begin{cases} 1, & 0 \leq t \leq T, \\ 0, & t > T. \end{cases} \]

The second factorial moment \( M_Z(t, T) \), defined as

\[ M_Z(t, T) = \langle Z(Z - 1) \rangle = \frac{\partial^2 G(x, z, t)}{\partial z^2} \bigg|_{x=z=1}, \]

is given by

\[ M_Z(t, T) = \int_0^t Q_Z(t', T) N(t - t') \, dt', \]

with

\[ Q_Z(t, T) = \lambda_f \langle v(v - 1) \rangle Z^2(t, T), \]

where \( Z(t, T) \) is the single-particle induced detector count, given in Eq. (15).

The corresponding formulae for the source induced factorial moments are obtained from (5) or (6) and (7) as follows. Here, we still keep notations on the random variable \( \xi \), although it can be neglected when treating the deterministic case. Expectation values w.r.t. \( \xi \) will be denoted by brackets as in Eq. (6), whereas \( \bar{N} \) and \( \tilde{Z} \) stand for the first moments and \( \tilde{M}_Z \) for the second factorial moment of the corresponding random variables. The basic formulae for the first moments are then

\[ \tilde{N}(t) = \langle \tilde{N}(t|\xi) \rangle = \left\langle \left. \frac{\partial^2 G(x, z, t|\xi)}{\partial x} \right|_{x=z=1} \right\rangle = \int_0^t \langle S(t'|\xi) \rangle N(t - t') \, dt', \]

where

\[ \tilde{N}(t|\xi) = \int_0^t S(t'|\xi) N(t - t') \, dt', \]

and

\[ \tilde{Z}(t, T) = \langle \tilde{Z}(t, T|\xi) \rangle = \left\langle \left. \frac{\partial^2 G(x, z, t|\xi)}{\partial z} \right|_{x=z=1} \right\rangle = \int_0^t \langle S(t'|\xi) \rangle Z(t - t', T) \, dt' \]

\[ = \lambda_d \int_0^t \Delta(t', T) \tilde{N}(t - t') \, dt', \]

with

\[ \tilde{Z}(t, T|\xi) = \int_0^t S(t'|\xi) Z(t - t', T) \, dt' = \lambda_d \int_0^t \Delta(t', T) \tilde{N}(t - t'|\xi) \, dt'. \]
The reason for the fact that the source induced detector count can be written in two different ways is that it is a double convolution between three different functions which, in turn, is due to a commutation relation (Bell, 1965; Muñoz-Cobo et al., 1987). By using previous definitions and expressions, one out of two convolutions can be re-denoted as an autonomous function/notation, and this can be done by two different ways. This dual representation gives a possibility to choose the form that suits the actual computations better, which fact will be utilized below.

The second factorial moment of the source induced distributions,

\[ \tilde{M}_Z(t, T) = \langle \tilde{Z}(\tilde{Z} - 1) \rangle = \left( \frac{\partial^2 G(x, z, t|\xi)}{\partial z^2} \right)_{z=1}, \tag{24} \]

can also be expressed in two equivalent ways. For simplicity and later reference, we give it here before the expected value w.r.t. \( n \) is calculated. One has then

\[ \tilde{M}_Z(t, T|\xi) = \int_0^t S(t'|\xi)M_Z(t - t', T) \, dt' + \tilde{Z}^2(t, T|\xi) \]

\[ = \int_0^t Q_Z(t', T)\tilde{N}(t - t', \xi) \, dt' + \tilde{Z}^2(t, T|\xi). \tag{25} \]

One often uses the modified variance instead, which from the definition (2) and (24) above can be written as

\[ \tilde{\mu}_Z(t, T) = \tilde{M}_Z(t, T) - \tilde{Z}^2(t, T). \tag{26} \]

From the above this leads to

\[ \tilde{\mu}_Z(t, T) = \langle \tilde{\mu}_Z(t, T|\xi) \rangle \]

\[ = \int_0^t \langle S(t'|\xi)M_Z(t - t', T) \rangle \, dt' + \langle \tilde{Z}^2(t, T|\xi) \rangle - \tilde{Z}^2(t, T) \]

\[ = \int_0^t Q_Z(t', T)\tilde{N}(t - t') \, dt' + \langle \tilde{Z}^2(t, T|\xi) \rangle - \tilde{Z}^2(t, T). \tag{27} \]

Here the term which is next last in the last two equalities, \( \langle \tilde{Z}^2(t, T|\xi) \rangle \), requires some attention. Its kernel is given by Eq. (23) which has to be squared and then the expected value w.r.t. \( \xi \) is to be taken. Due to the convolution in (23), it is easy to see that with the probability distribution as in (12), i.e., the stochastic method, the squaring and the expectation value w.r.t. \( \xi \) do not commute, hence the last two terms of (27) will be different. With the distribution of (13), i.e., the case of deterministic pulsing, those two operations commute, just as with a constant source. Hence in that case the last two terms of (27) cancel each other, and then again, with neglecting notation on \( \xi \), the convolution can be rearranged to

\[ \tilde{\mu}_Z(t, T) = \int_0^t S(t')M_Z(t - t') \, dt' = \int_0^t Q_Z(t', T)\tilde{N}(t - t') \, dt', \tag{28} \]

with \( Q_Z(t, T) \) being given by (19). This is what is used in principle in the deterministic method.
3. General calculation of the variance-to-mean with arbitrary pulse shapes and pulsing methods

As was seen in the previous section, the task is to calculate the kernels $\tilde{N}(t|\xi), \tilde{Z}(t, T|\xi)$, and with their help the second order quantities $M_Z(t, T|\xi)$ and/or $\langle \tilde{\mu}_Z(t, T|\xi) \rangle$. These need to be obtained for the case $t \to \infty$ and then the expectation value of these quantities w.r.t. $\xi$ needs to be taken. The formal solution for these can be given for arbitrary pulse shapes, with the pulse shape only affecting certain terms of a Fourier expansion. In this section the pulse will be described by its “mother function” $f(t)$, as defined in (9).

The starting point is the Laplace transform of Eq. (21), which reads as

$$\tilde{N}(s|\xi) = S(s|\xi)N(s).$$  \hfill (29)

By virtue of (10) and (11) (or (8)) and (14), Eq. (29) can be written as

$$\tilde{N}(s|\xi) = \frac{S(s|\xi)}{s + \alpha} = \frac{S_0 e^{-\xi s} f(s)}{(s + \alpha)(1 - e^{-sT_0})}. \hfill (30)$$

Here, $f(s)$ is not a full Laplace transform, rather it is defined as

$$f(s) \equiv \int_0^{T_0} f(t) e^{-st} \, dt,$$  \hfill (31)

whereas the term $(1 - e^{-sT_0})$ in the denominator arises from the summing up of the geometric series, given rise by the shifted integrals for the terms $f(t - nT_0)$. Since the function $f(s)$ does not have any singularities of its own, for times $t > T_0$, the inverse transform of (30) is entirely determined by the residues of the function

$$\frac{e^{-\xi s} e^{st} f(s)}{(s + \alpha)(1 - e^{-sT_0})}. \hfill (32)$$

The actual form of the pulse shape only affects the numerical values of the residues, but not the residue structure, which is as follows:

1. A pole at $s = -\alpha$, which describes the transient after switching on the source at $t = 0$.
2. A pole at $s = 0$; the corresponding residue gives the asymptotic mean value of the oscillating function $\tilde{N}(t)$.
3. An infinite number of complex conjugate roots on the imaginary axis at the values $s = \pm i\omega_n$, yielding harmonic functions in the time domain, representing a Fourier series expansion of the oscillating part of $\tilde{N}(t)$ in the form $\sin(\omega_n t)$ and $\cos(\omega_n t)$. Here the notation

$$\omega_n \equiv \frac{2n\pi}{T_0}, \quad n = 1, 2, \ldots$$  \hfill (33)

was introduced.

Because we are only interested in the asymptotic values, the terms arising from the residue at $s = -\alpha$ can be neglected. The residue at $s = 0$ is simply given as
\[ \text{Res}_{s=0} \frac{e^{-s^2}e^{st}f(s)}{(s + \alpha)(1 - e^{-sT_0})} \equiv a_0 = \frac{f(s = 0)}{\alpha T_0}, \]  

whereas \( \text{Res}\{i\omega_n\} + \text{Res}\{-i\omega_n\} \) can be written as

\[ \begin{align*}
\text{Res}_{s=i\omega_n} & \frac{e^{-s^2}e^{st}f(s)}{(s + \alpha)(1 - e^{-sT_0})} + \text{Res}_{s=-i\omega_n} \frac{e^{-s^2}e^{st}f(s)}{(s + \alpha)(1 - e^{-sT_0})} \\
& = \{a_n \sin(\omega_n t) + b_n \cos(\omega_n t)\} \cos(\omega_n \xi) + \{b_n \sin(\omega_n t) - a_n \cos(\omega_n t)\} \\
& \times \sin(\omega_n \xi),
\end{align*} \]

where the parameters \( a_n \) and \( b_n \) are defined as

\[ a_n = \frac{2}{T_0} \frac{\omega_n \Re\{f(i\omega_n)\} - \alpha \Im\{f(i\omega_n)\}}{\omega_n^2 + \alpha^2}, \]

and

\[ b_n = \frac{2}{T_0} \frac{\alpha \Re\{f(i\omega_n)\} + \omega_n \Im\{f(i\omega_n)\}}{\omega_n^2 + \alpha^2}. \]

Then, \( \tilde{N}(t|\xi) \) is given as

\[ \begin{align*}
\tilde{N}(t|\xi) &= S_0 \left\{ a_0 + \sum_{n=1}^{\infty} \{\{a_n \sin(\omega_n t) + b_n \cos(\omega_n t)\} \cos(\omega_n \xi) \\
& + \{b_n \sin(\omega_n t) - a_n \cos(\omega_n t)\} \sin(\omega_n \xi)\} \right\}.
\end{align*} \]

The actual value of the parameters \( a_0, a_n \) and \( b_n \) will depend on the pulse form. Formally, however, all subsequent formulae including the variance-to-mean can be calculated formally from the above expression.

Eq. (38) shows that in the stochastic case, where averaging w.r.t. \( \xi \) incur integrations of the trigonometric functions over their period, all oscillating terms disappear in \( \tilde{N}(t) \), as is expected and as was already shown in previous works. In the deterministic case the oscillating terms remain present.

For the calculation of the asymptotic value of the detector counts

\[ \tilde{Z}(T|\xi) = \lim_{t \to \infty} \tilde{Z}(t, T|\xi), \]

the last equality of Eq. (23) is the more suitable, by simply integrating (38) term by term. The execution of the limit \( t \to \infty \) requires, however, some attention. Since in case of the deterministic pulsing the integrand is oscillating, the limit does not exist for an arbitrary \( t \). However, deterministic pulsing means per definition that the condition \( t = KT_0 + T \) has to be fulfilled, and it is obvious that letting \( t \to \infty \) equals to letting \( K \to \infty \). This procedure is not necessary for the stochastic pulsing, where the asymptotic number of neutrons becomes a constant after the averaging w.r.t. \( \xi \), but for the sake of uniform treatment, the limit will be calculated for both cases according to the above. With that, we have:
\[
\tilde{Z}(T|\xi) = \lambda_d \int_0^{KT_0 + T} \Delta(t', T) \tilde{N}(t - t'|\xi) \, dt' = \lambda_d \int_0^{KT_0 + T} \tilde{N}(t|\xi) \, dt.
\]

The last step above results from the periodic character of \( \tilde{N}(t) \). Performing the integral will yield the result

\[
\tilde{Z}(T|\xi) = S_0 \lambda_d [a_0 T + \sum_{n=1}^{\infty} \{ \mathcal{C}_n(T) \cos(\omega_n \xi) + \mathcal{S}_n(T) \sin(\omega_n \xi) \}],
\]

where

\[
\mathcal{C}_n(T) \equiv \frac{a_n \{1 - \cos(\omega_n T)\}}{\omega_n},
\]

\[
\mathcal{S}_n(T) \equiv \frac{b_n \{1 - \cos(\omega_n T)\} - a_n \sin(\omega_n T)}{\omega_n}.
\]

This is the general expression for arbitrary pulse shapes and pulsing techniques. The case of the deterministic pulsing is then obtained by substituting \( \xi = 0 \), whereas the stochastic pulsing is obtained by integrating (40) between zero and \( T_0 \) w.r.t. \( \xi \) which leads to the disappearance of all oscillating parts.

The calculation of the second order moments goes on similar lines. One finds it again that, due to the simple explicit expression for \( \tilde{N}(t|\xi) \), it is more convenient to use the last identity of (25) or (28). We shall use here the last equality of (25), where, from (15) and (19), one has

\[
Q(T,t) \equiv \begin{cases} 
\frac{\lambda_d^2 \lambda_f \langle v(v-1) \rangle}{\alpha^2} (1 - e^{-\alpha t})^2, & 0 < t \leq T, \\
\frac{\lambda_d^2 \lambda_f \langle v(v-1) \rangle}{\alpha^2} (e^{\alpha T} - 1)^2 e^{-2\alpha t}, & T < t.
\end{cases}
\]

As (43) shows, one needs to evaluate the integrals over trigonometric functions multiplied by the zeroth, first and second power of \( e^{-\alpha t} \). These integrals can be jointly evaluated in a formal manner. The execution of the limit \( t \to \infty \) is performed the same way as in connection with Eq. (39). After lengthy but straightforward calculations, the result is given in the following form:

\[
\tilde{\mu_Z}(T|\xi) = S_0 \lambda_d^2 \lambda_f \langle v(v-1) \rangle \frac{a_0 T \left(1 - \frac{1 - e^{-\alpha T}}{\alpha T}\right)}{\alpha^2} + S_0 \lambda_d^2 \lambda_f \langle v(v-1) \rangle \left(1 - e^{-\alpha T}\right)^2 \frac{a_n (-\omega_n) + b_n (2\alpha)}{\omega_n^2 + (2\alpha)^2}
\]

\[
\times \sum_{n=1}^{\infty} \{a_n \mathcal{C}_n(T) + b_n \mathcal{S}_n(T) + (1 - e^{-\alpha T}) \frac{a_n (-\omega_n) + b_n (2\alpha)}{\omega_n^2 + (2\alpha)^2}\}
\]

\[
\times \cos(\omega_n \xi) + \frac{S_0 \lambda_d^2 \lambda_f \langle v(v-1) \rangle}{\alpha^2} \sum_{n=1}^{\infty} \{b_n \mathcal{C}_n(T) - a_n \mathcal{S}_n(T)\}
\]

\[
+ \left(1 - e^{-\alpha T}\right)^2 \frac{b_n (-\omega_n) - a_n (2\alpha)}{\omega_n^2 + (2\alpha)^2}\} \sin(\omega_n \xi) + \tilde{Z}^2(T|\xi) - \tilde{Z}^2(T).
\]
Here the following notations were introduced:

\[ \mathcal{A}_n(T) \equiv p_n(0, T) - 2p_n(z, T) + p_n(2z, T), \]
with

\[ p_n(z, T) \equiv e^{-zT} \int_0^T e^{zt} \sin(\omega_n t) \, dt = \frac{z \sin(\omega_n T) + \omega_n \{e^{-zT} - \cos(\omega_n T)\}}{\omega_n^2 + z^2}, \]

and

\[ \mathcal{B}_n(T) \equiv q_n(0, T) - 2q_n(z, T) + q_n(2z, T), \]
with

\[ q_n(z, T) \equiv e^{-zT} \int_0^T e^{zt} \cos(\omega_n t) \, dt = \frac{\omega_n \sin(\omega_n T) - z\{e^{-zT} - \cos(\omega_n T)\}}{\omega_n^2 + z^2}. \]

Eqs. (40)--(42) and (44)--(48), together with (1) and (2), serve as the formal solution for the Feynman-alpha formulae with arbitrary pulse shapes and arbitrary pulsing statistics. In the following we turn now to the concrete cases. Regarding the pulse shapes, we shall give concrete values of the parameters \(a_0, a_n\) and \(b_n\) for two different shapes, i.e. square and Gaussian. Regarding the pulsing statistics, the cases of deterministic and stochastic pulsings will be treated.

### 4. Treatment of the pulse shapes and pulsing methods

#### 4.1. Square pulses

The sequence of square pulses is described by

\[ f(t) = H(t) - H(t - W), \]

or, through (10), as

\[ S(t) = S_0 \sum_{n=0}^{\infty} \{H(t - nT) - H(t - nT - W)\}, \]

where \(H\) is Heaviside’s step function, \(T_0\) the pulse period, and \(W\) the pulse width. In that case, one has

\[ f(s) = \frac{1 - e^{-sW}}{s}. \]

Putting this into (34), (36) and (37) will yield the following results for \(a_0, a_n\) and \(b_n\):

\[ a_0 = \frac{W}{2T_0}, \]

\[ a_n = \frac{\omega_n \sin(\omega_n W) + z\{1 - \cos(\omega_n W)\}}{n\pi(\omega_n^2 + z^2)}, \]
and
\[ b_n = \frac{x \sin(\omega_n W) - \omega_n \{1 - \cos(\omega_n W)\}}{n\pi(\omega_n^2 + x^2)}. \]  

(54)

4.2. Gaussian pulses

In this case one has
\[ f(t) = \exp \left\{ -\frac{(t - \frac{W}{2})^2}{2\sigma^2} \right\}, \]  

(55)

and
\[ S(t) = S_0 \sum_{n=0}^{\infty} f_n(t) = S_0 \sum_{n=0}^{\infty} \exp \left\{ -\frac{(t - \frac{W}{2} - nT_0)^2}{2\sigma^2} \right\}, \]  

(56)

where the parameter \( \sigma \) has to be chosen such that the overwhelming area of the Gaussian falls within the pulse period. A good practical choice is \( \sigma = W/4 \). The normalization of the pulse shape \( f(t) \) was chosen such that the maximum value of the pulse function is unity, i.e., the same as with the square shape. This choice has no other motivation than easy comparison of the two pulse shape forms in the plots.

Strictly speaking, the form (55) does not fulfill the condition (9). One can of course append the condition that (55) is valid only for \( 0 \leq t < T_0 \) and that \( f(t) \) is zero otherwise. However, when calculating \( f(s) \) according to (31), the integration boundaries will be extended to \( \pm \infty \), in order to get an analytic result. The error of this approximation is very small since \( f(t) \) decays very fast outside \( 0 \leq t < T_0 \).

With this, \( f(s) \) is given as
\[ f(s) = \int_0^{T_0} e^{-st} \exp \left\{ -\frac{(t - \frac{W}{2})^2}{2\sigma^2} \right\} \approx \int_{-\infty}^{\infty} e^{-st} \exp \left\{ -\frac{(t - \frac{W}{2})^2}{2\sigma^2} \right\} \]  

(57)

where \( \tilde{s} = \sqrt{2\pi\sigma} \). Substituting (57) into (34), (36) and (37) yields
\[ a_0 = \frac{\tilde{s}}{2\sigma T_0}, \]  

(58)

\[ a_n = \frac{\tilde{s}\omega_n e^{\frac{\left(\omega_n e^2\right)^2}{2}}}{n\pi(\omega_n^2 + x^2)} \left\{ x \sin \left( \frac{\omega_n}{2} W \right) + \omega_n \cos \left( \frac{\omega_n}{2} W \right) \right\}, \]  

(59)

\[ b_n = \frac{\tilde{s}\omega_n e^{\frac{\left(\omega_n e^2\right)^2}{2}}}{n\pi(\omega_n^2 + x^2)} \left\{ x \cos \left( \frac{\omega_n}{2} W \right) - \omega_n \sin \left( \frac{\omega_n}{2} W \right) \right\}. \]  

(60)
The above two sets of parameters $a_0$, $a_n$ and $b_n$ can now be used in the formulae with deterministic and stochastic pulsings, derived below.

4.3. Deterministic pulsing

The treatment of the pulsing methods means taking the expected values of the expressions for the first and second moments of the detector count w.r.t. the random variable $\xi$. As mentioned earlier and is also seen from (6) and (13), for the deterministic case this is achieved by substituting $\xi = 0$ into the relevant formulae (40) and (44). This means that the terms containing the $\sin(\omega_n \xi)$ terms disappear, and likewise the last two terms of (44) cancel out. The results for the mean and the modified variance will become as follows (for simplicity, the expectation value sign, indicating the averaging w.r.t. $\xi$ will be omitted):

\[
\bar{Z}(T) = S_0 \lambda_d \left[ a_0 T + \sum_{n=1}^{\infty} a_n \frac{1 - \cos(\omega_n T)}{\omega_n} + b_n \sin(\omega_n T) \right], \quad (61)
\]

and

\[
\tilde{\mu}_Z(T) = \frac{S_0 \lambda_d^2 \lambda_f (v(v-1))}{\alpha^2} \times \left[ a_0 T \left( 1 - \frac{1 - e^{-xT}}{xT} \right) + \sum_{n=1}^{\infty} \left\{ a_n \mathcal{A}_n(T) + b_n \mathcal{B}_n(T) \right\} \right.

\left. + \left( 1 - e^{-xT} \right)^2 \sum_{n=1}^{\infty} \frac{a_n (-\omega_n) + b_n (2x)}{\omega_n^2 + (2x)^2} \right]. \quad (62)
\]

The particular cases of square and Gaussian pulses can be then obtained by substituting (52)–(54) or (58)–(60), respectively, into (61) and (62).

Since the Feynman $Y(T)$-function is given as $\bar{Y}(T) = \bar{\mu}_Z(T)/\bar{Z}(T)$, the source strength $S_0$ cancels out from the formula, similarly as in the case with continuous sources. Since both $\bar{Z}(T)$ and $\tilde{\mu}_Z(T)$ are linearly dependent on the measurement time $T$, the $Y(T)$ curve goes into saturation, again similarly as in the case with continuous sources. The formulae above also show that the oscillating deviations from the continuous Feynman curve tend to zero asymptotically with the gate time $T$.

The relative weight of the oscillations depends on the relationship between the pulse angular frequency $\omega = 2\pi/T_0$ and the prompt neutron decay constant $x$. For the case $\omega \gg x$, i.e., high pulse repetition frequency, the oscillations are small. Furthermore, with increasing pulse width $W$ the relative magnitude of the oscillations decreases. For short pulses with low pulse frequency, $\omega \leq x$, the deviations from the smooth $Y(T)$ become quite significant. The magnitude of the oscillations depends rather weakly on the level of subcriticality, through the parameter $x$ in the formulae, and it increases with increasing subcriticalities.

The above qualitative analysis is illustrated quantitatively in Fig. 1, which shows deterministic Feynman $Y(T)$ functions for various pulse frequencies, pulse forms and pulse widths. For comparison, the traditional Feynman $Y(T)$-curves are also shown, designated in the figures as “stationary” (i.e., with a constant source). All curves are
normalized to the asymptotic values. More illustrative figures for all pulse forms and pulsing methods can be found in Wright and Pázsit (2004).

4.4. Stochastic pulsing

In that case, as (13) shows, one needs to integrate (40) and (44) over the pulse period. Then all harmonic functions disappear in (40), and also the ones seen explicitly in (44). The only non-trivial case is the integration of the second last term in (44), \( \bar{Z}(T|\xi) \), which contains second order products of the sine and cosine functions, i.e., requires the evaluation of

\[
\frac{1}{T_0} \int_0^{T_0} \left[ \sum_{n=1}^{\infty} \left( \mathcal{C}_n(T) \cos(\omega_n \xi) + \mathcal{S}_n(T) \sin(\omega_n \xi) \right) \right]^2 d\xi.
\] (63)

However, substituting the definitions of \( \mathcal{C}_n \) and \( \mathcal{S}_n \), and making use of the orthogonality relationships between the trigonometric functions, this integral is readily shown to be equal to
\[
\sum_{n=1}^{\infty} \frac{\mathcal{C}_2^n(T) + \mathcal{S}_2^n(T)}{2} = 2 \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{\omega_n^2} \sin^2 \left( \frac{\omega_n}{2} T \right).
\] 

(64)

Because the expected value of the detector counts is an especially simple smooth function for the case of stochastic pulsing, i.e.,

\[
\tilde{Z}(T) = S_0 \lambda_d a_0 T,
\]

(65)

it is practical to give directly the \( Y(T) \) function. It will have the relatively simple form

\[
Y(T) = \frac{\lambda_d \lambda_f \langle \nu(\nu - 1) \rangle}{x^2} \left( 1 - \frac{1 - e^{-xT}}{xT} \right) + \frac{2S_0 \lambda_d}{a_0 T} \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{\omega_n^2} \sin^2 \left( \frac{\omega_n}{2} T \right).
\]

(66)

Again, the particular cases of square and Gaussian pulses are obtained by substituting (52)–(54) or (58)–(60), respectively, into (66). This formula, for the specific case of square pulses, has been already calculated by Ceder and Pázsit (2003), although with a different and less general solution method which explicitly utilized the properties of the square pulse.

Although the Feynman \( Y(T) \) function for stochastic pulsing has hence appeared in the literature earlier, some of its important properties were not discussed. One significant difference compared to the deterministic case concerns the relative weight of the oscillating part to the smooth part. Most notably, the oscillating part is linear in the source strength, that is the source strength does not disappear from the relative variance. This is a clear consequence of the “randomizing” of the pulse, which leads to a qualitatively different dispersion of the source neutrons, as remarked in Pázsit et al. (2004); and this property is then transferred to the statistics of the neutron chain and that of the detector counts. It is interesting that the Diven factor of fission, which is present in all formulae of the continuous source and deterministic pulsing, is absent from the oscillating part of the Feynman formula of the stochastic pulsing. This can be expressed as if the oscillating part is controlled by the source statistics instead of the statistics of the multiplication of the fission chain.

This also means that with a strong source, the relative oscillations become large. This is presumably the case in the MUSE experiments, where the evaluation of the stochastic Feynman measurements showed a very large oscillating part (Rugama et al., 2004).

A second feature is that the smooth part of (66) is proportional to \( 1/x^2 \), whereas the oscillating part behaves approximately as \( 1/x \), at least for small pulse widths. This means that the relative weight of the oscillating part will increase with increasing subcriticalities. This can be interpreted to be a consequence that the oscillating part is influenced mostly by the source properties, whose significance increases in deep subcritical systems.

Some representative calculated values of the stochastic pulsed Feynman-alpha curves are shown in Fig. 2, to illustrate the above qualitative analysis.
5. Experimental

A large series of Feynman-alpha measurements were made at the KUCA reactor (Kyoto University Critical Assembly) by using a pulsed neutron generator. The measurements were made with various subcriticality levels, by various pulse repetition frequencies (or periods), and pulse widths. Four subcritical reactivities will be analyzed in the paper, namely 0.65$, 1.30$, 2.07$, and 2.72$, which correspond to alpha values $266 \pm 2 \text{s}/C_0$, $369 \pm 3 \text{s}/C_0$, $494 \pm 3 \text{s}/C_0$, and $598 \pm 4 \text{s}/C_0$, respectively. These reference reactivities and corresponding alpha values were determined by the rod drop and the pulsed neutron technique, respectively. The pulse periods $T_0$ used were about 1, 10 and 20 ms. The $1/\alpha$ values varied between 4 and 1.7 ms. Especially with the longest $1/\alpha$ value, the fastest pulsing (1 ms repetition time) led to a Feynman $Y$-curve which did hardly deviate at all from a continuous Feynman curve. Therefore, in this analysis only the lowest repetition frequency, corresponding to the 20 ms pulse period, will be discussed. The pulse width used is 2 $\mu$s throughout. More details of the measurements can be found in Kitamura et al. (2004b).

Fig. 2. Theoretical Feynman $Y$-curves for stochastic pulsing, with square pulses.
The data could be evaluated both by the deterministic and the stochastic methods. This gave us a possibility to test the above derived formulae both qualitatively and quantitatively, by trying to determine the prompt neutron decay constant with fitting the above formulae to the measured $Y$-curves. The analysis will be described for the deterministic and stochastic pulsings below. Corresponding to the known pulse properties of the neutron generator used, a square pulse form with a width $W = 2 \mu s$ was used in the curve fitting.

Fig. 3 shows the measurement results for the four different subcriticality levels for the case of deterministic pulsing, for the case of $T_0 = 20$ ms. The experimentally determined $Y$ values are shown as symbols together with the experimental errors, and the fitted curves are shown as continuous lines. These fitted curves were obtained by the least square fitting calculation where the number of oscillating terms used in the expansion was 50. The reference alpha values and the ones obtained from the curve fitting are shown in all four sub-figures. It is seen that for the deep subcriticalities, the fitted values
obtain with the reference ones quite well. Correspondingly, the fitted $Y(T)$ curves agree well with the experimental ones, although they do not coincide completely. With weak subcriticalities, the fitting procedure yields alpha values and fitted curves that deviate from the reference alpha value and measured curve appreciably.

Fig. 4 shows the same cases for the stochastic pulsing. For determining the alpha values, it was found that the least square fitting calculation with the first 10 oscillating terms was sufficient. It can be seen that both the fitted alpha values and the curves agree with the reference alpha values and the measured curves quite well, for all subcriticalities. This is in a sharp contrast with the previous conclusions (e.g., Pázsit et al., 2004) where the stochastic method was judged to be less suitable, due to the larger oscillations of the $Y(T)$ curve around (or rather above) the continuous one.

The reference reactivity and alpha values, and the fitted alpha values from both the deterministic and the stochastic pulsings are shown in Table 1.

![Fig. 4. Measured and fitted results for the stochastic pulsing.](image-url)
There may be several reasons for the efficiency of the stochastic method. One reason for the previous judgement, i.e., that the deterministic method is more applicable, was the comparison with the Feynman-alpha measurements in the MUSE project. Since the MASURCA reactor on which the MUSE measurements are made is a typical fast reactor core, the $1/\alpha$ values are much smaller, hence only a very few pulse periods occur before the saturation of the $Y(T)$ curve. Such a case is clearly disadvantageous for the application of the Feynman method. In the present measurements, the pulse repetition frequency is higher in comparison to the alpha value, and the statistics of the measurements is significantly better. It can also be noted that the oscillations of the $Y(T)$ curve are smaller in the present measurements than in the MUSE ones (Rugama et al., 2004). This can be explained by the fact, noted in the previous Section, that the source strength (intensity of the neutron generator) might be larger in the MUSE experiments as in the KUCA ones. With the stochastic method, the relative magnitude of the oscillations depends on both the source intensity and the level of subcriticality.

Another circumstance, when comparing the two methods is that, although the oscillations of the Feynman $Y$-curve may be larger (even much larger) for the stochastic method than for the corresponding deterministic one, yet the oscillations are quite smooth. With short pulses, the deterministic curve has sharp "edges", which has the consequence that far more many terms are needed in the numerical evaluations to reconstruct the deterministic $Y$ function in (61) and (62), than for the stochastic one, Eq. (66). This has also an effect on the accuracy of the curve fitting procedure.

6. Conclusion

The method of solving the backward master equations for the Feynman-alpha in case of pulsed sources proved to be very effective with the technique developed in this
paper. Various pulse shapes can easily be treated, and it was possible to give a common frame for the derivations in the cases of stochastic and deterministic pulsings. The results showed good agreement with a large range of experiments performed recently at the Kyoto University Critical Assembly. Unfolding of the prompt neutron decay constant by fitting the theoretical expressions to the experimental results was possible with good accuracy. Despite the larger oscillatory part, and in contrast to earlier judgements and expectations, the stochastic pulsing method proved to be definitely superior to the deterministic one, regarding the agreement with experiments and the accuracy of recovering the prompt neutron decay constant.

The flexibility of the algorithm used extends even to the treatment of the Rossi-alpha formula and to the inclusion of delayed neutrons into the formalism. Work has actually been performed in this direction, and the results will shortly be reported in forthcoming communications (Kitamura et al., 2005a,b).

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References


Paper III
Calculation of the pulsed Feynman- and Rossi-alpha formulae with delayed neutrons

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Abstract

In previous works, the authors have developed an effective solution technique for calculating the pulsed Feynman- and Rossi-alpha formulae. Through derivation of these formulae, it was shown that the technique can easily handle various pulse shapes of the pulsed neutron source. Furthermore, it was also shown that both the deterministic (i.e., synchronizing with the pulsing of neutron source) and stochastic (non-synchronizing) Feynman-alpha formulae can be obtained with this solution technique. However, for mathematical simplicity and the sake of insight, the formal derivation was performed in a model without delayed neutrons. In this paper, to demonstrate the robustness of the technique, the pulsed Feynman- and Rossi-alpha formulae were re-derived by taking one group of delayed neutrons into account. The results show that the advantages of this technique are retained even by inclusion of the delayed neutrons. Compact explicit formulae are derived for the Feynman- and Rossi-alpha methods for various pulse shapes and pulsing methods.

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1. Introduction

In recent years, the necessity of determination of the subcriticality in future accelerator driven systems (ADSs) has became obvious. Due to technical reasons, ADSs are currently planned to be operated in a pulsed mode, hence the pulsed neutron (or area-ratio) method is expected to be a strong candidate for this purpose (Sjöstrand, 1956; Simmons and King, 1958). Nevertheless, theoretical and experimental studies on the Feynman- (or variance-to-mean) and Rossi-alpha (or correlation) methods in a pulsed mode have also been pursued, since these methods are regarded as attractive complements of the pulsed neutron method (Degweker,
In previous papers of the present authors (Pázsit et al., 2005; Kitamura et al., 2005b), an effective solution technique was developed for the calculation of the pulsed Feynman- and Rossi-alpha formulae, based on the Laplace transform and complex function techniques (Ceder and Pázsit, 2003; Pázsit et al., 2004; Wright and Pázsit, 2004). One particular advantage of this technique lies in the feasibility of treating various shape functions of the pulsed neutron source. Using this technique, only minimum extra calculations are required to take various shape functions into account. Hence, if necessary, one can easily derive various formulae for different shape functions, although the pulsed Feynman- and Rossi-alpha formulae for two most probable shape functions (i.e., square and Gaussian) were already given in our previous papers. Furthermore, one finds that the solution technique has yet another advantage in that it provides a common treatment for deriving the formulae for the two variants of Feynman-alpha method, i.e., the deterministic (or synchronizing) and stochastic (or non-synchronizing) methods (Pázsit et al., 2004, 2005).

However, for mathematical simplicity and the sake of insight, our previous works were performed in a model without delayed neutrons (Pázsit et al., 2005; Kitamura et al., 2005b). At the same time, it was expected that these advantages of the solution technique would not be lost by inclusion of delayed neutrons. The main subject of this paper lies in this point. Namely, the previous treatment was extended to the case when one group of delayed neutrons is taken into account. This way also the robustness of our solution technique is demonstrated. In this paper, explicit solutions for the pulsed Feynman- and Rossi-alpha formulae will be given by rigorously taking the prompt–prompt, prompt–delayed, and delayed–delayed correlations between prompt and one group of delayed neutrons into account.

In the following section, calculation of the pulsed Feynman-alpha formula with delayed neutrons will be described. Calculation of the pulsed Rossi-alpha formula is given in Section 3. In Section 4, specific calculations with respect to square and Gaussian pulse shapes are performed. The conclusions are summarized in Section 5.

2. Calculation of pulsed Feynman-alpha formulae

2.1. General theory

In Feynman-alpha experiments, neutron counts with respect to a certain gate width are repeatedly measured, and the correlation index \( Y \) is calculated from the mean and variance of the neutron counts thus obtained (de Hoffmann, 1949; Feynman et al., 1956). The \( Y \) value is defined as variance-to-mean ratio minus unity, i.e.
\[ Y(T) \equiv \frac{\sigma_Z^2(T)}{Z(T)} - 1, \]  

(1)

where \( \bar{Z}(T) \) and \( \sigma_Z^2(T) \) are the mean and variance of neutron counts with respect to a gate width \( T \). From the gate width dependence of \( Y \) measured for various gate widths, some important reactor kinetic parameters can be evaluated.

For Feynman-alpha experiments with a pulsed neutron source, two alternatives have been developed so far, i.e., the deterministic and the stochastic pulsing methods (Pázsit et al., 2004, 2005). In the deterministic method, neutron counting is always started so as to synchronize with the pulsing of neutron source. On the other hand, in the stochastic method, this synchronization between the neutron counting and pulsing is not established.

In this section, we will derive the pulsed Feynman-alpha formulae for both the deterministic and stochastic methods by introducing a solution technique developed by the authors (Pázsit et al., 2005; Kitamura et al., 2005b). The calculations will be performed by assuming one neutron energy group and a one-point reactor approximation model throughout; also, one group of delayed neutrons will be rigorously taken into account. For this goal, a zero-power subcritical reactor system that is coupled with a pulsed neutron source is supposed, and the backward master equation technique for the following probability distribution is utilized (Pál, 1958):

\[ P(N, C, Z, T; t) \text{; } \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z=0}^{\infty} P(N, C, Z, T; t) = 1. \]  

(2)

Here, \( P(N, C, Z, T; t) \) is the probability of finding \( N \) neutrons and \( C \) delayed neutron precursors at time \( t \) in the system and detecting \( Z \) counts in the time interval \([t - T, t]\), due to one initial neutron injected at time \( t = 0 \) (see Fig. 1). The master equation for the probability generating function of \( P(N, C, Z, T; t) \), i.e.

\[ G(x, y, z, T; t) \equiv \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z=0}^{\infty} x^N y^C z^Z P(N, C, Z, T; t), \]  

(3)

is written as

![Fig. 1. Timing diagram for \( P(N, C, Z, T; t) \).](image)
\[
\frac{\partial G(x, y, z, T; t)}{\partial t} = -(\lambda_c + \lambda_f + \lambda_d)G(x, y, z, T; t) + (\lambda_c + \lambda_d)
\]
\[
+ \lambda_f \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p(n, m)G^n(x, y, z, T; t)
\]
\[
\times \left\{ \lambda \int_0^t dt' G(x, y, z, T; t') e^{-\lambda(t-t')} + ye^{-\lambda t} \right\}^m
\]
\[
+ \lambda_d (z - 1) \Delta(T; t),
\]

where \(\lambda_c\) is the probability that one neutron is captured per unit time, \(\lambda_f\) the probability that one neutron undergoes a fission reaction per unit time, \(\lambda_d\) the probability that one neutron is detected per unit time by a neutron detector, \(\lambda\) the delayed neutron time constant, and \(p(n, m)\) the probability that \(n\) prompt neutrons and \(m\) delayed neutron precursors are emitted in one fission reaction (Pázsit and Yamane, 1999). The function \(\Delta(T; t)\) in the last term of Eq. (4) is defined as

\[
\Delta(T; t) = \begin{cases} 
1, & 0 < t \leq T, \\
0, & T < t.
\end{cases}
\]

This function is introduced to express the status of the counting gate, i.e., the number of neutrons captured by the neutron detector are registered when \(\Delta(T; t)\) is equal to unity. Eq. (4) will be differentiated to derive single-particle induced moments later.

The backward master equation technique usually involves also another probability

\[
\tilde{P}(N, C, Z; T; t, t_0),
\]

which is the probability of having \(N\) neutrons and \(C\) delayed neutron precursors in the system at time \(t\) and \(Z\) counts in the time interval \([t - T, t]\), due to an extraneous neutron source switched on at time \(t = t_0\). Introducing the corresponding probability generating function,

\[
\tilde{G}(x, y, z, T; t, t_0) \equiv \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z=0}^{\infty} x^N y^C z^Z \tilde{P}(N, C, Z; T; t, t_0),
\]

one obtains the so-called “Bartlett formula” (that was originally derived by Sevast’yanov in 1951) that connects the single-particle induced probability generating function with the source-induced one (Pázsit, 1999; Sevast’yanov, 1951):

\[
\tilde{G}(x, y, z, T; t) = \exp \left[ \int_0^t dt' S(t') \{ G(x, y, z, T; t - t') - 1 \} \right],
\]

where \(S(t)\) is the intensity of extraneous neutron source. Here, the notation on \(t_0\) was suppressed, since substitution of \(t_0 = 0\) was carried out in deriving Eq. (8). To express the periodic nature of source intensity \(S(t)\), the pulse shape function of pulsed neutron source \(f(t)\) that is non-vanishing only within \(0 \leq t < T_0\) is introduced, and with its help the pulse train can be written as
Here, \( S_0 \) is the source intensity factor (constant value), and \( T_0 \) the pulse period. The use of the parameter \( S_0 \) makes it possible to normalize the pulse shape function arbitrarily.

In this section, to discuss both the deterministic and stochastic types of the pulsed Feynman-alpha formulae, a generalized expression of Eq. (8) is employed (Ceder and Pázsit, 2003; Pázsit et al., 2004, 2005). This generalization is made by introducing a random variable \( \bar{n} \) \((0 \leq \bar{n} < T_0):\)

\[
G(x, y, z; T; t) = \int_0^{T_0} \mathrm{d} \bar{n} p(\bar{n}) \bar{G}(x, y, z; T; t; t), \tag{10}
\]

with

\[
\bar{G}(x, y, z; t; \bar{n}) = \exp \left[ \int_0^t \mathrm{d} \tau \, S(\tau; \bar{n}) \{ G(x, y, z; t; t') - 1 \} \right]. \tag{11}
\]

The modified source intensity \( S(t; \bar{n}) \) is simply defined as

\[
S(t; \bar{n}) = S(t - \bar{n}). \tag{12}
\]

The treatment of the two different experimental methods will be realized by specifying the probability distribution \( p(\bar{n}) \). For the deterministic method, one will set

\[
p(\bar{n}) = \delta(\bar{n}); \quad \text{[deterministic method]}, \tag{13}
\]

with

\[
t = k T_0 + T; \quad k = 0, 1, \ldots, \tag{14}
\]

whereas for the stochastic method,

\[
p(\bar{n}) = \frac{1}{T_0}; \quad \text{[stochastic method]}. \tag{15}
\]

2.2. Single-particle induced moments

As mentioned above, one can calculate the single-particle induced moments by differentiating \( G(x, y, z; T; t) \). Here, for the later reference, the following moments will be prepared:

\[
Z(T; t) = \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z=0}^{\infty} Z P(N, C, Z; T; t) = \frac{\partial G(x, y, z, T; t)}{\partial z} \bigg|_{x=y=z=1}, \tag{16}
\]

\[
M_Z(T; t) = \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z=0}^{\infty} Z(Z - 1) P(N, C, Z; T; t) = \frac{\partial^2 G(x, y, z, T; t)}{\partial z^2} \bigg|_{x=y=z=1}. \tag{17}
\]

By substituting \( x = y = z = 1 \) after differentiating Eq. (4) with respect to \( z \), one obtains the following equation for the first-order moment \( Z(T; t):\)
\[
\frac{dZ(T; t)}{dt} = -\alpha Z(T; t) + \frac{\lambda \beta}{A} \int_0^t dt' Z(T; t') e^{-\lambda(t-t')} + \lambda_d \Delta(T; t),
\]

(18)

where the definitions of \( \alpha, \beta, \) and \( A \) are listed in Nomenclature. Using the initial condition of \( Z(T; t) \), i.e.

\[
Z(T; 0) = 0,
\]

(19)

one obtains the temporal Laplace transform of Eq. (18) as

\[
Z(T; s) = \lambda_d \Delta(T; s) \frac{s + \lambda}{s^2 + (\alpha + \lambda)s - \frac{\lambda \rho}{A}} = \lambda_d \Delta(T; s) \sum_{j=p,d} \frac{A_j}{s + \alpha_j},
\]

(20)

with

\[
\alpha_p = \frac{(\alpha + \lambda)A + \sqrt{(\alpha + \lambda)^2A^2 + 4\lambda \rho}}{2A},
\]

(21)

\[
\alpha_d = \frac{(\alpha + \lambda)A - \sqrt{(\alpha + \lambda)^2A^2 + 4\lambda \rho}}{2A},
\]

(22)

\[
A_j = \frac{\alpha_j - \lambda}{\alpha_j - \alpha_{j'}}; \quad j = p, d; \quad j' \neq j.
\]

(23)

Here, \( \alpha_p \) and \( \alpha_d \) are the roots of the inhour equation,

\[
s^2 + (\alpha + \lambda)s - \frac{\lambda \rho}{A} = 0.
\]

(24)

Hence, with the definition\(^1\)

\[
N(s) = \sum_{j=p,d} \frac{A_j}{s + \alpha_j} \quad \text{or} \quad N(t) = \sum_{j=p,d} A_j e^{-\alpha_j t},
\]

(25)

one obtains the solution of Eq. (18) as

\[
Z(T; t) = \lambda_d \int_0^t dt' \Delta(T; t') N(t - t').
\]

(26)

Performing the integral in the above equation, one obtains the result

\[
Z(T; t) = \begin{cases}
\lambda_d \sum_{j=p,d} \frac{A_j}{\alpha_j^2} (1 - e^{-\alpha_j t}), & 0 < t \leq T,
\lambda_d \sum_{j=p,d} \frac{A_j}{\alpha_j^2} (e^{\alpha_j T} - 1) e^{-\alpha_j t}, & T < t.
\end{cases}
\]

(27)

\(^1\) It is easily seen that \( N(s) \) is the temporal Laplace transform of the single-particle induced neutron number \( N(t) \), which also serves as the Green’s function of the mean and higher moments of the detector counts.
For the second-order factorial moment $M_Z(T; t)$, one obtains the following equation from Eq. (4):

$$\frac{dM_Z(T; t)}{dt} = -\alpha M_Z(T; t) + \frac{\lambda \beta}{A} \int_0^t dt' M_Z(T; t') e^{-\lambda(t-t')} + Q_Z(T; t).$$  (28)

Here, the definition of $Q_Z(T; t)$ is written as

$$Q_Z(T; t) \equiv \lambda \langle v_p(v_p - 1) \rangle Z^2(T; t) + 2\lambda \langle v_p v_d \rangle Z(T; t) \left\{ \lambda \int_0^t dt' Z(T; t') e^{-\lambda(t-t')} \right\}$$

$$+ \lambda \langle v_d(v_d - 1) \rangle \left\{ \lambda \int_0^t dt' Z(T; t') e^{-\lambda(t-t')} \right\}^2,$$  (29)

where the three terms on the right-hand side are referred to as the prompt–prompt, prompt–delayed, and delayed–delayed correlations, respectively. The latter two usually give negligible contribution to the final result, and hence they are usually neglected in the derivations of the traditional Feynman-alpha formula, but we keep them here for completeness. Using the initial condition of $M_Z(T; t)$, i.e.

$$M_Z(T; 0) = 0,$$  (30)

Eq. (28) is solved as

$$M_Z(T; t) = \int_0^t dt' Q_Z(T; t') N(t - t').$$  (31)

However, we will not perform this integral, because the later discussions will not require it.

2.3. Source-induced moments

The next step is to derive the source-induced moments due to the pulsed neutron source that is switched on at time $t = 0$. Such moment quantities are to be calculated, since pulsed Feynman-alpha experiments are performed by using the pulsed neutron source.

The source-induced moment of the first-order is derived from Eqs. (10) and (11) as

$$\tilde{Z}(T; t) \equiv \int_0^T d\xi \, p(\xi) \tilde{Z}(T; t|\xi),$$  (32)

where

$$\tilde{Z}(T; t|\xi) = \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z=0}^{\infty} \tilde{P}(N, C, Z, T; t|\xi) = \frac{\partial \tilde{G}(x, y, z, T; t|\xi)}{\partial z} \bigg|_{x=y=z=1}$$

$$= \int_0^t dt' S(t' | \xi) Z(T; t - t').$$  (33)
Because of the above expression being a double convolution, the temporal Laplace transform of Eq. (33) becomes

\[ Z(T; s|\xi) = S(s|\xi)Z(T; s) = \lambda_d \Delta(T; s)S(s|\xi)N(s), \]

(34)

where \( S(s|\xi) \) is the temporal Laplace transform of the source intensity \( S(t|\xi) \) and is written as follows (Pázsit et al., 2005):

\[ S(s|\xi) = \frac{S_0 e^{-\xi J s}}{1 - e^{-s\tau_0}}. \]

(35)

Here, \( f(s) \) is not a full Laplace transform, rather it is defined as

\[ f(s) = \int_0^{\tau_0} dt \ e^{-st} f(t). \]

(36)

Hence, if one uses

\[ \tilde{N}(s|\xi) \equiv S(s|\xi)N(s) = \frac{S_0 e^{-\xi J s}}{1 - e^{-s\tau_0}} \sum_{j=p,d} A_j \frac{1}{s + \xi_j}, \]

(37)

Eq. (33) is solved as

\[ Z(T; t|\xi) = \lambda_d \int_0^t dt' \ \Delta(T; t') \tilde{N}(t' - \xi) . \]

(38)

The Laplace inversion of Eq. (37) is readily obtained by calculating three kinds of residues at \( s = 0, s = -\xi_j (j = p,d), \) and \( s = \pm \frac{2n\pi}{T_0} (n = 1,2,\ldots) \) (Ceder and Pázsit, 2003; Pázsit et al., 2004, 2005). By taking \( t > W + \xi, \) the inversion is given as follows (Pázsit et al., 2005):

\[ \tilde{N}(t|\xi) = \frac{S_0 A f(s = 0)}{(-\rho)T_0} - S_0 \sum_{j=p,d} A_j f(s = -\xi_j) e^{-\xi_j(t - \xi)} \]

\[ + S_0 \sum_{n=1}^{\infty} \left\{ \{a_n \sin(\omega_n t) + b_n \cos(\omega_n t)\} \cos(\omega_n \xi) \right\} \]

\[ + \{a_n \sin(\omega_n t) - a_n \cos(\omega_n t)\} \sin(\omega_n \xi), \]

(39)

with

\[ \omega_n = \frac{2n\pi}{T_0}; \quad n = 1,2,\ldots \]

(40)

\[ a_n = \frac{2}{T_0} \sum_{j=p,d} A_j \frac{\omega_n \Im\{f(s = i\omega_n)\} - \xi_j \Re\{f(s = i\omega_n)\}}{\xi_j^2 + \omega_n^2}; \quad n = 1,2,\ldots, \]

(41)

\[ b_n = \frac{2}{T_0} \sum_{j=p,d} A_j \frac{\omega_n \Im\{f(s = i\omega_n)\} + \omega_n \Re\{f(s = i\omega_n)\}}{\xi_j^2 + \omega_n^2}; \quad n = 1,2,\ldots \]

(42)
Therefore, by substituting Eq. (39) into Eq. (38) and taking \( T < t \), one obtains

\[
\tilde{Z}(T; t|\zeta) = \frac{\lambda_s S_0 Af(s = 0)}{(-\rho)T_0} T - \lambda_s S_0 \sum_{j=p,d} A_j f(s = -x_j) e^{x_j T} - 1 e^{-x_j (t - \zeta)} + \lambda_s S_0 \sum_{n=1}^{\infty} \{ \mathcal{C}_n(T; t) \cos(\omega_n \tilde{\zeta}) + \mathcal{S}_n(T; t) \sin(\omega_n \tilde{\zeta}) \},
\]

where

\[
\mathcal{C}_n(T; t) = \frac{a_n}{\omega_n} [\cos(\omega_n (t - T)) - \cos(\omega_n t)] - \frac{b_n}{\omega_n} [\sin(\omega_n (t - T)) - \sin(\omega_n t)],
\]

\( n = 1, 2, \ldots \) \hspace{1cm} (44)

\[
\mathcal{S}_n(T; t) = \frac{b_n}{\omega_n} [\cos(\omega_n (t - T)) - \cos(\omega_n t)] + \frac{a_n}{\omega_n} [\sin(\omega_n (t - T)) - \sin(\omega_n t)],
\]

\( n = 1, 2, \ldots \) \hspace{1cm} (45)

For the calculation of asymptotic value of \( \tilde{Z}(T; t) \), one has to evaluate the expression

\[
\tilde{Z}(T) \equiv \lim_{t \to \infty} \tilde{Z}(T; t) = \int_0^{T_0} \int_0^t d\zeta \, p(\zeta) \tilde{Z}(T|\zeta),
\]

where

\[
\tilde{Z}(T|\zeta) \equiv \lim_{t \to \infty} \tilde{Z}(T; t|\zeta).
\]

From Eq. (43), one obtains

\[
\tilde{Z}(T|\zeta) = \frac{\lambda_s S_0 Af(s = 0)}{(-\rho)T_0} T + \lambda_s S_0 \sum_{n=1}^{\infty} \{ \mathcal{C}_n(T) \cos(\omega_n \tilde{\zeta}) + \mathcal{S}_n(T) \sin(\omega_n \tilde{\zeta}) \}.
\]

Here, to calculate \( \mathcal{C}_n(T) \) and \( \mathcal{S}_n(T) \) that are the asymptotic values of \( \mathcal{C}_n(T; t) \) and \( \mathcal{S}_n(T; t) \), we took the limit for \( k \to \infty \) after substituting Eq. (14) into Eqs. (44) and (45):

\[
\mathcal{C}_n(T) \equiv \lim_{k \to \infty} \mathcal{C}_n(T; kT_0 + T) = \frac{a_n}{\omega_n} \{ 1 - \cos(\omega_n T) \} + \frac{b_n}{\omega_n} \sin(\omega_n T),
\]

\( k \to \infty \) \hspace{1cm} (49)

\[
\mathcal{S}_n(T) \equiv \lim_{k \to \infty} \mathcal{S}_n(T; kT_0 + T) = \frac{b_n}{\omega_n} \{ 1 - \cos(\omega_n T) \} - \frac{a_n}{\omega_n} \sin(\omega_n T).
\]

In the following subsection, specific expressions of \( \tilde{Z}(T) \) for the deterministic and stochastic methods will be given.

The asymptotic value of the source-induced moment of the second-order is derived from Eqs. (10) and (11) as

\[
\tilde{M}_Z(T) \equiv \int_0^{T_0} d\zeta \, p(\zeta) \tilde{M}_Z(T|\zeta),
\]

\( k \to \infty \) \hspace{1cm} (51)
with
\[
\tilde{M}_Z(T|\xi) \equiv \lim_{t \to -\infty} \tilde{M}_Z(T; t|\xi),
\]
and
\[
\tilde{M}_Z(T; t|\xi) \equiv \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z=0}^{\infty} \text{Z}(Z-1) \tilde{P}(N, C, Z, T; t|\xi) = \frac{\partial^2 \tilde{G}(x, y, z, T; t|\xi)}{\xi^2} \bigg|_{x=y=z=1} = \int_0^t \text{d}'S(t'|\xi)M_Z(T; t-t') + \tilde{Z}^2(T; t|\xi).
\]
Due to a double convolution again, one obtains two equivalent expressions for the temporal Laplace transform of the first term of the last line of Eq. (53):
\[
S(s|\xi)M_Z(T; s) = Q_Z(T; s)\tilde{N}(s|\xi),
\]
where the definition of \(Q_Z(T; t)\) is given in Eq. (29). Hence, Eq. (52) can be written as follows:
\[
\tilde{M}_Z(T|\xi) = \lim_{t \to -\infty} \left\{ \int_0^t \text{d}'Q_Z(T; t')\tilde{N}(t-t'|\xi) \right\} + \tilde{Z}^2(T|\xi)
\]
\[
= \lim_{k \to -\infty} \left\{ \int_{kT_0}^{kT_0+T} \text{d}'Q_Z(T; t')\tilde{N}(t-t'|\xi) \right\} + \tilde{Z}^2(T|\xi),
\]
where \(\tilde{Z}(T|\xi)\) is given in Eq. (48). After lengthy but straightforward calculations, \(\tilde{M}_Z(T|\xi)\) is given as follows:
\[
\tilde{M}_Z(T|\xi) = \frac{\lambda_d S_0 Af(s = 0)}{(-\rho)T_0} \frac{T}{Y_j} \left[ 1 - \frac{1 - e^{-\gamma T}}{\gamma J} \right]
\]
\[
+ \lambda_d S_0 \sum_{n=1}^{\infty} \left\{ a_n A_n(T) + b_n B_n(T) \right\} \cos(\omega_n \xi)
\]
\[
+ \lambda_d S_0 \sum_{n=1}^{\infty} \left\{ \sum_{j=p,d} \sum_{k=p,d} \alpha_{j,k}^2 \left( 1 - e^{-\gamma_i T} \right)(1 - e^{-\gamma_k T}) \right\} \cos(\omega_n \xi)
\]
\[
\times \frac{a_n(-\omega_n) + b_n(\omega_n + \omega_k)}{(\omega_k + \omega_n)^2 + \omega_n^2} \sin(\omega_n \xi)
\]
\[
+ \lambda_d S_0 \sum_{n=1}^{\infty} \left\{ a_n A_n(T) - a_n B_n(T) \right\} \sin(\omega_n \xi)
\]
\[
+ \lambda_d S_0 \sum_{n=1}^{\infty} \left\{ \sum_{j=p,d} \sum_{k=p,d} \alpha_{j,k}^2 \left( 1 - e^{-\gamma_i T} \right)(1 - e^{-\gamma_k T}) \right\} \sin(\omega_n \xi) + \tilde{Z}^2(T|\xi),
\]
where

\[
Y_j = \frac{\hat{\lambda}_d \hat{\lambda}_f \langle v_p (v_p - 1) \rangle}{(x_j + x_{j'}) (x_j - x_{j'})} \left\{ 1 - \frac{\langle v (v - 1) \rangle}{\langle v_p (v_p - 1) \rangle} \frac{\hat{\lambda}^2}{x_j^2} \right\}; \quad j = p, d; \quad j' \neq j, \quad (57)
\]

\[
\mathcal{A}_n(T) \equiv \mathcal{A}^{(0)} p_n(0, T) - 2 \sum_{j=p, d} \mathcal{A}^{(1)} p_n(x_j, T) + \sum_{j=p, d} \sum_{k=p, d} \mathcal{A}^{(2)}_{j,k} p_n(x_j + x_k, T), \quad (58)
\]

\[
\mathcal{B}_n(T) \equiv \mathcal{B}^{(0)} q_n(0, T) - 2 \sum_{j=p, d} \mathcal{B}^{(1)} q_n(x_j, T) + \sum_{j=p, d} \sum_{k=p, d} \mathcal{B}^{(2)}_{j,k} q_n(x_j + x_k, T), \quad (59)
\]

\[
\mathcal{A}^{(0)} = \frac{\hat{\lambda}_d \hat{\lambda}_f \langle v (v - 1) \rangle}{x_p x_d} \left( \frac{\hat{\lambda}^2}{x_p} \right), \quad (60)
\]

\[
\mathcal{A}^{(1)} = \frac{\hat{\lambda}_d \hat{\lambda}_f \langle v_p (v_p - 1) \rangle}{x_f (x_j - x_{j'})} \left\{ 1 + \frac{\langle v_p v_{d} \rangle}{\langle v_p (v_p - 1) \rangle} \right\} \frac{\hat{\lambda}}{x_j} - \frac{\langle v (v - 1) \rangle}{x_j} \frac{\hat{\lambda}^2}{x_j^2}; \quad j = p, d; \quad j' \neq j, \quad (61)
\]

\[
\mathcal{A}^{(2)}_{j,k} = \frac{\hat{\lambda}_d \hat{\lambda}_f \langle v_p (v_p - 1) \rangle}{(x_j - x_{j'}) (x_k - x_{k'})} \left[ 1 - \left\{ 1 + \frac{\langle v_p v_{d} \rangle}{\langle v_p (v_p - 1) \rangle} \right\} \frac{\hat{\lambda}}{x_j} + \frac{\langle v (v - 1) \rangle}{x_j} \frac{\hat{\lambda}^2}{x_j^2} \right] + \frac{\langle v (v - 1) \rangle}{x_j x_k} \frac{\hat{\lambda}^2}{x_j^2}; \quad j = p, d; \quad j' \neq j; \quad k = p, d; \quad k' \neq k, \quad (62)
\]

\[
p_n(x, T) \equiv \frac{\alpha \sin(\omega_n T) + \omega_n \{ e^{-xT} - \cos(\omega_n T) \}}{x^2 + \omega_n^2}, \quad (63)
\]

\[
q_n(x, T) \equiv \frac{\omega_n \sin(\omega_n T) - \alpha \{ e^{-xT} - \cos(\omega_n T) \}}{x^2 + \omega_n^2}. \quad (64)
\]

Specific expressions of \( \tilde{M}_Z(T) \) for the deterministic and stochastic methods will be given in the following section.

2.4. Treatment of the two pulsing methods

As mentioned earlier, one particular advantage of the solution technique used in this paper lies in a common treatment for deriving the respective formulae for the deterministic and stochastic Feynman-alpha methods (Pázsit et al., 2005). This subsection will be devoted to confirm this feature of the present solution technique.

By substituting Eqs. (13) and (48) into Eq. (46), one obtains the first-order moment \( \tilde{Z}(T) \) for the deterministic method,

\[
\tilde{Z}(T) = \frac{\hat{\lambda}_d S_0 A f(s = 0)}{(-\rho) T_0} T + \hat{\lambda}_d S_0 \sum_{n=1}^{\infty} C_n(T); \quad \text{[deterministic method]}, \quad (65)
\]

whereas that for the stochastic method is obtained by substituting Eq. (15) instead of Eq. (13) as
\[
\tilde{Z}(T) = \frac{\lambda_d S_0 A f(s = 0)}{(-\rho) T_0} T; \quad \text{[stochastic method].} \tag{66}
\]

As could be expected, these expressions show a very close similarity to their counterparts without delayed neutrons that were obtained in our previous paper (Pázsit et al., 2005), and formally even a complete equivalence. The case of the stochastic method is exactly equivalent, while, for the deterministic method, Eq. (65) is also formally equivalent to its counterpart with no delayed neutrons. Further, even the definition of the quantity \( \mathcal{C}_n(T) \) in Eq. (49) is formally equivalent to the case of no delayed neutrons (i.e., Eq. (41) in Pázsit et al., 2005) with the difference that in the present case, the coefficients \( a_n \) and \( b_n \), i.e., Eqs. (41) and (42), consist now of two terms, the prompt and delayed terms.

On the other hand, by substituting Eqs. (13) and (56) into Eq. (55), one obtains \( \bar{M}_Z(T) \) for the deterministic method as

\[
\bar{M}_Z(T) = \frac{\lambda_d S_0 A f(s = 0)}{(-\rho) T_0} T \sum_{j=p,d} Y_j \left( 1 - \frac{1 - e^{-\gamma_j T}}{\gamma_j T} \right)
+ \lambda_d S_0 \left\{ a_n \mathcal{A}_n(T) + b_n \mathcal{B}_n(T) \right\}
+ \lambda_d S_0 \left\{ \sum_{j=p,d} \sum_{k=p,d} \gamma_j^{(2)} (1 - e^{-\gamma_j T})(1 - e^{-\gamma_k T}) \frac{a_n (-\omega_n) + b_n (\gamma_j + \gamma_k)}{(\gamma_j + \gamma_k)^2 + \omega_n^2} \right\}
+ \tilde{Z}^2(T) \quad \text{[deterministic method],} \tag{67}
\]

where \( \tilde{Z}(T) \) is that of the deterministic method, i.e., Eq. (65). Furthermore, for the stochastic method, the following result is obtained:

\[
\tilde{M}_Z(T) = \frac{\lambda_d S_0 A f(s = 0)}{(-\rho) T_0} T \sum_{j=p,d} Y_j \left( 1 - \frac{1 - e^{-\gamma_j T}}{\gamma_j T} \right) + \tilde{Z}^2(T)
+ 2 \lambda_d S_0 \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{\omega_n^2} \sin^2 \left( \frac{\omega_n T}{2} \right); \quad \text{[stochastic method],} \tag{68}
\]

where \( \tilde{Z}(T) \) is that for the stochastic method, i.e., Eq. (66). Therefore, using

\[
\sigma_Z^2(T) = \tilde{M}_Z(T) - \tilde{Z}^2(T) + \tilde{Z}(T), \tag{69}
\]

the respective Feynman-alpha formulae are finally calculated from Eq. (1) as follows:

\[
Y(T) = \frac{\lambda_d S_0 A f(s = 0)}{\tilde{Z}(T)(-\rho) T_0} T \sum_{j=p,d} Y_j \left( 1 - \frac{1 - e^{-\gamma_j T}}{\gamma_j T} \right)
+ \lambda_d S_0 \sum_{n=1}^{\infty} \left\{ a_n \mathcal{A}_n(T) + b_n \mathcal{B}_n(T) \right\}
+ \lambda_d S_0 \left\{ \sum_{j=p,d} \sum_{k=p,d} \gamma_j^{(2)} (1 - e^{-\gamma_j T})(1 - e^{-\gamma_k T}) \frac{a_n (-\omega_n) + b_n (\gamma_j + \gamma_k)}{(\gamma_j + \gamma_k)^2 + \omega_n^2} \right\}; \quad \text{[deterministic method],} \tag{70}
\]
where $\tilde{Z}(T)$ must be that of the deterministic method, i.e., Eq. (65), and

$$Y(T) = \sum_{j=p,d} Y_j \left(1 - \frac{1 - e^{-\kappa_j T}}{\kappa_j T}\right) + \frac{2 \lambda_d S_0(\rho) T_0}{\lambda f(s = 0) T} \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{\alpha_n^2} \sin^2 \left(\frac{\omega_n T}{2}\right);$$

[stochastic method].

The formal similarity with the case of only prompt neutrons is again clear and the generalization for the inclusion of delayed neutrons is clearly seen in the above formulae. For instance, in the both cases (i.e., the deterministic and stochastic methods), the Feynman $Y$-function, corresponding to a static source but now also including delayed neutrons, can be recognized as one term or factor in the expressions above. Like in the case with no delayed neutrons, the stochastic case is given as a traditional Feynman $Y$-function plus an added non-negative oscillating modulation, such that the traditional Feynman $Y$-function constitutes a lower envelope of the stochastically pulsed formula. In the case of the deterministically pulsed formula, the modulation is oscillating around zero, hence the traditional Feynman $Y$-function is confined in the oscillating one of the deterministically pulsed formula.

3. Calculation of pulsed Rossi-alpha formula

3.1. General theory

In Orndoff’s type) Rossi-alpha experiments, the prompt behavior of neutron fission chains in a subcritical nuclear system is directly observed by measuring the time distribution of neutron pulses following every triggering pulse (Orndoff, 1957). According to the traditional definition, the formula for this distribution, i.e., the Rossi-alpha formula, is obtained as the ratio of the two-point (in time) neutron detection probability $I(2)(t_1, t_2)dt_1dt_2$ to the one-point neutron detection probability $I(1)(t_1)dt_1$. The latter is the probability that one neutron is detected in an infinitesimal time interval $dt_1$ around $t_1$. On the other hand, $I(2)(t_1, t_2)dt_1dt_2$, that is often referred to as the compound neutron detection probability (de Hoffmann, 1949), means the probability that one neutron is detected in $dt_1$ around $t_1$ and another neutron is detected in $dt_2$ around $t_2$. Therefore, to derive the Rossi-alpha formula, one has to introduce the master equation technique for a two-point distribution (Pázsit and Yamane, 1999; Kitamura et al., 2005b):

$$\frac{\partial G(x, y, z_1, T_1, z_2, T_2, \tau; t)}{\partial t} = -\left(\lambda_c + \lambda_f + \lambda_d\right) G(x, y, z_1, T_1, z_2, T_2, \tau; t) + \left(\lambda_c + \lambda_d\right)$$

$$+ \lambda_f \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p(n, m) G^*(x, y, z_1, T_1, z_2, T_2, \tau; t)$$

$$\times \left\{ \lambda \int_0^t df G(x, y, z_1, T_1, z_2, T_2, \tau; t') e^{-\lambda(t-t')} + y e^{-2t} \right\}^m$$

$$+ \lambda d \sum_{i=1}^{2} (z_i - 1) \Delta_i (T_i; t),$$

(72)
where \( G(x,y,z_1,T_1,z_2,T_2,\tau;t) \) is the single-particle induced generating function of the probability \( P(N,C,Z_1,T_1,Z_2,T_2,\tau;t) \). This latter, in turn, is the probability of finding \( N \) neutrons and \( C \) delayed neutron precursors at time \( t \) and \( Z_1 \) and \( Z_2 \) counts in mutually non-overlapping time intervals \([t-T_1-\tau\delta_{i,1},t-\tau\delta_{i,1}) (i = 1,2)\) due to one initial neutron injected at time \( t = 0 \) (see Fig. 2). The functions for describing the counting gates, i.e., \( \Delta_i(T_i;t) \), are defined as follows:

\[
\Delta_i(T_i;t) = \begin{cases} 
0, & t \leq \tau\delta_{i,1}, \\
1, & \tau\delta_{i,1} < t \leq T_i + \tau\delta_{i,1}, \quad i = 1,2. \\
0, & T_i + \tau\delta_{i,1} < t,
\end{cases}
\] (73)

On the other hand, the Bartlett formula for the two-point distribution is written as (Pázsit, 1999):

\[
\tilde{G}(x,y,z_1,T_1,z_2,T_2,\tau;t) = \int_0^{T_0} d\zeta \ p(\zeta) \tilde{G}(x,y,z_1,T_1,z_2,T_2,\tau;t|\zeta)
= \frac{1}{T_0} \int_0^{T_0} d\zeta \tilde{G}(x,y,z_1,T_1,z_2,T_2,\tau;t|\zeta),
\] (74)

with

\[
\tilde{G}(x,y,z_1,T_1,z_2,T_2,\tau;t|\zeta) = \exp \left[ \int_0^t d\tau' S(t' | \zeta) \left( G(x,y,z_1,T_1,z_2,T_2,\tau;t - \tau') - 1 \right) \right].
\] (75)

Here, \( \tilde{G}(x,y,z_1,T_1,z_2,T_2,\tau;t) \) stands for the source-induced generating function of the probability \( \tilde{P}(N,C,Z_1,T_1,Z_2,T_2,\tau;t) \), which is the probability of finding \( N \) neutrons and \( C \) delayed neutron precursors at time \( t \) and \( Z_1 \) and \( Z_2 \) counts in mutually non-overlapping time intervals \([t-T_1-\tau\delta_{i,1},t-\tau\delta_{i,1}) (i = 1,2)\) due to an extraneous neutron source being switched on at time \( t = 0 \). We have to note here that the synchronization between the neutron counting and pulsing of neutron source is not established in pulsed Rossi-alpha experiments, so that substitution of \( p(\zeta) = 1/T_0 \), i.e., Eq. (15), was performed into Eq. (74).
3.2. Single-particle induced moments

With regard to the pulsed Rossi-alpha formula, the following single-particle induced moments will be prepared:

\[ Z_i(T; t) = \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z_1=0}^{\infty} \sum_{Z_2=0}^{\infty} Z_i P(N, C, Z_1, T_1, Z_2, T_2, \tau; t) \]

\[ = \left. \frac{\partial G(x, y, z_1, T_1, z_2, T_2; \tau; t)}{\partial z_i} \right|_{x=y=z_1=z_2=1}; \quad i = 1, 2, \quad (76) \]

\[ M_{Z_1Z_2}(T_1, T_2, \tau; t) = \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z_1=0}^{\infty} \sum_{Z_2=0}^{\infty} Z_1Z_2 P(N, C, Z_1, T_1, Z_2, T_2, \tau; t) \]

\[ = \left. \frac{\partial^2 G(x, y, z_1, T_1, z_2, T_2; \tau; t)}{\partial z_1 \partial z_2} \right|_{x=y=z_1=z_2=1}. \quad (77) \]

By substituting \( x = y = z_1 = z_2 = 1 \) after differentiating Eq. (72) with respect to \( z_i \), one obtains the equation for \( Z_i(T_i; t) \):

\[ \frac{dZ_i(T_i; t)}{dt} = -\alpha Z_i(T_i; t) + \frac{\lambda \beta}{A} \int_0^t dt' Z_i(T_i; t') e^{-\lambda(t-t')} + \lambda_d \Delta_i(T_i; t); \quad i = 1, 2. \quad (78) \]

Because of the initial conditions, i.e., \( Z_i(T_i; 0) = 0 \), this equation is readily solved with the result

\[ Z_i(T_i; t) = \lambda_d \int_0^t dt' \Delta_i(T_i; t') N(t - t'); \quad i = 1, 2, \quad (79) \]

where the definition of \( N(t) \) is given in Eq. (25). Hence, performing the following integral in Eq. (79), one obtains the following results:

\[ Z_i(T_i; t) = \begin{cases} 
0, & t \leq \tau \delta_{i,1}, \\
\lambda_d \sum_{j=p,d} \frac{A_j}{x_j} \left\{ 1 - e^{-s_j(t-\tau \delta_{i,1})} \right\}, & \tau \delta_{i,1} < t \leq T_i + \tau \delta_{i,1}, \\
\lambda_d \sum_{j=p,d} \frac{A_j}{x_j} (e^{s_j/T_i} - 1)e^{-s_j(t-\tau \delta_{i,1})}, & T_i + \tau \delta_{i,1} < t.
\end{cases} \quad (80) \]

On the other hand, for the second-order factorial moment \( M_{Z_1Z_2}(T_1, T_2, \tau; t) \), one obtains

\[ \frac{dM_{Z_1Z_2}(T_1, T_2, \tau; t)}{dt} = -\alpha M_{Z_1Z_2}(T_1, T_2, \tau; t) + \frac{\lambda \beta}{A} \int_0^t dt' M_{Z_1Z_2}(T_1, T_2, \tau; t') e^{-\lambda(t-t')} + Q_{Z_1Z_2}(T_1, T_2, \tau; t), \quad (81) \]
where
\[ Q_{Z_1Z_2}(T_1, T_2, \tau; t) \equiv \lambda f(v_p(v_p - 1))Z_1(T_1; t)Z_2(T_2; t) \]
\[ + \lambda f(v_p v_d)Z_1(T_1; t) \left\{ \lambda \int_0^t dt' Z_2(T_2; t')e^{-i(t-t')} \right\} \]
\[ + \lambda f(v_p v_d)Z_2(T_2; t) \left\{ \lambda \int_0^t dt' Z_1(T_1; t')e^{-i(t-t')} \right\} \]
\[ + \lambda f(v_d(v_d - 1)) \prod_{n=1}^2 \left\{ \lambda \int_0^t dt' Z_i(T_i; t')e^{-i(t-t')} \right\}. \] (82)

Hence, the initial condition \( M_{Z_1Z_2}(T_1, T_2, \tau; t) = 0 \) yields the solution of Eq. (81) as
\[ M_{Z_1Z_2}(T_1, T_2, \tau; t) = \int_0^t dt' Q_{Z_1Z_2}(T_1, T_2, \tau; t')N(t - t'). \] (83)

However, again, we will not need to perform this integral.

### 3.3. Source-induced moments

From the Bartlett formula, the first-order source-induced moments are derived as
\[ \tilde{Z}_i(T_i; t) \equiv \frac{1}{T_0} \int_0^{T_0} d\xi \tilde{Z}_i(T_i; t|\xi); \quad i = 1, 2, \] (84)
where
\[ \tilde{Z}_i(T_i; t|\xi) \equiv \sum_{N=0}^{\infty} \sum_{C=0}^{\infty} \sum_{Z_1=0}^{\infty} \sum_{Z_2=0}^{\infty} Z_i \tilde{N}(N, C, Z_1, T_1, Z_2, T_2, \tau; t|\xi) \]
\[ = \frac{\partial \tilde{G}(x, y, z_1, T_1, z_2, T_2, \tau; t|\xi)}{\partial z_i} \bigg|_{x=y=z_1=z_2=1} \]
\[ = \int_0^t dt' S(t' | \xi) Z_i(T_i; t - t'); \quad i = 1, 2. \] (85)

By virtue of a double convolution and the definition of \( \tilde{N}(s|\xi) \) in Eq. (37), Eq. (85) can be re-written as follows:
\[ \tilde{Z}_i(T_i; t|\xi) = \lambda_d \int_0^t dt' \Delta_i(T_i; t') \tilde{N}(t - t' | \xi); \quad i = 1, 2, \] (86)
where \( \tilde{N}(t|\xi) \) is given in Eq. (39). The solutions for \( T_i + \tau \delta_{i,1} < t \) are hence obtained as follows:
\[ \tilde{Z}_i(T_i; t|\xi) = \tilde{Z}(T_i; t - \tau \delta_{i,1} | \xi); \quad i = 1, 2, \] (87)
where \( \tilde{Z}(T; t|\xi) \) is given in Eq. (43). Therefore, the asymptotic values of \( \tilde{Z}_i(T_i; t) \) are calculated as
\[
\bar{Z}_i(T_i) \equiv \lim_{t \to \infty} \bar{Z}_i(T_i; t) = \lim_{t \to \infty} \left\{ \frac{1}{T_0} \int_0^{T_0} d\xi \bar{Z}_i(T_i; t|\xi) \right\} = \frac{\lambda_0 S_0 A f(s = 0)}{(-\rho) T_0} T_i; \quad i = 1, 2.
\]

(88)

As mentioned above, since \( P^{(1)}(t_1) dt_1 \) stands for the one-point neutron detection probability that one neutron is detected in \( dt_1 \) around \( t_1 \), one obtains the following result:

\[
P^{(1)}(t_1) dt_1 = \lim_{t_1 \to 0} \bar{Z}_1(T_1) = \frac{\lambda_0 S_0 A f(s = 0)}{(-\rho) T_0} dt_1.
\]

(89)

Furthermore, from the Bartlett formula, the asymptotic value of the second-order moment, \( \bar{M}_{Z_1 Z_2}(T_1, T_2, \tau) \), is calculated as

\[
\bar{M}_{Z_1 Z_2}(T_1, T_2, \tau) \equiv \lim_{t \to \infty} \left\{ \frac{1}{T_0} \int_0^{T_0} d\xi \frac{\partial^2 \bar{G}(x, y, z_1, T_1, z_2, T_2, \tau; t|\xi)}{\partial z_1 \partial z_2} \right\}_{x = y = z_1 = z_2 = 1} = \lim_{t \to \infty} \left\{ \frac{1}{T_0} \int_0^{T_0} d\xi \bar{M}_{Z_1 Z_2}(T_1, T_2, \tau; t|\xi) \right\},
\]

(90)

with

\[
\bar{M}_{Z_1 Z_2}(T_1, T_2, \tau; t|\xi) = \int_0^t dt' S(t' | \xi) M_{Z_1 Z_2}(T_1, T_2, \tau; t - t') + \bar{Z}_1(T_1; t|\xi) \bar{Z}_2(T_2; t|\xi) = \int_0^t dt' Q_{Z_1 Z_2}(T_1, T_2, \tau; t') \bar{N}(t - t' | \xi) + \bar{Z}(T_1; t - \tau | \xi) \bar{Z}(T_2; t|\xi).
\]

(91)

Since \( P^{(2)}(t_1, t_2) dt_1 dt_2 \) means the probability that one neutron is detected in \( dt_1 \) around \( t_1 \) and another neutron is detected in \( dt_2 \) around \( t_2 = t_1 + \tau \), it is obtained by letting \( T_1 \to 0 \) and \( T_2 \to 0 \) in Eq. (90):

\[
P^{(2)}(t_1, t_2) dt_1 dt_2 = \lim_{T_1, T_2 \to 0} \bar{M}_{Z_1 Z_2}(T_1, T_2, \tau).
\]

(92)

After a lengthy but straightforward calculation with Eqs. (39), (43), (82), and (90), one obtains

\[
P^{(2)}(t_1, t_2) dt_1 dt_2 = \left( \frac{\lambda_0 S_0 A f(s = 0)}{(-\rho) T_0} \right)^2 dt_1 dt_2 + \frac{\lambda_0^2 S_0^2}{2 (-\rho) T_0} \sum_{j = p, d} x_j Y_j e^{-\nu \tau} dt_1 dt_2 + \left( \frac{\lambda_0 S_0 A f(s = 0)}{(-\rho) T_0} \right)^2 dt_1 dt_2
\]

(93)

where \( Y_j \) are identical to those defined in Eq. (57). Therefore, using the probabilities given in Eqs. (89) and (93), the pulsed Rossi-alpha formula, \( R(\tau) d\tau \), becomes as follows:
where $dt_2$ was replaced with $ds$. As could be expected again, this expression shows a very close similarity to that obtained in our previous paper (Kitamura et al., 2005b).

4. Treatment of pulse shapes

In Eqs. (70), (71), and (94), we have derived the pulsed Feynman- (the deterministic and stochastic) and the Rossi-alpha formulae, respectively. However, specific parameters that depend on the pulse shape function $f(t)$ have not been determined yet. Hence, in this section, the parameters for the two most probable shape functions, i.e., square and Gaussian, will be calculated to demonstrate the feasibility of treating various shape functions of pulsed neutrons.

The sequence of square pulses is described by the pulse shape function

$$f(t) = H(t) - H(t - W),$$

where $H(t)$ is the Heaviside’s step function and $W$ the pulse width. In that case, one has

$$f(s) = \int_0^{T_0} dt \ e^{-st} f(t) = \int_0^{\infty} dt \ e^{-st} f(t) = \frac{1 - e^{-sW}}{s}.$$  

Hence, the specific parameters for square pulses are obtained as follows:

$$a_n = \sum_{j=p,d} A_j \omega_n \sin(\omega_n W) + x_j \{1 - \cos(\omega_n W)\}; \quad n = 1,2,\cdots,$$

$$b_n = \sum_{j=p,d} A_j x_j \sin(\omega_n W) - \omega_n \{1 - \cos(\omega_n W)\}; \quad n = 1,2,\cdots,$$

$$a_n^2 + b_n^2 = \frac{4(\lambda^2 + \omega_n^2)}{n^2 \pi^2(x_p^2 + \omega_n^2)(x_d^2 + \omega_n^2)} \sin^2\left(\frac{\omega_n W}{2}\right); \quad n = 1,2,\cdots$$

On the other hand, when the pulse shape is expressed as the Gaussian function, one can use

$$f(t) = \exp\left\{ -\left(\frac{t - \frac{W}{2}}{2\sigma^2}\right)^2\right\},$$
where \( \sigma \) is a parameter for the width of Gaussian pulses. In that case, the temporal Laplace transform of \( f(t) \) is given as

\[
f(s) = \int_0^{T_0} dt \ e^{-st} f(t) \approx \int_{-\infty}^{\infty} dt \ e^{-st} f(t) = \tilde{W} e^{-\frac{W^2}{2(W^2 - s^2)}},
\]

where \( \tilde{W} = \sqrt{2\pi\sigma} \). It is to be noted here that the boundaries of the above integral were extended to \( \pm \infty \) to obtain an analytic result. However, the error of this extension is very small, since the pulse shape function \( f(t) \) decays very fast (Pázsit et al., 2005). Hence, the parameters for Gaussian pulses are obtained as follows:

\[
f(s = 0) = \tilde{W},
\]

\[
a_n = \sum_{j=p,d} \frac{A_j \tilde{W} \omega_n e^{-\frac{\omega_n^2}{2}}}{n\pi(x_j^2 + \omega_n^2)} \left\{ \chi_j \sin \left( \frac{\omega_n}{2} W \right) + \omega_n \cos \left( \frac{\omega_n}{2} W \right) \right\}; \quad n = 1, 2, \ldots,
\]

\[
b_n = \sum_{j=p,d} \frac{A_j \tilde{W} \omega_n e^{-\frac{\omega_n^2}{2}}}{n\pi(x_j^2 + \omega_n^2)} \left\{ \chi_j \cos \left( \frac{\omega_n}{2} W \right) - \omega_n \sin \left( \frac{\omega_n}{2} W \right) \right\}; \quad n = 1, 2, \ldots,
\]

\[
a_n^2 + b_n^2 = \frac{\tilde{W}^2 \omega_n^2 (2 + \omega_n^2)}{n^2\pi^2(x_j^2 + \omega_n^2)(x_d^2 + \omega_n^2)} e^{-\omega_n^2 s^2}; \quad n = 1, 2, \ldots
\]

5. Conclusion

In this paper, by using a solution technique developed by the authors (Pázsit et al., 2005; Kitamura et al., 2005b), derivation of the pulsed Feynman- and Rossi-alpha formulae was performed by explicitly taking the prompt–prompt, prompt–delayed, and delayed–delayed correlations between prompt and one-group delayed neutrons into account.

Through derivation of these formulae, the following two advantages of this solution technique were demonstrated:

1. The solution technique can easily incorporate various shape functions of the pulsed neutron source into the pulsed Feynman- and Rossi-alpha formulae.
2. The solution technique can provide a common treatment for deriving the formulae for the two versions of the pulsed Feynman-alpha method, i.e., the deterministic (i.e., synchronizing with the pulsing of neutron source) and stochastic (non-synchronizing) methods.
These two advantages are identical to those demonstrated through our formal derivations that were performed in a model without delayed neutrons (Pázsit et al., 2005; Kitamura et al., 2005b). The present treatment shows that these advantages are retained even if delayed neutrons are accounted for. Compact explicit solutions were derived for various pulse shapes and pulsing methods.

Some comparison with measurements, and the use of the formulae for evaluating pulsed Feynman-alpha measurements will be given in another forthcoming publication.

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References


Neutron kinetics in subcritical cores with application to the source modulation method

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Abstract

The main objective of this paper is an investigation of the performance of the so-called source modulation technique for the measurement of reactivity in subcritical, source-driven cores. Methods of measuring reactivity by a single detector, including the source modulation method, are based on the assumption of point kinetic behaviour of the core. Hence, first, the conditions of point kinetic behaviour in subcritical source driven cores are revisited. In addition to the known conditions for such behaviour, which have an analogy to those in critical cores, some additional cases of such behaviour are found which only exist in subcritical cores. Then the performance of the source modulation technique is investigated. Somewhat surprisingly, it is found that the error of the method, arising from the deviation of the local detector signal from the amplitude factor of point kinetics, remains finite and non-zero even in the limit of exact point kinetic behaviour (e.g. with low frequencies). This is demonstrated and explained by analytical formulae. Some remedies for this shortcoming of the method are also suggested and discussed.

1 Introduction

The neutron kinetic characteristics of subcritical, source driven cores, as well as the mathematical methods to treat their temporal behaviour, are markedly different from those of critical cores. This property has received some new attention recently in connection with the increased interest in the so-called accelerator-driven systems (ADS) (Pázsit and Arzhanov, 1999; Takahashi et al., 1998; Pázsit, 2003; Cacuci, 2004; Dulla et al., 2004, 2005; Ravetto et al., 2004; Eriksson et al., 2005). As is known, such systems are planned to be used for the incineration of nuclear waste through transmutation as well as for energy production from fertile nuclides such as U-238 and Th-232, and some experi-
mental programs are being performed in support of the development of such systems (Soule et al., 2004; Imel et al., 2004; Rubbia et al., 2004).

As the above mentioned experimental programs also underline, on-line monitoring of the subcritical reactivity is one of the central operational and safety issues of a future ADS. In the MUSE project a large number of reactivity measurement methods were investigated. Since that project was performed on the research reactor MASURCA, the emphasis was on methods applicable in zero or low power systems, such as the Feynman- and the Rossi-alpha methods, and pulse decay methods, including the area ratio method of Sjöstrand (1959). However, it is obvious that at the higher operating level of a power producing ADS, zero noise methods need to be replaced by other ones, just as in an operating critical reactor inverse kinetic methods are used for reactivity measurements instead of the Feynman alpha or pulse decay methods. In an ADS, on the other hand, one can take advantage of the fact that the flux is maintained by an extraneous source, and the relationship between the static source intensity or its fluctuation on one hand, and the static flux or its fluctuation on the other hand, contains the reactivity as a parameter. The expectation is that the relative (normalized with the mean value) fluctuations of the neutron flux and the source intensity can be used for a calibration-free determination of the reactivity.

This is the basis of the so-called source modulation technique, suggested recently by Carta and D’Angelo (1999). Using the point kinetic equations, they have shown how by making measurements at both the so-called plateau frequency and at very low frequencies, both the reactivity and the effective delayed neutron fraction can be derived. Carta and D’Angelo suggest a harmonic modulation of the source intensity, but the principle they suggest is just as well applicable with a stochastic modulation or even with the inherent fluctuations of the source intensity, taken at a selected frequency or frequencies of the power spectra.

However, while the theory is derived in the point kinetic approximation, the measurement is performed by using a local neutron signal. This fact can lead to an inaccuracy of the method in case of deviations from point kinetic behaviour. This deviation can nevertheless be easily investigated and quantified in model calculations. We shall here utilize the fact that spatial and frequency behaviour of the neutron flux fluctuations in a subcritical core, induced by the fluctuations of the external source, has already been studied in detail in a simple 1-D model by Pázsit and Arzhanov (1999). Due to the frequent references to it, this paper will be referred to as Part I in the continuation. In particular, in Part I, the full space-dependent solution was given in a closed analytic form. From those results, one can use local flux values, calculated in the model, in the formula of Carta and D’Angelo and compare the value given by the formula with the true value, which is known from the core model.
The work leading to the present paper started out with the modest objective of quantifying the performance of the source modulation method as a function of subcriticality, system size and frequency. A similar analysis, although not for exactly the same reasons, was already suggested by Carta and D’Angelo, who considered an eigenfunction expansion solution to the space-dependent problem. The expectation was that, by driving the system towards point kinetic behaviour by the conditions known from both the theory of critical systems as well as from that for subcritical systems, the method would provide reactivity values that are exact in the limit.

In order to do these investigations, we also had to extend the tools used in two ways. First, regarding the conditions of point kinetic behaviour, it was found that in addition to the ones given in Part I, that are analogous to those in critical systems (low frequency or small system size, i.e. $\infty \gg 1$), there exists another possibility, which exists only in subcritical cores. This is the case of deeply subcritical systems of any size, including large ones, such that $\infty < 0$ and $|\infty| \gg 1$. This possibility was not discussed in Part I, where the kinetics was investigated only as a function of frequency, but not as a function of the system size. The other extension concerns the source modulation method. In the suggestion by Carta and D’Angelo, the plateau frequency approximation of the zero reactor transfer function was used. In order to investigate the point kinetic limit for vanishing frequencies, the method had to be extended by keeping the full frequency dependence in the formulae.

The investigation of the accuracy of the source modulation method, including the limiting case of point kinetic behaviour in the system, showed the surprising result that there is a "built-in bug" in the method which means that it never yields an exact result, not even asymptotically. It is easily seen that when the core behaves in a perfect point kinetic way, such as when formally substituting $\omega = 0$, the equation becomes singular and the reactivity drops out from the expression. One would expect that by letting $\omega \to 0$ but keeping $\omega$ finite, the error of the formula would converge to zero, such that the singularity only exists when the zero value is substituted for the frequency. However, not even this expectation is fulfilled. It will be shown by an asymptotic analysis of the relevant formulae, and illustrated in a quantitative way, that the formula of expressing the reactivity in terms of the relative flux fluctuations and relative source fluctuations has a finite, non-negligible error, even when the system behaviour asymptotically approaches point kinetics. The reactivity, as determined from the formula, is easily in error with a factor of close to two in the range of a power ADS with $k_{eff} = 0.95$ even in the case of low frequencies.

This disadvantage could, in principle, be counteracted by recalling that the point kinetic equations refer to the amplitude factor and not the flux in a given point. The amplitude factor is defined as a weighted integral (with the critical adjoint) of the flux fluctuations over the core. This integral can be
approximated with a sum over a few selected detectors if they are chosen "representatively" such as described in Karlsson and Pázsit (1998). This possibility is also investigated and it is shown that its application is only feasible by making use of detailed information on the exact flux shape and selecting detector positions freely, which in practice is not possible.

These findings constitute a warning what regards the choice of reactivity monitoring in ADS. The source modulation method, and various related methods such as the proton importance method, are currently being judged as the most suitable ones for the reactivity monitoring of an operating power ADS. The present work shows that all these methods suffer from the same intrinsic shortcoming, which means that they need to be calibrated for on-line use in each particular case by other methods, and hence are not suitable for absolute measurement of the reactivity.

2 Basic Principles

2.1 General Principles

In this work the same one-dimensional bare reactor model will be used with one group of delayed neutrons as in Part I, but with some slight change of the notations. This change is motivated by trying to apply a more consistent set of notations that provide better clarity than in Part I, for the case of the subcritical kinetics. This latter is, in a way, more involved than the kinetics of critical systems. Namely, in the latter case, i.e. when inducing neutron fluctuations by perturbing a previously critical system, one only has to distinguish between the static (critical or unperturbed) and the time-dependent (perturbed) quantities. This is usually effected by denoting the static (critical) quantities with a zero subscript whereas the time-dependent quantities have no subscript. In case of the kinetics of a subcritical core, one actually has to distinguish between three different quantities. In addition to the static (and now subcritical) equilibrium state, one has to consider not only the time-dependent quantities, but also the fundamental flux or adjoint, with corresponding buckling, which belongs to a hypothetical critical system. We have therefore chosen the convention that the static core quantities, belonging to the subcritical case, will be denoted by a subscript \( \rho \); the fundamental mode eigenfunction with a subscript zero, and the time dependent quantities still have no subscript. This notation convention is logical in the sense that in a critical system \( \rho = 0 \), and then, the static and the critical flux are identical. The only exception from the above conventions is the source, which only has a static value denoted by a zero subscript, and a time-dependent deviation, but is unaffected by the reactivity of the system.
With these notations the model of Part I can be summarized as follows. In a one-dimensional homogeneous one-group diffusion model the static equation with a steady source reads as

$$D \nabla^2 \phi(x) + (\nu \Sigma_f - \Sigma_a) \phi(x) + S_0(x) = 0.$$  \hspace{1cm} (1)

with the usual diffusion theory boundary condition

$$\phi(x_B) = 0,$$ \hspace{1cm} (2)

where $x_B$ stands for any of the two boundaries of the system. The other symbols have their usual meaning.

The system is assumed to be subcritical, i.e. $k_{eff} < 1$, and the subcriticality can be calculated from the eigenvalue equation

$$D \nabla^2 \phi_0(x) + \left( \frac{\nu \Sigma_f}{k_{eff}} - \Sigma_a \right) \phi_0(x) = 0,$$ \hspace{1cm} (3)

or

$$\nabla^2 \phi_0(x) + B^2_0 \phi_0(x) = 0,$$ \hspace{1cm} (4)

where the zero subscript denotes the critical flux (fundamental eigenvalue), and $B^2_0$ is given by

$$B^2_0 = \frac{\nu \Sigma_f/k_{eff} - \Sigma_a}{D}.$$ \hspace{1cm} (5)

Eq. (1) can be solved with the Green’s function technique by defining the equation of the static Green’s function as

$$\nabla^2 G(x, x') + B^2_0 G(x, x') + \frac{1}{D} \delta(x - x') = 0,$$ \hspace{1cm} (6)

with

$$B^2_0 = \frac{\nu \Sigma_f - \Sigma_a}{D}.$$ \hspace{1cm} (7)

The solution of Eq. (1) is then given as:

$$\phi(x) = \int G(x, x') S_0(x') \, dx'$$ \hspace{1cm} (8)

The space-time dependent fluctuations of the flux in the system are assumed to be induced by the temporal and spatial variations of the external source. These are given by the one-dimensional space-time dependent diffusion equations as

$$\frac{1}{v} \frac{\partial \phi_{\rho}(x, t)}{\partial t} = D \nabla^2 \phi_{\rho}(x, t) + [(1 - \beta) \nu \Sigma_f - \Sigma_a] \phi_{\rho}(x, t) + \lambda C(x, t) + S(x, t)$$

and

$$\frac{\partial C(t)}{\partial t} = \beta \nu \Sigma_f \phi_{\rho}(x, t) - \lambda C(x, t)$$ \hspace{1cm} (9)
with the boundary conditions
\[ \phi_{\rho}(x_B, t) = C(x_B, t) = 0 \] (10)

Inserting the time-dependent quantities split up into stationary values and fluctuations in Eq. (9), subtracting the static equations and eliminating the fluctuations of delayed neutrons by a temporal Fourier-transform, we obtain for the neutron noise in frequency domain:
\[ \nabla^2 \delta \phi(x, \omega) + B_{\rho}^2(\omega) \delta \phi(x, \omega) + \frac{1}{D} \delta S(x, \omega) = 0, \] (11)

where
\[ B_{\rho}^2(\omega) = B_{\rho}^2 \left( 1 - \frac{1}{\rho_{\infty} G_0(\omega)} \right). \] (12)

It is of interest to note that in (12), \( G_0(\omega) \) has the form of the well-known zero power reactor transfer function of a critical core, given by
\[ G_0(\omega) = \frac{1}{i\omega \left[ \Lambda + \frac{\beta}{i\omega + \lambda} \right]}, \] (13)

It is seen that, in contrast to the zero power reactor transfer function of a subcritical core (cf. Eq. (32) later), it does not contain the reactivity, and as a consequence, it tends to infinity for vanishing frequencies. However, the corresponding solution of (11) does exist even for \( \omega = 0 \), because
\[ B_{\rho}^2(\omega = 0) = B_{\rho}^2 \neq B_0^2 \] (14)

On the other hand, the material parameters appearing in (13) are that of the actual subcritical system. So the formal identity with the transfer function of a critical core only refers to the property that the \( G_0(\omega) \) of (13) diverges with \( \omega \rightarrow 0 \).

Equation (11) can also be solved by the Green’s function technique. The corresponding Green’s function equation is:
\[ \nabla^2 G(x, x', \omega) + B_{\rho}^2(\omega) G(x, x', \omega) + \frac{1}{D} \delta(x - x') = 0, \] (15)

and with this, the solution of Eq. (11) is given as
\[ \delta \phi(x, \omega) = \int G(x, x', \omega) \delta S(x', \omega) \, dx'. \] (16)
2.2 Linearised Reactor Kinetic Approximations

In this subsection we briefly summarize the principles of flux factorisation and the derivation of the point kinetic equations, as it is applied in subcritical cores, along the same lines as in Part I. The factorisation assumption into an amplitude function $P(t)$ and a shape function $\psi(x, t)$ is expressed by

$$\phi(x, t) = P(t)\psi(x, t),$$

(17)

with the normalisation condition

$$\frac{\partial}{\partial t} \int \psi(x, t)\phi_0^\dagger(x)\,dx = 0,$$

(18)

Again of interest to note is that in order to obtain point kinetic equations that are consistent with those of the critical cores, one has to use the critical adjoint function $\phi_0^\dagger(x)$ in (77) as the weight function. Here $\phi_0^\dagger(x)$ is the solution of the adjoint eigenvalue equation

$$D \nabla^2 \phi_0^\dagger(x) + \left(\frac{\nu \Sigma f}{k_{ef}} - \Sigma_a\right)\phi_0^\dagger(x) = 0.$$

(19)

Since in this paper we only use one-group diffusion theory, it is equal to the critical flux, but we keep this notation for more generality.

Assuming that the perturbation was switched on at $t = -\infty$ before which the system was in its stationary state yields

$$\phi(x, t = -\infty) = \phi_\rho(x),$$

(20)

which gives:

$$P(t = -\infty) \equiv P_0 = 1,$$

(21)

and

$$\int \psi(x, t)\phi_0^\dagger(x)\,dx = \int \phi_\rho(x)\phi_0^\dagger(x)\,dx$$

(22)

We still have a freedom to normalise the critical adjoint $\phi_0^\dagger(x)$ arbitrarily, and for practical reasons we choose the normalisation

$$\int_{-a}^{a} \phi_0^\dagger(x)\phi_\rho(x)\,dx = 1.$$  

(23)

In line with Part I, the point reactor approximation is defined as

$$\phi(x, t) = P(t)\phi_\rho(x),$$

(24)
i.e. one assumes
\[ \psi(x, t) = \phi_p(x) \quad \forall \ t. \] (25)
where \( \phi_p(x) \) is the static subcritical flux.

To derive equations for the fluctuating part, and in particular to derive the linearized point kinetic equations, we split up all the time-dependent quantities into mean values and fluctuations:

\[
\begin{align*}
\phi(x, t) &= \phi_p(x) + \delta \phi(t) \\
P(t) &= 1 + \delta P(t) \\
\psi(x, t) &= \phi_p(x) + \delta \psi(t) \\
Q(t) &= Q_0 + \delta q(t).
\end{align*}
\] (26)

Inserting Eq. (26) into Eq. (17) and neglecting the second order terms gives
\[ \delta \phi(x, t) = \phi(x) \cdot \delta P(t) + \delta \psi(x, t). \] (27)

Using the above in (9), multiplying by \( \phi_p^\dagger(x) \) and integrating, finally subtracting the static equations leads to the point kinetic equations

\[
\begin{align*}
\frac{d \delta P(t)}{dt} &= \frac{\rho - \beta}{\Lambda} \delta P(t) + \lambda \delta C(t) + \delta q(t) \\
\frac{d \delta C(t)}{dt} &= \frac{\beta}{\Lambda} \delta P(t) - \lambda \delta C(t)
\end{align*}
\] (28)

where \( \delta q(t) \) is defined as
\[ \delta q(t) = \int \frac{\delta S(x, t) \phi_p^\dagger(x)}{\phi_p(x) \phi_p^\dagger(x)} \, dx. \] (29)

The solution in the frequency domain is
\[ \delta P(\omega) = \Lambda G_p(\omega) \delta q(\omega), \] (30)

and hence
\[ \delta \phi_{pk}(x, \omega) = \Lambda G_p(\omega) \delta q(\omega) \phi_p(x). \] (31)

Here now \( G_p(\omega) \) is the zero reactor transfer function of the subcritical system:
\[ G_p(\omega) = \frac{1}{i \omega \left( \Lambda + \frac{\beta}{i \omega + \lambda} \right) - \rho}. \] (32)

Due to the appearance of the reactivity in the denominator of (32), it is seen that the point kinetic solution (31) does not diverge for \( \omega \to 0 \), as is expected.
3 Kinetics of Subcritical Cores

Two basic model perturbations were investigated in Part I: the variations in the source strength, and variations in the source impact point (source position). Out of these two, the first one corresponds to the source modulation method, hence in this paper only the variable strength source will be investigated. To give a wider survey of the performance of the point kinetic approximation, both a small system and a large system will be considered. The concepts "small" and "large" are of course arbitrary and are always to be interpreted in comparison with some characteristic length. In traditional reactor physics, a system is interpreted as large if its dimensions are much larger than the diffusion length, and small in the opposite case. Consequently, the leakage rate will be small in the large system (only a few percent compared to the number of neutrons lost in reactions per unit time) and large in a small system. For critical reactors, a small system can also be characterised as having a large $k_\infty$ such that its $\rho_\infty(\$) \gg 1$, because the condition $k_\text{eff} = 1$ establishes a monotonic one-to-one correspondence between system size and $k_\infty$ or $\rho_\infty$. Hence in all neutron kinetics studies in critical reactors in the past, the parameter $\rho_\infty(\$)$ was used to characterise the size of the system.

For subcritical systems, however, there is no such direct relationship between system size and $\rho_\infty(\$)$, since the $k_\text{eff}$ of the system is a free parameter. Hence we shall choose two system sizes that would count as small and large if they were critical. Then we shall change their $k_\text{eff}$ such that they become subcritical, by changing the macroscopic fission cross section. This procedure will nevertheless not change the ratio of the leakage rate and the absorption rate significantly, hence these systems can be still regarded as small and large, respectively, even in the subcritical state.

As in Part I, a one-dimensional bare homogeneous system will be used with boundaries at $x = \pm a$. The system is described by the material parameters $D, \Sigma_a, \nu \Sigma_f, \beta, \lambda$. These data were taken, with some modifications, from Garis et al. (1996). The two different sizes used are $a = 30 \text{ cm}$ and $a = 150 \text{ cm}$, with corresponding values $\rho_\infty(\$) = 3.401$ and $\rho_\infty(\$) = 0.136$. To obtain the desired subcriticality level for both reactor sizes, the macroscopic fission cross section was modified.

It was pointed out in Part I and also by Carta and D’Angelo (1999), that for source driven subcritical cores, the validity region of the point kinetic approximation depends, apart from the frequency of the perturbation and the system size, also on the spatial and temporal variation of the source fluctuation. In particular, in order that the point kinetic behaviour shall dominate for low frequencies or small system sizes, it is necessary that the time and space dependence of the source fluctuations is factorised into an arbitrary time func-
tion and a spatial dependence which is identical with that of the static flux. This condition is actually fulfilled with the source modulation technique. We shall here also investigate the validity of the point kinetic approximation for this particular perturbation. It has though to be kept in mind that the results below are only valid for this type of source fluctuations.

In the present one-dimensional model, the static source will be represented by an infinitely thin beam as

\[ S_0(x) = S_0 \delta(x - x_p) \]  

where \( x_p \) is the position of the beam impact point. In this case the static equation can be written as:

\[
D \nabla^2 \phi_{\rho}(x) + (\nu \Sigma_f - \Sigma_{\alpha}) \phi(x) + S_0(x) = 0 \\
\phi_{\rho}(\pm a) = 0
\]

The corresponding Green’s function is

\[
G(x, x') = \begin{cases} 
\frac{\sin[B_{\rho}(a - x') \cdot \sin[B_{\rho}(a + x)]}{DB_{\rho} \sin(2B_{\rho}a)} & x \leq x' \\
\frac{\sin[B_{\rho}(a + x') \cdot \sin[B_{\rho}(a - x)]}{DB_{\rho} \sin(2B_{\rho}a)} & x \geq x'
\end{cases}
\]

where \( B_{\rho} \) is defined by (7).

For the source (33), the static flux can be expressed as:

\[
\phi_{\rho}(x) = \int_{-a}^{+a} S_0(x') G(x, x') dx' = S_0 G(x, x_p).
\]

The adjoint critical flux is defined by equation

\[
D \nabla^2 \phi_{0}^\dagger(x) + \left(\frac{\nu \Sigma_f}{k_{\text{eff}}} - \Sigma_{\alpha}\right) \phi_{0}^\dagger(x) = 0 \\
\phi_{0}^\dagger(\pm a) = 0
\]

which has the solution

\[
\phi_{0}^\dagger(x) = C \cdot \cos(B_0 x),
\]

where C is an arbitrary factor, \( B_0 = \frac{\pi}{2a} \), and

\[
k_{\text{eff}} = \frac{\nu \Sigma_f}{DB_0^2 + \Sigma_{\alpha}} \leq k_{\infty} = \frac{\nu \Sigma_f}{\Sigma_{\alpha}}.
\]

C is derived from the normalisation condition (23).
In the frequency domain, the dynamic Green’s function is given by

\[
G(x, x', \omega) = \begin{cases} 
\frac{\sin[B(\omega)(a - x')] \cdot \sin[B(\omega)(a + x)]}{DB(\omega) \sin(2B(\omega)a)} & x \leq x' \\
\frac{\sin[B(\omega)(a + x')] \cdot \sin[B(\omega)(a - x)]}{DB(\omega) \sin(2B(\omega)a)} & x \geq x'
\end{cases}
\tag{40}
\]

The source is time-dependent and hence the source fluctuations are described by

\[
S(x, t) = S(t)\delta(x - x_p) = [S_0 + \delta S(t)]\delta(x - x_p) \tag{41}
\]

It is easy to see that the induced neutron noise in the frequency domain will be proportional to the Greens function:

\[
\delta \phi(x, \omega) = \delta S(\omega)G(x, x_p, \omega), \tag{42}
\]

where \(\delta S(\omega)\) is the Fourier transform of \(\delta S(t)\). In this case the exact dynamic transfer function for the amplitude fluctuations is

\[
\frac{\delta \phi(x, \omega)}{\delta S(\omega)} = G(x, x_p, \omega) \tag{43}
\]

The point kinetic solution in this case reads, using Eq. (31), as

\[
\delta \phi_{pk}(x, \omega) = \Lambda v G_\rho(\omega)\delta S(\omega)\phi_0^\dagger(x_p)\phi_\rho(x), \tag{44}
\]

from which the point kinetic dynamic transfer function is obtained as

\[
\frac{\delta \phi_{pk}(x, \omega)}{\delta S(\omega)} = \Lambda v G_\rho(\omega)\phi_0^\dagger(x_p)\phi_\rho(x) = \Lambda v G_\rho(\omega)\phi_0^\dagger(x_p)S_0G(x, x_p) \tag{45}
\]

Here in the last step, the Green’s function solution to the static equation, (36), was used.

A comparison of the space dependence of the exact solution (43) with the point kinetic one, Eq. (45), together with the corresponding expressions for the respective Green’s functions, (40) and (35) shows that the full space-frequency dependent solution converges to the point kinetic one when

\[
B_\rho^2(\omega) = B_\rho^2 \left(1 - \frac{1}{\rho_\infty G_0(\omega)}\right) \approx B_\rho^2 \tag{46}
\]

i.e. when

\[
| \rho_\infty G_0(\omega) | \gg 1, \tag{47}
\]

This can be shown more rigorously by expanding the full solution (43) in a Taylor series w.r.t.

\[
\epsilon = \frac{1}{\rho_\infty G_0(\omega)}, \quad | \epsilon | \ll 1 \tag{48}
\]
and keeping the leading term only. Such an expansion will actually be used in the next Section; here, however, we omit the details, and only discuss how the condition (47) can be fulfilled. One possibility, known from the case of the critical systems, is when $\omega \to 0$, since then $G_0(\omega) \to \infty$. It is obvious that with sufficiently low frequencies, for the source modulation method, both small and large systems will behave in a point kinetic manner.

The other possibility is for non-vanishing frequencies, e.g. for plateau frequencies when

$$G_0(\omega) \approx \frac{1}{\beta},$$

such that the point kinetic behaviour is ensured by the material and geometrical properties of the system. For the practical applications of reactivity measurement, this case is more interesting, because measurements with a very low frequency are time consuming and slow, and a fast measurement at plateau frequencies is much more useful. For critical systems, as is known, point kinetic behaviour in such a case can be achieved by fulfilling the already mentioned condition

$$\rho_{\infty}(\$) \gg 1 \quad (49)$$

This condition is equivalent with a small system size, usually associated with a critical facility or a research reactor. Not surprisingly, for systems that are slightly subcritical, the same condition will still ensure point kinetic behaviour even at plateau frequencies in a physically small system (i.e. with the same geometrical size as a critical reactor fulfilling (49)). This is shown in Fig. 1.

In the case of subcritical systems, however, the situation becomes more complex. This is because with the increasing subcriticality of an originally critical system fulfilling (49), the $\rho_{\infty}$ can decrease such that (49) will not be fulfilled. Actually even in a small system, one can have either

$$|\rho_{\infty}G_0(\omega)| \approx 1 \quad (50)$$

or

$$\rho_{\infty}(\$) \approx 0 \quad (51)$$

In these cases, the factor

$$\left(1 - \frac{1}{\rho_{\infty}G_0(\omega)^{-}}\right)$$

is either much smaller or much larger than unity, which means that there will be a substantial difference between $B^2(\rho)$ and $B^2_{\rho}$. Hence, in principle, one could expect that for such systems the system behaviour will deviate from point kinetics very significantly even at plateau or low frequencies. However, a closer look also reveals that for these cases, $B^2_{\rho}$ is very small, and thus, except for very high frequencies, the absolute value of $B^2_{\rho}(\omega)$ will still be small, and the large relative difference between the dynamic and static buckling does not play a role at all. That is, even for cases satisfying conditions (50) or (51),
the system will behave point-kinetically. This is illustrated in Fig. 2 for a case with $a = 30\text{ cm}$ and $\rho_\infty(\$) = 1.03$.

In a deeply subcritical system of any size, $\rho_\infty(\$)$ becomes negative. With a sufficiently deep subcriticality, condition (47) can be fulfilled at plateau frequencies by

$$|\rho_\infty(\$)| \gg 1,$$

This condition can be fulfilled by both small and large systems with $\rho_\infty < 0$; in fact, it is easier to fulfil it in a large system than in a small one, with the same subcriticality. This means that point kinetic behaviour also exists at plateau frequencies in deeply subcritical systems, especially in large cores. Such a behaviour is illustrated in Fig. 3.

It can also be noted by passing that although a small, tightly coupled critical system indeed behaves point kinetically at plateau frequencies, showing that (49) can be satisfied with realistic systems, formally the condition (52) is easier to fulfil than (49). This is because $\rho_\infty(\$)$ cannot be arbitrarily large, rather there exist an upper limit on it since by definition, reactivity cannot exceed unity. This circumstance is vastly overlooked in most books on reactor physics, which plot the solution of the inhour equation with the vertical axis for the reactivity going from minus infinity to infinity. On the other hand, no lower
Fig. 2. The noise induced by a fluctuating strength beam at various frequencies and $\rho_\infty(\$)=1.03$ for a small reactor. The solid line and the dashed line denotes the exact solution and the point kinetic approximation, respectively.

limit exist for the reactivity, which approaches $-\infty$ with diminishing the fissile material in the system, and so (52) can always be fulfilled for any reactor of any geometrical size.

Summarizing, in contrast to critical systems where the only two factors determining the kinetic behaviour of the system, i.e. the frequency of the perturbation and the size of the system, in subcritical cores even the static subcritical reactivity of the system influences the kinetic behaviour, and in some cases it also alters the dependence on the other two parameters, as compared to the kinetic behaviour of critical systems. In the next section these conditions will be made use of in the analysis of the source modulation method of measuring reactivity.
4 Analysis of the Performance of the Source Modulation Method

4.1 Principles of the method

The starting point is the linearized point kinetics equations, Eqs. (28), and their solution in the frequency domain for the fluctuation of the amplitude factor $\delta P(\omega)$, Eqs. (30) and (32),

$$\delta P(\omega) = \Lambda G_\rho(\omega) \delta q(\omega)$$

and

$$G_\rho(\omega) = \frac{1}{i\omega \left[ \Lambda + \frac{\beta}{i\omega + \lambda} \right] - \rho}.$$  

Then, Carta and D’Angelo (1999) use the assumption of being at the plateau frequency region $\lambda \ll \omega \ll (\beta - \rho)/\Lambda$, in which case the subcritical zero power transfer function, $G_\rho$ can be approximated as:

$$G_\rho(\omega) = \frac{1}{\beta - \rho}, \quad (53)$$

Fig. 3. The noise induced by a fluctuating strength beam at various frequencies $\rho_\infty(\omega) = -3.00$ for a large reactor. The solid line and the dashed line denote the exact solution and the point kinetic approximation, respectively.

$\text{k}_{\text{eff}}=0.98$, $\rho_\infty(\omega)=-3.00$.
which leads to
\[ \delta P(\omega) = \frac{\Lambda}{\beta - \rho} \delta q(\omega). \] (54)

Since in practice, whenever possible one uses relative variations for all measured quantities to get rid of the detector efficiency, one also needs to utilize the static equation, which reads as
\[ P_0 = -\frac{\Lambda}{\rho} Q_0. \] (55)

It might appear as if (55) contradicts with (21). This is, however, only an apparent contradiction. Namely, in this formalism, \( Q_0 \) is a quantity, derived from the original space-dependent static source as
\[ Q_0 = \frac{\int S_0(x) \phi_0^\dagger(x) \, dx}{\int \phi_0(x) \phi_0^\dagger(x) \, dx}. \] (56)

Using the normalisation condition (23) for \( \phi_0^\dagger(x) \) and the concrete form for \( \phi_\rho(x) \), it can easily be shown that the r.h.s. of (55) is indeed equal to unity. We shall however use at this point (55) without making use of this simplification, to maintain the formal similarity with the notations in the literature.

Dividing Eq. (54) with Eq. (55) yields
\[ \frac{\delta P(\omega)}{P_0} = \frac{\rho(\$)}{\rho(\$) - 1} \frac{\delta q(\omega)}{Q_0}. \] (57)

or, introducing the normalized (by the static values) fluctuations \( \tilde{\delta P}(\omega) \) and \( \tilde{\delta q}(\omega) \)
\[ \tilde{\delta P}(\omega) = \frac{\rho(\$)}{\rho(\$) - 1} \tilde{\delta q}(\omega) \] (58)

This is the formula suggested by Carta and D’Angelo (1999) for reactivity measurements. From here the reactivity can be expressed as
\[ \rho(\$) = \frac{\tilde{\delta P}(\omega)}{\tilde{\delta P}(\omega) - \tilde{\delta q}(\omega)}. \] (59)

However, in reality one can only measure \( \delta \phi(x, \omega) \) and \( \phi_\rho(x) \) and not \( \delta P(\omega) \) and \( P_0 \). Hence in the formula, corresponding to the measurement, one has to replace \( \tilde{\delta P}(\omega) \) with the normalised space-dependent flux \( \tilde{\delta \phi}(x, \omega) = \delta \phi(x, \omega)/\phi_\rho(x) \) taken at some position \( x \). Then one obtains
\[ \tilde{\delta \phi}(x, \omega) = \frac{\rho(\$)}{\rho(\$) - 1} \tilde{\delta S}(\omega), \] (60)
which gives for the estimated reactivity:

\[ \rho_{\text{est}}(\$, x) = \frac{\delta\phi(x, \omega)}{\delta\phi(x, \omega) - \delta S(\omega)}. \]  

(61)

This is the expression for the reactivity, to be used with measured values. Because in evaluating this expression one uses the local relative flux fluctuation instead of the amplitude factor of the theory, this only yields an estimate of the reactivity. Also, it is seen that due to the estimation through the space dependent neutron fluctuations, the estimation will yield a space-dependent value, which is indicated in the argument.

The accuracy of such an estimation can be investigated in the present model, in which the calculated values of the relative noise can be taken from the solutions derived earlier. This is quantified by inserting Eq. (42) into (61), also utilising Eq. (36) which leads to

\[ \rho_{\text{est}}(\$, x) = \frac{G(x, x_p, \omega)}{G(x, x_p, \omega) - G(x, x_p)}. \]  

(62)

The quantitative work below will be based on the analysis of this formula, such that \( \rho_{\text{est}} \) is evaluated from (62) for various system sizes, subcriticalities and frequencies, also as a function of the detector position, and the result compared with the exact or true value which is known from the parameters of the system.

Since, as mentioned earlier, we want to investigate the performance of the method in the point kinetic limit, we need to be able to change not only the system size, but also the frequency. This means that we need to abandon the restriction to plateau frequencies, expressed by Eq. (53).

For the case when \( \omega \to 0 \) we need to generalize Carta’s formula. Instead of using the plateau frequency approximation in Eq. (53), we re-arrange the exact subcritical transfer function of Eq. (32) as

\[ G_\rho(\omega) = \frac{1}{G_0^{-1}(\omega) - \rho}, \]  

(63)

where \( G_0(\omega) \) is the critical zero power transfer function as defined in Eq. (13). Then, using Eqs. (30) and (55), Eq. (57) takes the form

\[ \delta P(\omega) = \frac{\rho}{\rho - G_0^{-1}(\omega)} \cdot \tilde{\delta q}(\omega) \]  

(64)

and for the estimated reactivity we obtain:

\[ \rho_{\text{est}}(x) = \frac{G_0^{-1}(\omega) \cdot G(x, x_p, \omega)}{G(x, x_p, \omega) - G(x, x_p)}. \]  

(65)
Equation (64) can now be used to illustrate the principal difficulties of the method. On one hand, one expects that it will work well when a measurement of the flux fluctuations in one point approximates the point kinetic amplitude function well. This is when the system behaves in a point kinetic way. On the other hand, we have shown that point kinetic behaviour exists in the limit of either \( |\rho_{\infty}| \to \infty \), and hence also \( |\rho| \to \infty \), or for \( \omega \to \infty \). In either case, the reactivity disappears from Eq. (64) asymptotically, and hence the formula for its determination becomes ill-conditioned, and indeed singular, in the cases when physical intuition would expect it working best, which might hinder the usefulness of the method. These suspicions will be confirmed and quantified in the next Section.

4.2 Quantitative Analysis of the Performance of the Source Modulation Method

The estimated reactivity, as obtained from (62) or (65) was calculated for both the small and the large system for several subcritical reactivities. In the first run, the performance was investigated at plateau frequencies, corresponding to how the method would be used in practice. Since the accuracy of the estimation is space-dependent, the results show the ratio of the estimated to the true reactivity for all positions in the core within one figure. The results are summarized in the figures below.

For the small reactor the results are shown in Fig. (4). In all figures we chose \( \omega = 20 \text{ rad/s} \), which is a typical plateau frequency. In Fig. (5) we show the ratio of the estimated reactivity and the real reactivity for the relatively large reactor.

The figures shows that nearly irrespectively of the level of subcriticality of the system, the accuracy of the method is strongly dependent on the detector position. For both cases, subcritical reactivities corresponding to \( k_{eff} \) being equal to 0.9996, 0.99, 0.95, and 0.90, were chosen, or in dollars \( -0.061, -1.55, -8.10, -17.9 \). For the smaller system, with \( a = 30 \text{ cm} \), this corresponds to \( \rho_{\infty} \) values 3.34, 1.88, -4.52, and -13.31, respectively. For the cases with mild subcriticality, the performance is reasonably good. It is interesting to note, however, that with subcriticality levels that are more likely to be used in planned power ADS, the method breaks down even in such a small core at plateau frequencies, which is really remarkable.

For the large reactor with \( a = 150 \text{ cm} \), more interesting from the practical point of view, the corresponding \( \rho_{\infty} \) values are 0.0075, -1.42, -7.95, and -17.09, respectively. Here the estimated reactivity value shows large deviations from the reference value for all four subcriticalities. The conclusion is that for the large system, at plateau frequency, the performance of the method
is surprisingly poor. One could think that the reason for this fact is the large system size and the deviation from point kinetics of the flux fluctuations. Hence in the following we investigate the performance of the method in the point kinetic limit.

4.3 Analytical and numerical analysis of the method in the point kinetic limit

Since in the foregoing we have generalized the reactivity estimation formula for all frequencies instead of only plateau frequencies, we are in the position of being able to investigate the performance of the method when perfect point kinetics prevails, i.e. when $|\rho_{\infty}| \rightarrow \infty$ or $\omega \rightarrow 0$.

We start with the investigation of the original formula when the plateau frequency approximation is used. In that case, point kinetic behaviour is achieved by letting $|\rho_{\infty}| \rightarrow \infty$. We expand the dynamic transfer function, Eq. (40), in a Taylor series letting

$$\varepsilon(\omega) = \frac{\beta}{2\rho_{\infty}}; \quad |\varepsilon| \ll 1.$$
For simplicity a central source will be used, i.e. \( x_p = 0 \). One obtains for \( x \leq x_p \):

\[
G(x, 0, \omega) = \frac{\sin[B(\omega)a] \sin[B(\omega)(x + a)]}{DB(\omega) \sin[2B(\omega)a]} = \\
= \frac{\sin[B(\omega)(x + a)]}{2DB(\omega) \cos[B(\omega)a]} \approx \\
\approx \frac{\sin[B_\rho(x + a)]}{2DB_\rho \cos[B_\rho(a + x)]} [1 + \varepsilon - \varepsilon B_\rho a \cdot \tan(B_\rho a) - \varepsilon B_\rho(a + x) \cdot \tan(B_\rho(a + x))] \\
(66)
\]

From here, when \( \varepsilon \to 0 \), using Eq. (35) with \( x_p = 0 \) one obtains

\[
G(x, 0, \omega) \xrightarrow{\varepsilon \to 0} \frac{\sin[B_\rho(x + a)]}{2DB_\rho \cos[B_\rho(a + x)]} = G(x, 0). \\
(67)
\]

The reactivity can be expressed using the above Taylor series expansion, (66) of the dynamic transfer function in Eq. (62) and keeping the leading terms
\[ \rho_{est}(\$, x) = \frac{1}{\varepsilon} - 1 - B_\rho a \cdot \tan(B_\rho a) - B_\rho (a + x) \cot(B_\rho (a + x)) \]

\[ \frac{1}{\rho(\$)} = 2\varepsilon, \]

(68)

Since, from the definition of \( \varepsilon \), one has

\[ \frac{1}{\rho(\$)} = 2\varepsilon, \]

(69)

one can write the asymptotic value of the ratio of the estimated and true reactivity in the form

\[ \frac{\rho_{asy}(x)}{\rho} = f(x) \cdot \frac{\rho_\infty}{\rho}, \]

(70)

where the function \( f(x) \) is defined as

\[ f(x) = \frac{2}{1 - B_\rho [a \cdot \tan(B_\rho a) - (a + x) \cdot \cot(B_\rho (a + x))]}. \]

(71)

For \( x > x_p \) we obtain analogously:

\[ f(x) = \frac{2}{1 - B_\rho [a \cdot \tan(B_\rho a) - (a - x) \cdot \cot(B_\rho (a - x))]]. \]

(72)

By using expressions for \( \rho_\infty \) and \( \rho \) with material and geometrical constants, (70) can be put into the form

\[ \frac{\rho_{asy}(x)}{\rho} = \frac{\rho_\infty}{\rho} \cdot f(x) = \frac{\nu \Sigma f - \Sigma_a}{\nu \Sigma f - \Sigma_a - DB_0^2} \cdot f(x). \]

(73)

In this form the limit \( |\rho_\infty| \to \infty \) and \( |\rho| \to \infty \) can be formally enforced by letting \( \nu \Sigma f \to 0 \) which leads to

\[ \frac{\rho_{asy}(x)}{\rho} = \frac{\Sigma_a}{\Sigma_a + DB_0^2} \cdot f(x) \neq const. \]

(74)

This shows that even if point kinetic behaviour prevails, the error of the method is non-zero.

The investigation of the performance of the method in the alternative case of enforcing point kinetic behaviour by letting the frequency tend to zero goes on similar lines and leads to similar results. One re-defines \( \varepsilon \) as

\[ \varepsilon = \frac{1}{2\rho_\infty G_0(\omega)}, \]

then, again keeping \( x_p = 0 \), one obtains for the asymptotic estimated reactiv-
ity, \( \rho_{\text{asy}} \), the result

\[
\rho_{\text{asy}}(x) = \frac{G_0^{-1}(\omega)2\rho_\infty G_0(\omega)}{1 - B_p \left[ a \cdot \tan(B_p a) - (a + x) \cdot \cot(B_p(a + x)) \right]} = f(x) \cdot \rho_\infty, \tag{75}
\]

where \( f(x) \) is identical with what we obtained for \( f(x) \) in Eq. (70) for the case when \( |\rho_\infty| \to \infty \). So similarly to Eq. (70) we arrive at

\[
\frac{\rho_{\text{asy}}(x)}{\rho} = f(x) \cdot \frac{\rho_\infty}{\rho}. \tag{76}
\]

In other words, the estimated reactivity has a finite error even when the system asymptotically approaches point kinetic behaviour with vanishing frequencies. Also, as could be expected from the asymptotic derivation, it does not matter from the point of view of point kinetic behaviour how the limit is reached, i.e. by \( |\rho_\infty| \to \infty \) or by \( \omega \to 0 \).

The above results are illustrated quantitatively for a few cases. Figures 6 and 7 show the asymptotic formula, Eqs. (70)-(72) (symbols), and the generalized (non-asymptotic) formula (65) (solid lines), for \( \omega = 0.001 \text{ rad/s} \), for the small and the large core, respectively. The two formulae agree well with each other. Except for the extreme case of a very small system with rather low frequencies, the error of the estimation is non-negligible.

5 Possible Methods for Improving the Performance

The above investigations showed that using a single neutron detector, unless it is by chance placed to a position where the theoretical error of the method is zero or very small, will give an error which will not disappear even in the limit of exact point kinetic behaviour of the system. To alleviate this problem, one could resort to the fact that the original formula was derived for the amplitude factor of the factorised neutron noise, i.e. \( \delta P(\omega) \) and not for the space-dependent noise \( \delta \phi(x, \omega) \). For this purpose we need to use the formula (59), i.e.

\[
\rho(\$) = \frac{\overline{\delta P}(\omega)}{\delta P(\omega) - \delta q(\omega)}
\]

Recalling that

\[
\delta \phi(x, \omega) = \phi(\rho) \delta P(\omega) + \delta \psi(x, \omega) \tag{77}
\]

and the fact that \( \delta \psi(x, \omega) \) and the critical adjoint \( \phi^{\dagger}(x) \) are orthogonal, the amplitude factor \( \delta P(\omega) \) can be recovered from \( \delta \phi(x, \omega) \) as

\[
\delta P(\omega) = \int \delta \phi(x, \omega) \phi^{\dagger}(x) dx. \tag{78}
\]
Equation (78) suggests that instead of using the local signals, a weighted integral, yielding the amplitude factor, could be used in the applications of the method. It is easy to show that there are no conceptual bugs in this procedure, because when using the correct \( \delta P(\omega) \) instead of the local flux fluctuations, there is no need for the system to behave in a point kinetic way. Hence one can apply the method also at plateau frequencies in a large system.

The feasibility of this approach can also be proven through the theory. Namely, one can use the known solutions for the noise, Eq. (42) together with (40), to construct \( \delta P(\omega) \) through (78), and then put it into Eq. (59) and check the validity of the result. After some algebra one finds the formula for the estimated reactivity which reads as

\[
\rho_{\text{est}}(\omega) = \frac{D(B_0^2 - B_p^2) \int G(x, x_p, \omega) \cos(B_0 x) dx}{D(B_0^2 - B_p^2) \int G(x, x_p, \omega) \cos(B_0 x) dx - \cos(B_0 x_p)}
\]  

(79)

Using this formula we obtain for all subcriticalities that

\[
\frac{\rho_{\text{est}}(\omega)}{\rho(\omega)} \approx 1
\]

which is exactly what is expected.
Fig. 7. The ratio of estimated and real reactivity and the ratio of asymptotic and real reactivity for $\omega = 0.001$ rad/s and different subcriticalities, for a large reactor with the source at the center of the core. The solid line and the line with x denotes the estimated ratio and the asymptotic ratio, respectively.

It remains to find a way of approximating the procedure of reconstructing $\delta P(\omega)$ from (78). This can be done by using several detectors at different positions in the core, distributed relatively evenly, and then trying to approximate the integral as a sum. Such an approach was used to separate the point kinetic and space-dependent components in other noise problems earlier (Karlsson and Páżsit, 1998). That is, one uses the approximation

$$
\delta P(\omega) = \int \delta \phi(x, \omega) \phi_0^\dagger(x) dx \approx \sum_i \delta \phi(x_i, \omega) \phi_0^\dagger(x_i) dx_i
$$

(80)

Applying (80) in Eq. (79) leads to the formula of estimating the reactivity from a number of detector signals as

$$
\rho_{\text{est}}(\omega) = \frac{D(B_0^2 - B_p^2) \sum_i G(x_i, x_p, \omega) \cos(B_0 x_i) dx_i}{D(B_0^2 - B_p^2) \sum_i G(x_i, x_p, \omega) \cos(B_0 x_i) dx_i - \cos(B_0 x_p)}
$$

(81)

Obviously, the quality of this approximation depends on the number of the terms used in the sum (i.e. number of detectors in the measurement), and on the relative positioning of the detectors. It is obvious that detectors both close to, and far away from the source, are needed to be able to eliminate the space dependent term $\delta \psi(x, \omega)$ from (77) through its orthogonality with the critical adjoint.
The applicability of this procedure was also investigated quantitatively. Here now we only consider the large system, as it is the one which is interesting from the practical point of view. Figure 8 shows the ratio of the approximating sum and the integral as a function of number of terms in the sum the frequency $\omega = 20 \text{ rad/s}$ and $x_p = 0$, for four different subcritical values. It is seen that the approximation converges to the value of the integral quite slowly, especially for the deeper subcriticalities.

![Graphs showing approximations for different subcriticalities](image_url)

**Fig. 8.** Approximation of the integral for $\omega = 20 \text{ rad/s}$ and different subcriticalities, for a large reactor as a function of number of terms in the sum.

What is more important is how the reactivity, as estimated from Eq. (81) approximates the true reactivity. This is illustrated in Fig. 9, where the ratio is plotted as a function of number of detectors. Since the ratio is calculated for fixed detector positions, there is no explicit space dependence indicated in the results; however it is clear that the detector positions are selected somewhat arbitrarily, and one could plot such figures for many different detector position choices. The results shown here can though be considered as representative. They indicate that in practice, where the number of detectors available will unlikely to exceed 10-15 for the purpose of reactivity monitoring, the method of summing up the detector signal fluctuations does not give the expected measure of improvement of the method. The situation may be even more disadvantageous in a real case which, in contrast to the investigations here that used a one-dimensional model, the dynamics is three-dimensional, which amplifies the space-dependent effects further.
6 Conclusions

The investigation of the source modulation method of determining reactivity in a source driven subcritical system led to the somewhat unexpected conclusion that in practical applications it encounters difficulties when it is implemented by using single detector signals to approximate the point kinetic amplitude factor. The difficulty, in addition, does not lie in the large deviations of the behaviour of the system from point kinetics. It was shown that a large power ADS, with the subcritical reactivities that are being considered for future applications, behaves in a fairly point-kinetic way. The difficulties of the application of the method lie deeper than that; namely, just in the limit when the system behaviour is point kinetic, the formula used becomes ill-conditioned for the determination of the reactivity. In other words, even in the case of rather good point kinetic behaviour of the system, the error of the reactivity formula is still significant. Even methods of using several detectors, which in other cases were helpful in order to separate the point kinetic component from the total noise, perform poorly in this case. Hence, most likely, the source modulation method, as well as related methods such as the proton importance method, only can be used in a relative way, after having been calibrated by other methods.
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References


