Parameterization of Information matrix for MIMO systems with input process with finite dimensional spectrum parameterization

Christian Larsson

Consider the parameterized model

$$y_t = G(q,\theta)u_t + H(q,\theta)e_t, \tag{1}$$

where $\theta \in \mathbf{R}^n$, $u_t \in \mathbf{R}^m$, $y_t \in \mathbf{R}^p$ and $e_t \in \mathbf{R}^p$. The input is filtered Gaussian process while the noise is a white, Gaussian process with covariance matrix Λ. The inverse per-sample covariance matrix of the parameter estimates is

$$\mathcal{I} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma(e^{i\omega})\left(\Lambda^{-1}\otimes\phi_{\chi_o}\right)\Gamma^*(e^{i\omega})d\omega, \tag{2}$$

where

$$\phi_{\chi_o}(\omega) \triangleq \begin{bmatrix}\phi_u(\omega) & \phi_{ue}(\omega)\\ \phi_{eu}(\omega) & \Lambda\end{bmatrix}, \qquad \Gamma(e^{i\omega}) \triangleq \begin{bmatrix}(\mathrm{vec}\,\mathbf{F}_1^T)^T\\ \vdots \\ (\mathrm{vec}\,\mathbf{F}_n^T)^T\end{bmatrix}, \qquad \mathbf{F}_i \triangleq \begin{bmatrix}H^{-1}(q,\theta)\frac{\partial G}{\partial\theta_i} & H^{-1}(q,\theta)\frac{\partial H}{\partial\theta_i}\end{bmatrix}.$$

The spectrum is parameterized using a finite-dimensional parameterization given by

$$\phi_{\chi_o}(\omega) = \sum_{k=-M}^{M} C_k\mathcal{B}_k(e^{i\omega}), \tag{3}$$

where $\mathcal{B}_{-k}(z) = \mathcal{B}_k(z^{-k})$, $C_k \in \mathbf{R}^m$ and $C_{-k} = C_k^T$.

Vectorizing (3) gives

$$\begin{aligned}
\mathrm{vec}\,\mathcal{I} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\left(\Gamma^*(e^{i\omega})\right)^T\otimes\Gamma(e^{i\omega})\right]\mathrm{vec}\left(\Lambda^{-1}\otimes\phi_{\chi_o}(\omega)\right)d\omega\\
&= \frac{1}{2\pi}\sum_{k=-M}^{M}\int_{-\pi}^{\pi}\left[\Gamma(e^{-i\omega})\otimes\Gamma(e^{i\omega})\right]\mathcal{B}_k(e^{i\omega})d\omega\,\mathrm{vec}\left(\Lambda^{-1}\otimes C_k\right)\\
&= \begin{bmatrix}R_{-M} & \cdots & R_{-1} & R_0 & R_1 & \cdots & R_M\end{bmatrix}\begin{bmatrix}\mathrm{vec}\,(\Lambda^{-1}\otimes C_M^T)\\ \vdots \\ \mathrm{vec}\,(\Lambda^{-1}\otimes C_1^T)\\ \mathrm{vec}\,(\Lambda^{-1}\otimes C_0)\\ \mathrm{vec}\,(\Lambda^{-1}\otimes C_1)\\ \vdots \\ \mathrm{vec}\,(\Lambda^{-1}\otimes C_M)\end{bmatrix},
\end{aligned}$$

where

$$R_k \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\Gamma(e^{-i\omega})\otimes\Gamma(e^{i\omega})\right]\mathcal{B}_k(e^{i\omega})d\omega.$$

The R_ks are not covariance matrices, in fact they are in general not square. However, the elements can be obtained from the covariances of a system formed by vectorizing $\Gamma(e^{i\omega})$. First, let's consider the above Kronecker product, i.e.,

$$\left(\Gamma^*(e^{i\omega})\right)^T\otimes\Gamma(e^{i\omega}) = \begin{bmatrix}\Gamma_{1,1}(e^{-i\omega})\Gamma(e^{i\omega}) & \Gamma_{1,2}(e^{-i\omega})\Gamma(e^{i\omega}) & \cdots & \Gamma_{1,p(p+m)}(e^{-i\omega})\Gamma(e^{i\omega})\\ \Gamma_{2,1}(e^{-i\omega})\Gamma(e^{i\omega}) & \Gamma_{2,2}(e^{-i\omega})\Gamma(e^{i\omega}) & \cdots & \Gamma_{2,p(p+m)}(e^{-i\omega})\Gamma(e^{i\omega})\\ \vdots & \vdots & \ddots & \vdots\\ \Gamma_{n,1}(e^{-i\omega})\Gamma(e^{i\omega}) & \Gamma_{n,2}(e^{-i\omega})\Gamma(e^{i\omega}) & \cdots & \Gamma_{n,p(p+m)}(e^{-i\omega})\Gamma(e^{i\omega})\end{bmatrix}. \tag{4}$$

Now consider the matrix

$$
\begin{aligned}
&\operatorname{vec}\Gamma(e^{i\omega})\left[\operatorname{vec}\Gamma(e^{i\omega})\right]^* \\
&\quad= \big[\operatorname{vec}(\Gamma(e^{i\omega}))\Gamma_{1,1}(e^{-i\omega}) \quad \operatorname{vec}(\Gamma(e^{i\omega}))\Gamma_{2,1}(e^{-i\omega} \quad \cdots \quad \operatorname{vec}(\Gamma(e^{i\omega}))\Gamma_{n,1}(e^{-i\omega})) \quad \operatorname{vec}(\Gamma(e^{i\omega}))\Gamma_{1,2}(e^{-i\omega}) \\
&\qquad\qquad \operatorname{vec}(\Gamma(e^{i\omega}))\Gamma_{2,2}(e^{-i\omega}) \quad \cdots \quad \operatorname{vec}(\Gamma(e^{i\omega}))\Gamma_{n,2}(e^{-i\omega}) \quad \cdots \quad \operatorname{vec}(\Gamma(e^{i\omega}))\Gamma_{n,p(p+m)}(e^{-i\omega})\big]. \qquad (5)
\end{aligned}
$$

The elements of (4) can be formed by suitably reshaping the columns of (5). The integrals

$$
\tilde{R}_k \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{vec}\Gamma(e^{i\omega})\left[\operatorname{vec}\Gamma(e^{i\omega})\right]^* \mathcal{B}_k(e^{i\omega})d\omega
$$

can be evaluated using numerically robust methods. Note that $\tilde{R}_{-k} = \tilde{R}_k$.