Lecture 3

Poisson's equation in $\mathbb{R}^2$ with non-homogeneous Dirichlet boundary conditions:

$$(D) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = g & \text{on } \Gamma \end{cases}$$

g is given boundary data.

Variational formulation: Find $u \in V_g$ s.t.

$$(V) \qquad (\nabla u, \nabla v) = (f, v) \quad \forall v \in V_0$$

where $V_g = \{ v : v = g \text{ on } \Gamma \ \& \ \int_\Omega (|\nabla v|^2 + v^2)\,dx < \infty \}$

$V_0 = \{ v : v = 0 \text{ on } \Gamma \ \& \ \int_\Omega (|\nabla v|^2 + v^2)\,dx < \infty \}$

V_g trial space, V_0 test space

The test space is choosen to be V_0 (and not V_g) so that the boundary integral disappears in the integration by parts:

$$(-\Delta u, v) = -\int_\Gamma \cancel{(\nabla u \cdot n) v}^{\,0}\, ds + (\nabla u, \nabla v)$$

When computing FEM approximation U, set nodes on the boundary to $\mathcal{N}_b$ and interior nodes to $\mathcal{N}_h$:

$$U = \sum_{N_j \in \mathcal{N}_b} \xi_j \phi_j + \sum_{N_j \in \mathcal{N}_h} \xi_j \phi_j$$

where $\xi_j = g(N_j)$ for $N_j \in \mathcal{N}_b$.

$\Rightarrow$ The resulting discrete system is:

$$\sum_{N_j \in \mathcal{N}_h} \xi_j (\nabla \phi_j, \nabla \phi_i) = (f, \phi_i) - \sum_{N_j \in \mathcal{N}_b} g(N_j)(\nabla \phi_j, \nabla \phi_i)$$

for all $N_i \in \mathcal{N}_h$

Robin & Neumann b.c.

$$(D) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \partial_n u + \varkappa u = g & \text{on } \Gamma_2 \end{cases} \qquad \begin{aligned} &\Gamma = \Gamma_1 \cup \Gamma_2 \\ &\varkappa \geq 0 \end{aligned}$$

Test & trial space should satisfy Dirichlet boundary conditions:

$$V = \{ v : v = 0 \text{ on } \Gamma_1 \ \& \ \int_\Omega (|\nabla v|^2 + v^2)\,dx < \infty \}$$

Find variational formulation: mult. (D) by test function $v \in V$ & integrate:

$$(f,v) = -\int_\Omega \Delta u \, v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Gamma \partial_n u \, v \, ds$$

$$= \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_2} \kappa u v \, ds - \int_{\Gamma_2} g v \, ds$$

Variational formulation: Find $u \in V$ s.t.

$$(*) \quad (\nabla u, \nabla v) + \int_{\Gamma_2} \kappa u v \, ds = (f,v) + \int_{\Gamma_2} g v \, ds \quad \forall v \in V$$

From $(*)$ we have that

$$\int_\Omega (-\Delta u - f) v \, dx + \int_{\Gamma_2} (\partial_n u + \kappa u - g) v \, ds = 0$$

Since $-\Delta u = f$ we have

$$\int_{\Gamma_2} (\partial_n u + \kappa u - g) v \, ds = 0 \quad \forall v \in V$$

- Robin (and Neumann) b.c. enforced weakly through the variational form.
- Dirichlet b.c. typically enforced strongly by the choice of space V

Weak implementation of Neumann/Dirichlet b.c. using Robin b.c.

$$a\,\partial_n u + \gamma(u - u_b) = g \qquad \text{on } \Gamma$$

$\gamma = 0 \Rightarrow$ Neumann $\quad a\,\partial_n u = g \quad$ on Γ

$\gamma = \infty \Rightarrow$ Dirichlet $\quad u = u_b \quad$ on Γ

This is used in Puffin.

Adaptivity & Error control

In lecture 4 we will prove the following a posteriori error estimate:

$$\| \nabla u - \nabla U \| \leq C_i \| h R(U) \|$$

with the residual $R(U) = |f + \Delta U|$

$C_i \approx 0.1$ and $h = h(x)$ is the mesh function

For p.w. linear basis V_h; $\Delta U = 0$.

How can we approximate ΔU for $U \in V_h$?

Computing the residual using the discrete Laplacian Δ_h

For a given $w \in V$, let $\Delta_h w$ be the unique function in V_h s.t.

$$(*) \qquad -(\Delta_h w, v) = (\nabla w, \nabla v) \quad \forall v \in V_h$$

How to compute $\Delta_h U$?

$U = \sum_{j=1}^{M} \eta_j \phi_j$ with $\eta_j = U(N_j)$ nodal values

$$\Delta_h U \in V_h \Rightarrow \Delta_h U = \sum_{j=1}^{M} \zeta_j \phi_j$$

$$(*) \Rightarrow -\sum_{j=1}^{M} \zeta_j (\phi_j, \phi_i) = \sum_{j=1}^{M} \eta_j (\nabla \phi_j, \nabla \phi_i)$$

$$i = 1, \dots, M$$

Corresponds to linear system of equations:

$$-M \zeta = A \eta$$

with $\zeta = (\zeta_j)$ $\eta = (\eta_j)$ $M = (\phi_j, \phi_i)$ $A = (\nabla \phi_j, \nabla \phi_i)$

<u>M mass matrix</u>, <u>A stiffness matrix</u>

$$\Rightarrow \zeta = -M^{-1} A \eta \Rightarrow \Delta_h U = \sum_{j=1}^{M} \zeta_j \phi_j$$

$$\Rightarrow R(U) = f + \Delta U \approx f + \Delta_h U$$

To minimize the error $\|\nabla u - \nabla U\|$ we want to minimize $\|h R(U)\|$: so that we choose $h = h(x)$ small where the residual $R(U)$ is large.

Simple adaptive algorithm

Start from initial (coarse) mesh τ_h^0. Set $i=1$.

(1) Compute solution $U \in V_h$ by FEM

(2) Compute $R(U) = f + \Delta U \approx f + \Delta_h U$

(3) Mark 50% of the elements for refinement which have the largest residual $R(U)$.

(4) Refine the mesh τ_h^{i-1}, which then gives a new mesh τ_h^{i}.

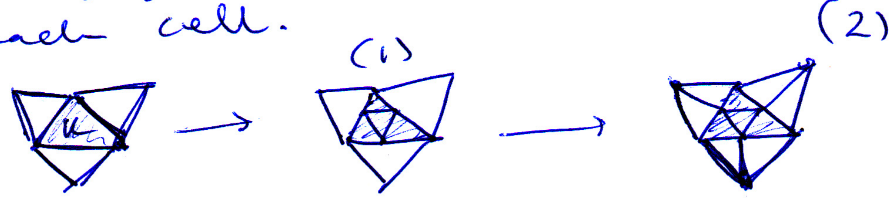
(5) Set $i = i+1$ then go to (1).

Red-green mesh refinement

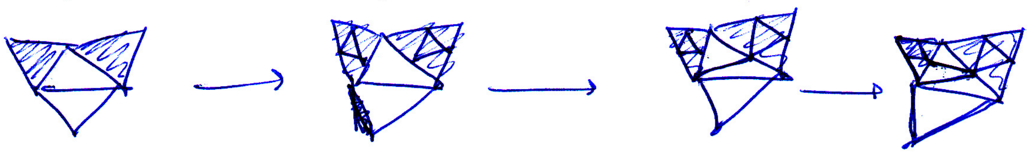
(1) loop over all marked cells: Insert new nodes at edge midpoints, and connect them by new edges $\Rightarrow$ 4 new cells.

(2) loop over all hanging nodes: connect ~~each~~ each hanging node with the node opposite in each cell.

Ex:



Ex:



Further reading:

CDE 15.1, 15.3, 15.4 (+ Roban b.c.)