Parameterization of spectrum constraints for MIMO systems with input process with finite dimensional spectrum parameterization

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Consider the system

$$y(t) = G(q)u(t) + H(q)e(t), \tag{1}$$

where $\theta \in \mathbf{R}^n$, $u_t \in \mathbf{R}^m$, $y_t \in \mathbf{R}^p$ and $e_t \in \mathbf{R}^p$. The input is a filtered Gaussian process while the noise is a white Gaussian process with covariance matrix Λ. In a closed-loop identification, using a controller $F_y(q)$, we have

$$u(t) = S_u(q)r(t) - S_u(q)F_y(q)H(q)e(t), \tag{2}$$

$$y(t) = S_y(q)G(q)r(t) + S_y(q)H(q)e(t), \tag{3}$$

where $S_u(q) = (I + G(q)F_y(q))^{-1}$ and $S_y(q) = (I + F_y(q)G(q))^{-1}$ in accordance to the set-up in Figure 1.

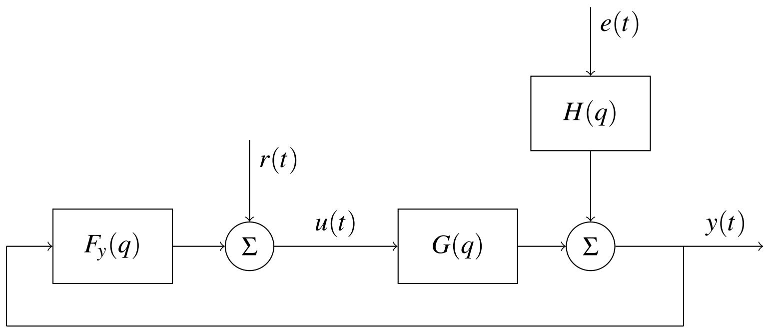

Figure 1. The system set-up. Here, $r(t)$ is the excitation signal, $u(t)$ is the input signal, $e(t)$ is the noise signal, $y(t)$ is the output signal, and $F_y(q)$, $G(q)$ and $H(q)$ are the transfer functions of the system.

The spectrum of the input and output signal are

$$\Phi_u(\omega) = S_u(e^{j\omega})\Phi_r(\omega)S_u^*(e^{j\omega}) + S_u(e^{j\omega})F_y(e^{j\omega})H(e^{j\omega})\Phi_e(\omega)(S_u(e^{j\omega})F_y(e^{j\omega})H(e^{j\omega}))^*, \tag{4}$$

$$\Phi_y(\omega) = S_y(e^{j\omega})G(e^{j\omega})\Phi_r(\omega)(S_y(e^{j\omega})G(e^{j\omega}))^* + S_y(e^{j\omega})H(e^{j\omega})\Phi_e(\omega)(S_y(e^{j\omega})H(e^{j\omega}))^*, \tag{5}$$

where q has been replaced by $e^{j\omega}$ and we used the fact that $r(t)$ and $e(t)$ are uncorrelated. The spectra sprung from the excitation signal $r(t)$ simply are

$$\Phi_{u_r}(\omega) = S_u(e^{j\omega})\Phi_r(\omega)S_u^*(e^{j\omega}), \tag{6}$$

$$\Phi_{y_r}(\omega) = S_y(e^{j\omega})G(e^{j\omega})\Phi_r(\omega)(S_y(e^{j\omega})G(e^{j\omega}))^*. \tag{7}$$

In the following, we will consider a general spectrum $\Phi_x = \tilde{G}(e^{j\omega})\Phi_r(\omega)\tilde{G}^*(e^{j\omega})$.

The spectrum of the excitation signal is parameterized using a finite-dimensional parameterization given by

$$\Phi_r(\omega) = \sum_{k=-M}^{M} C_k^r \mathscr{B}_k(e^{j\omega}), \tag{8}$$

where $\mathscr{B}_{-k}(z) = \mathscr{B}_k(z^{-k})$, $C_k^r \in \mathbf{C}^m$ and $C_{-k}^r = (C_k^r)^*$. The expression of $\Phi_x(\omega)$ then becomes

$$\Phi_x(\omega) = \tilde{G}(e^{j\omega}) \left(\sum_{k=-M}^{M} C_k^r \mathscr{B}_k(e^{j\omega}) \right) \tilde{G}^*(e^{j\omega}), \tag{9}$$

Vectorizing (9) gives

$$\operatorname{vec} \Phi_x(\omega) = \left(\left(\tilde{G}^*(e^{j\omega}) \right)^T \otimes \tilde{G}(e^{j\omega}) \right) \left(\sum_{k=-M}^{M} \operatorname{vec} C_k^r \mathscr{B}_k(e^{j\omega}) \right). \tag{10}$$

The Kronecker product in (10) is

$$\left(\tilde{G}^*(e^{j\omega}) \right)^T \otimes \tilde{G}(e^{j\omega}) = \begin{bmatrix} \tilde{G}_{1,1}(e^{-j\omega})\tilde{G}(e^{j\omega}) & \tilde{G}_{1,2}(e^{-j\omega})\tilde{G}(e^{j\omega}) & \cdots & \tilde{G}_{1,p(p+m)}(e^{-j\omega})\tilde{G}(e^{j\omega}) \\ \tilde{G}_{2,1}(e^{-j\omega})\tilde{G}(e^{j\omega}) & \tilde{G}_{2,2}(e^{-j\omega})\tilde{G}(e^{j\omega}) & \cdots & \tilde{G}_{2,p(p+m)}(e^{-j\omega})\tilde{G}(e^{j\omega}) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{G}_{n,1}(e^{-j\omega})\tilde{G}(e^{j\omega}) & \tilde{G}_{n,2}(e^{-j\omega})\tilde{G}(e^{j\omega}) & \cdots & \tilde{G}_{n,p(p+m)}(e^{-j\omega})\tilde{G}(e^{j\omega}) \end{bmatrix}. \tag{11}$$

Now consider the matrix

$$\begin{aligned} &\operatorname{vec} \tilde{G}(e^{j\omega}) \left[\operatorname{vec} \tilde{G}(e^{j\omega}) \right]^* \\ &= \big[\operatorname{vec}(\tilde{G}(e^{j\omega}))\tilde{G}_{1,1}(e^{-j\omega}) \quad \operatorname{vec}(\tilde{G}(e^{j\omega}))\tilde{G}_{2,1}(e^{-j\omega}) \quad \cdots \quad \operatorname{vec}(\tilde{G}(e^{j\omega}))\tilde{G}_{n,1}(e^{-j\omega}) \quad \operatorname{vec}(\tilde{G}(e^{j\omega}))\tilde{G}_{1,2}(e^{-j\omega}) \\ &\qquad \operatorname{vec}(\tilde{G}(e^{j\omega}))\tilde{G}_{2,2}(e^{-j\omega}) \quad \cdots \quad \operatorname{vec}(\tilde{G}(e^{j\omega}))\tilde{G}_{n,2}(e^{-j\omega}) \quad \cdots \quad \operatorname{vec}(\tilde{G}(e^{j\omega}))\tilde{G}_{n,p(p+m)}(e^{-j\omega}) \big], \end{aligned} \tag{12}$$

which can be approximated by

$$\operatorname{vec} \tilde{G}(e^{j\omega}) \left[\operatorname{vec} \tilde{G}(e^{j\omega}) \right]^* \approx \sum_{k=-M_g}^{M_g} C_k^g e^{-j\omega k},$$

using finite dimensional parametrization. To see this, consider $\operatorname{vec} \tilde{G}(e^{j\omega}) \left[\operatorname{vec} \tilde{G}(e^{j\omega}) \right]^*$ as a (power) spectrum. A spectrum must be (a) Hermitian, (b) symmetric with respect to $\omega = 0$, (c) positive semidefinite and (d) periodic with period 2π. All of these constraints are fulfilled for $\operatorname{vec} \tilde{G}(e^{j\omega}) \left[\operatorname{vec} \tilde{G}(e^{j\omega}) \right]^*$. We can then retrieve C_k^g from the inverse Fourier transform of the spectrum. That is

$$C_k^g \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{vec} \tilde{G}(e^{j\omega}) \left[\operatorname{vec} \tilde{G}(e^{j\omega}) \right]^* e^{i\omega k} d\omega.$$

We then get

$$\operatorname{vec} \tilde{G}(e^{j\omega}) \left[\operatorname{vec} \tilde{G}(e^{j\omega}) \right]^* = \sum_{k=-\infty}^{\infty} C_k^g e^{-j\omega k} \approx \sum_{k=-M_g}^{M_g} C_k^g e^{-j\omega k},$$

for some M_g. Here we have restricted ourselves to the exponential basis function, and consequently to an FIR-shaped spectrum Φ_x.

The elements of (11) can be formed by suitably reshaping the columns of (12), leading to

$$\operatorname{vec} \Phi_x(\omega) \approx \left(\sum_{k=-M_g}^{M_g} \tilde{C}_k^g e^{-j\omega k} \right) \left(\sum_{k=-M}^{M} \operatorname{vec} C_k^r e^{-j\omega k} \right), \tag{13}$$

where $\tilde{C}_k^g$ is obtained from C_k^g. We can then express $\Phi_x(\omega)$ approximately as

$$\sum_{k=-(M_g+M)}^{M_g+M} C_k^x e^{-j\omega k}, \tag{14}$$

where C_k^x is in turn obtained from C_k^g and C_k^r. Note that the decision variable C_k^r appears linearly in (14).

We consider upper and lower spectrum constraints of the form

$$\Phi_{con}^{low}(\omega) \leq \Phi_x(\omega) \leq \Phi_{con}^{high}(\omega) \text{ for all } \omega.$$

We can enforce them frequency-by-frequency using a grid, that is,

$$\Phi_{con}^{low}(\omega_i) \leq \sum_{k=-(M_g+M)}^{M_g+M} C_k^x e^{-j\omega_i k} \leq \Phi_{con}^{high}(\omega_i) \text{ for } i = 1, \ldots, N,$$

Or, for the lower constraint, we can use the approximation

$$\Phi_{con}^{low}(\omega) \approx \sum_{k=-M_g}^{M_g} C_k^{con} e^{-j\omega k},$$

and enforce

$$\sum_{k=-M_g}^{M_g} (C_k^x - C_k^{con}) e^{-j\omega k} \geq 0 \text{ for all } \omega,$$

using the KYP-lemma.

(Actually, MOOSE2 also handles non-Hermitian constraints. It then enforces the constraints element-wise and frequency-by-frequency instead of as linear matrix inequalities. Of course, the resulting spectrum is forced to be Hermitian.)