L4

<u>Polynomial approximation</u>: $\mathcal{P}^q(a,b)$ is the set of polynomials $p(x) = \sum_{i=0}^{q} c_i x^i$. ($\dim(\mathcal{P}^q(a,b)) = q+1$)

$c_i \in \mathbb{R}$ coefficients in <u>monomial basis</u> $\{x^i\}_{i=0}^q = \{1, x, x^2, \dots, x^q\}$

<u>Lagrange basis</u> $\{\lambda_i\}_{i=0}^q$: associated with $q+1$ nodes:

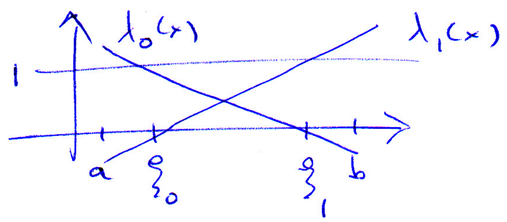
$\xi_0 < \xi_1 < \dots < \xi_q$ in (a,b).

Determined by: $\lambda_i(\xi_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$

$$\lambda_i(x) = \prod_{j \neq i} \frac{x-\xi_j}{\xi_i - \xi_j} = \frac{(x-\xi_0)(x-\xi_1)\cdots(x-\xi_{i-1})(x-\xi_{i+1})\cdots(x-\xi_q)}{(\xi_i-\xi_0)(\xi_i-\xi_1)\cdots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\cdots(\xi_i-\xi_q)}$$

Lagrange basis is a <u>nodal basis</u>: $p \in \mathcal{P}^q(a,b)$ can be written as $p(x) = \sum_{i=0}^{q} p_i \lambda_i(x)$ with $p_i = p(\xi_i)$.

$q=1$: linear Lagrange basis $\{\lambda_0(x), \lambda_1(x)\}$



$$\lambda_0(x) = \frac{x-\xi_1}{\xi_0-\xi_1}$$

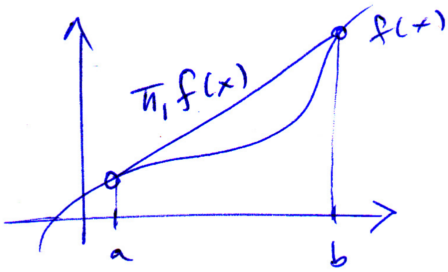
$$\lambda_1(x) = \frac{x-\xi_0}{\xi_1-\xi_0}$$

Polynomial interpolant $\pi_q f \in \mathcal{P}^q(a,b)$: defined by $\pi_q f(\xi_i) = f(\xi_i)$ for all $i = 0, \dots, q$.

$$\pi_q f(x) = f(\xi_0)\lambda_0(x) + \dots + f(\xi_q)\lambda_q(x) = \sum_{i=0}^{q} f(\xi_i)\lambda_i(x)$$

Linear interpolant $\pi_1 f(x) = f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}$

choosing $\xi_0 = a, \xi_1 = b$.



Theorem 5.1 : Assume $f(x)$ has $q+1$ continuous derivatives in (a,b). Then for $a \le x \le b$:

$$|f(x) - \pi_q f(x)| \le \left| \frac{(x-\xi_0)\cdots(x-\xi_q)}{(q+1)!} \right| \max_{[a,b]} |D^{q+1} f|$$

Proof : $q=1$

(1) $\pi_1 f(x) = f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x) = f(\xi_0)\frac{x-\xi_1}{\xi_0-\xi_1} + f(\xi_1)\frac{x-\xi_0}{\xi_1-\xi_0}$

Taylor's theorem for $x \in (\xi_0, \xi_1)$:

(2) $f(\xi_i) = f(x) + f'(x)(\xi_i - x) + \frac{1}{2}f''(\eta_i)(\xi_i - x)^2$

where η_i lies between x and ξ_i.

Putting (2) into (1) & using that

$$\lambda_0(x)+\lambda_1(x)=1 \quad \& \quad (\xi_0-x)\lambda_0(x)+(\xi_1-x)\lambda_1(x)=0$$

$$\Rightarrow \Pi_1 f(x) = \left(f(x)+f'(x)(\xi_0-x)+\tfrac{1}{2}f''(\eta_0)(\xi_0-x)^2\right)\lambda_0(x)$$
$$+\left(f(x)+f'(x)(\xi_1-x)+\tfrac{1}{2}f''(\eta_1)(\xi_1-x)^2\right)\lambda_1(x)$$
$$= f(x)(\underbrace{\lambda_0+\lambda_1}_{=1}) + f'(x)\left(\underbrace{(\xi_0-x)\lambda_0+(\xi_1-x)\lambda_1}_{=0}\right)$$
$$+\tfrac{1}{2}f''(\eta_0)(\xi_0-x)^2\lambda_0(x)+\tfrac{1}{2}f''(\eta_1)(\xi_1-x)^2\lambda_1(x)$$

$$\Rightarrow |f(x)-\Pi_1 f(x)| = \left|\tfrac{1}{2}\left(f''(\eta_0)(\xi_0-x)^2\lambda_0(x)+f''(\eta_1)(\xi_1-x)^2\lambda_1(x)\right)\right|$$

$$\leq \frac{1}{2}\left(\frac{|\xi_0-x|^2|x-\xi_1|}{|\xi_0-\xi_1|}|f''(\eta_0)|+\frac{|\xi_1-x|^2|x-\xi_0|}{|\xi_1-\xi_0|}|f''(\eta_1)|\right)$$

$$\leq \frac{1}{2}\left(\frac{|\xi_0-x|^2|x-\xi_1|}{|\xi_0-\xi_1|}+\frac{|\xi_1-x|^2|x-\xi_0|}{|\xi_1-\xi_0|}\right)\max_{[a,b]}|f''|$$

$$= \frac{1}{2}\left(\frac{(x-\xi_0)^2(\xi_1-x)}{(\xi_1-\xi_0)}+\frac{(\xi_1-x)^2(x-\xi_0)}{(\xi_1-\xi_0)}\right)\max_{[a,b]}|f''|$$

$$= \frac{1}{2}|x-\xi_0||x-\xi_1|\max_{[a,b]}|f''|$$

□

<u>Theorem 5.2</u>: For $\xi_0 \leq x \leq \xi_1$

$$|f'(x)-(\Pi_1 f)'(x)| \leq \frac{(x-\xi_0)^2+(x-\xi_1)^2}{2(\xi_1-\xi_0)}\max_{[a,b]}|f''|$$

Proof: $(\pi_1 f)'(x) = f(\xi_0)\lambda_0'(x) + f(\xi_1)\lambda_1'(x)$

Using Taylors formula &

$\lambda_0'(x) + \lambda_1'(x) = 0$ & $(\xi_0 - x)\lambda_0'(x) + (\xi_1 - x)\lambda_1'(x) = 1$

$$\Rightarrow \quad f'(x) - (\pi_1 f)'(x) = -\frac{1}{2}\left(f''(\eta_0)(\xi_0 - x)^2\lambda_0'(x) + f''(\eta_1)(\xi_1 - x)^2\lambda_1'(x)\right)$$

$$\Rightarrow |f'(x) - (\pi_1 f)'(x)| \leq \frac{1}{2}\,\frac{(x-\xi_0)^2 + (x-\xi_1)^2}{(\xi_1 - \xi_0)} \max_{[a,b]} |f''|$$

since $|\lambda_i'(x)| = \dfrac{1}{\xi_1 - \xi_0}$ $\square$

Maximumnorm $\|f\|_{L_\infty(a,b)} = \max_{x\in[a,b]} |f(x)|$

L_1-norm $\|f\|_{L_1(a,b)} = \int_a^b |f(x)|\,dx$

L_2-norm $\|f\|_{L_2(a,b)} = \left(\int_a^b f^2(x)\,dx\right)^{1/2}$

Theorem 5.1 & 5.2 $\Rightarrow$

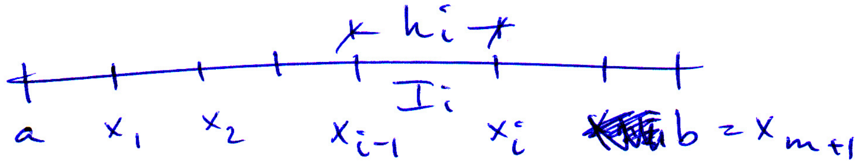
$$\|f - \pi_1 f\|_{L_\infty(a,b)} \leq \frac{1}{8}(b-a)^2 \|f''\|_{L_\infty(a,b)}$$

$$\|f' - (\pi_1 f)'\|_{L_\infty(a,b)} \leq \frac{1}{2}(b-a)\|f''\|_{L_\infty(a,b)}$$

($\frac{1}{8}$, $\frac{1}{2}$ "interpolation constants")

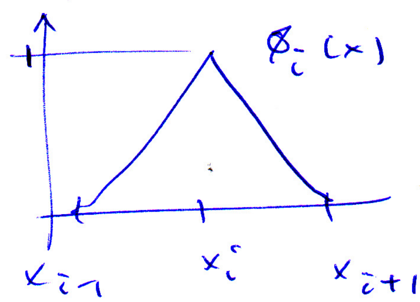
Piecewise linear functions on a mesh $\mathcal{T}_h = \{I_i\}$

$$V_h = V_h^{(1)} = \{ v \text{ cont} : v|_{I_i} \in \mathcal{P}^q(I_i),\ i = 1, \dots, m+1 \}$$



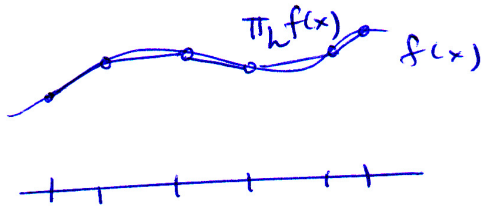
$$v \in V_h \Rightarrow v(x) = \sum_{i=0}^{m+1} v(x_i)\,\varphi_i(x)$$

$\varphi_i(x)$ hat functions



Continuous piecewise linear interpolant $\pi_h f \in V_h$

def. by $\pi_h f(x) = f(x_{i-1}) \dfrac{x - x_i}{x_{i-1} - x_i} + f(x_i) \dfrac{x - x_{i-1}}{x_i - x_{i-1}}$



for $x_{i-1} \le x \le x_i$

& $i = 1, \dots, m+1$

Theorem 5.4: For $p = 1, 2, \infty$ there are constants C_i such that

$$\|f - \pi_h f\|_{L_p(a,b)} \le C_i \|h^2 f''\|_{L_p(a,b)}$$
$$\|f - \pi_h f\|_{L_p(a,b)} \le C_i \|h f'\|_{L_p(a,b)}$$
$$\|f' - (\pi_h f)'\|_{L_p(a,b)} \le C_i \|h f''\|_{L_p(a,b)}$$

Interpolation in 2D

Theorem 14.2: There exist interpolation error constants C_i, depending only on the minimum angle in the mesh $\mathcal{T}_h$ and the order of the estimate m, such that the piecewise linear interpolant $\pi_h w \in V_h$ of a function w satisfies

$$\| D^m (w - \pi_h w) \| \leq C_i \| h^{2-m} D^2 w \|$$

for $m = 0, 1$, where $D^1 w = Dw = \nabla w$,

and $D^2 w = \left(\sum_{i,j=1}^{2} \left(\frac{\partial^2 v}{\partial x_i \partial x_j} \right)^2 \right)^{1/2}$, and

$$\| h^{-2+m} D^m (w - \pi_h w) \| + \left(\sum_{K \in \mathcal{T}_h} h_K^{-3} \| w - \pi_h w \|_{\partial K}^2 \right)^{1/2}$$
$$\leq C_i \| D^2 w \|$$

Further, there exist interpolant $\tilde{\pi}_h w_h \in V_h$ s.t.

$$\| h^{-1+m} D^m (w - \tilde{\pi}_h w) \| + \left(\sum_{K \in \mathcal{T}_h} h_K^{-1} \| w - \tilde{\pi}_h w \|_{\partial K}^2 \right)^{1/2} \leq C_i \| D w \|$$

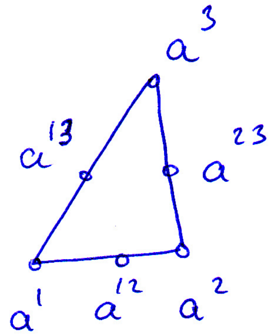
$\tilde{\pi}_h w$ defined using suitable averages of w around the nodal points. ($\| Dw \| < \infty$ does not guarantee ~~[illegible]~~ well defined nodal values; but $\| D^2 w \| < \infty$ does).

Continuous p.w. quadratic interpolation in 2D

$V_h = \{v ; v \text{ cont. in } \Omega, \; v \in \mathcal{P}^2(K), \; K \in \mathcal{T}_h\}$

$\mathcal{P}^2(K) = \{v : v(x) = \sum_{0 \le i+j \le 2} c_{ij} x_1^i x_2^j \text{ for } x = (x_1, x_2) \in K, \; c_{ij} \in \mathbb{R}\}$

$\dim(\mathcal{P}^2(K)) = 6 \quad (c_{00}, c_{10}, c_{01}, c_{20}, c_{11}, c_{02})$



Element nodes $a^1, \ldots, a^{23}$

element nodal basis functions $\{\psi_i\}_{i=1}^6$ for $\mathcal{P}^2(K)$ are given as combinations of the linear basis functions $\lambda_1, \lambda_2, \lambda_3$:

$\psi_1 = \lambda_1(2\lambda_1 - 1), \; \psi_2 = \lambda_2(2\lambda_2 - 1), \; \psi_3 = \lambda_3(2\lambda_3 - 1)$

$\psi_4 = 4\lambda_1\lambda_2, \; \psi_5 = 4\lambda_1\lambda_3, \; \psi_6 = 4\lambda_2\lambda_3$

$\Rightarrow v(x) = \sum_{i=1}^6 \xi_i \psi_i \quad \left(\begin{array}{ll} \xi_i = v(a^i) & i = 1,2,3 \\ \xi_k = v(a^{ij}) & 1 \le i < j \le 3 \\ & k = 4,5,6 \end{array} \right)$

Interpolation error:

$\|u - \pi_h u\| \le C_i \|h^3 D^3 u\|$ $\quad \pi_h u \in V_h$ interpolates u in all nodes

Error estimation

Variational formulation: Find $u \in V$ s.t.

$$(V) \quad \int_0^1 a u' v' \, dx = \int_0^1 f v \, dx \quad \forall v \in V$$

$$V = \{ v : \|v\|^2 + \|v'\|^2 < \infty, \; v(0) = v(1) = 0 \}$$

$$a_1 \le a(x) \le a_2 \quad \text{for} \quad 0 \le x \le 1$$

Galerkin FEM: Find $U \in V_h$ s.t.

$$(G) \quad \int_0^1 a U' v' \, dx = \int_0^1 f v \, dx \quad \forall v \in V_h$$

$$V_h = \{ v : \text{continuous \& p.w. linear, on subintervals } I_i, \; v(0) = v(1) = 0 \}$$

$$V_h \subset V \Rightarrow (V) - (G) : \quad \int_0^1 a (u - U)' v' \, dx = 0 \quad \forall v \in V_h$$

"Galerkin orthogonality"

Weighted L_2-norm: $\|w\|_a = \left(\int_0^1 a w^2 \, dx \right)^{1/2}$

Energy norm: $\|v\|_E = \|v'\|_a = \left(\int_0^1 a (v')^2 \, dx \right)^{1/2}$

A priori error estimate:

$$\|(u-U)'\|_a^2 = \int_0^1 a (u-U)'(u-U)' \, dx$$

$$= \int_0^1 a (u-U)'(u-v)' \, dx + \underbrace{\int_0^1 a \cancel{(u-U)'} (v-U)' \, dx}_{\text{G.O.}}$$

$$= \int_0^1 a (u-U)'(u-v)' \, dx \le \|(u-U)'\|_a \|(u-v)'\|_a$$

$$\Rightarrow \|(u-U)'\|_a \le \|(u-v)'\|_a \quad \forall v \in V_h$$

FEM solution $U \in V_h$ optimal in energy norm!

Now choose $v = \pi_h u \in V_h$

(the error estimate is true for all $v \in V_h$)

$$\Rightarrow \quad \| (u-U)' \|_a^2 \leq \| (u - \pi_h u)' \|_a$$

Interpolation error estimates then gives

that $\quad \| (u - \pi_h u)' \|_a \leq C_i \| h u'' \|_a$

$$\Rightarrow \quad \| u - U \|_E = \| u' - U' \|_a \leq C_i \| h u'' \|_a$$

(Theorem 8.1) <u>A priori error estimate</u>

since π depend on solution $u(x)$.

<u>A posteriori error estimates</u> :

$$e = u - U \Rightarrow \| e' \|_a^2 = \int_0^1 a e' e' \, dx$$

$$= \int_0^1 a u' e' \, dx - \int_0^1 a U' e' \, dx = \int_0^1 f e \, dx - \int_0^1 a U' e' \, dx$$

FEM $(*)$ with $v = \pi_h e \in V_h \Rightarrow$

$$\| e' \|_a^2 = \int_0^1 f (e - \pi_h e) \, dx - \int_0^1 a U' (e - \pi_h e)' \, dx$$

$$= \int_0^1 f (e - \pi_h e) \, dx - \sum_{j=1}^{M+1} \int_{I_j} a U' (e - \pi_h e)' \, dx$$

Integrate by parts over each subinterval I_j, and use that $(e-\pi_h e)(x_j)=0$ $\forall$ nodes x_j.

$$\Rightarrow \quad \|e'\|_a^2 = \int_0^1 R(U)\,(e-\pi_h e)\,dx$$

with (discontinuous) residual

$$R(U) = f + (aU')' \quad \text{on each } I_j$$

$$\Rightarrow \quad \|e'\|_a^2 \leq \|h R(U)\|_{a^{-1}} \, \|h^{-1}(e-\pi_h e)\|_a$$

Interpolation error est. $\|h^{-1}(e-\pi_h e)\|_a \leq C_i \|e'\|_a$

$$\Rightarrow \quad \|u'-U'\|_a \leq C_i \|h R(U)\|_{a^{-1}}$$

<u>2D Poisson</u>: FEM: find $U \in V_h$ s.t.

$$(\nabla U, \nabla v) = (f, v) \quad \forall v \in V_h$$

$V_h = \{ v : v \text{ p.w. linear cont. on mesh } \mathcal{T}_h \}$

Galerkin orthogonality : $(\nabla u - \nabla U, \nabla v) = 0 \quad \forall v \in V_h$

A priori error estimate of error $e = u - U$:

$$\|\nabla e\|^2 = (\nabla e, \nabla(u-U)) = (\nabla e, \nabla(u-U)) + (\nabla e, \nabla(\underbrace{U-v}_{\in V_h}))$$

$$= (\nabla e, \nabla(u-v)) \leq \|\nabla e\| \, \|\nabla(u-v)\|$$

$\Rightarrow \|\nabla e\| \leq \|\nabla(u-v)\| \quad \forall v \in V_h$ ($U \in V_h$ optimal w.r.t. energy norm)

Interpolation error estimate : choose $v = \pi_h u$

$$\Rightarrow \|\nabla(u-U)\| \leq \|\nabla(u - \pi_h u)\| \leq C_i \|h D^2 u\|$$

A posteriori error estimate :

$$\|\nabla e\|^2 = (\nabla(u-U), \nabla e) = (\nabla u, \nabla e) - (\nabla U, \nabla e)$$

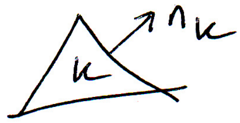
$$= (f, e) - (\nabla U, \nabla e) = (f, e - \tilde{\pi}_h e) - (\nabla U, \nabla(e - \tilde{\pi}_h e))$$

with $\tilde{\pi}_h e \in V_h$ interpolant based on weighted averages around node.

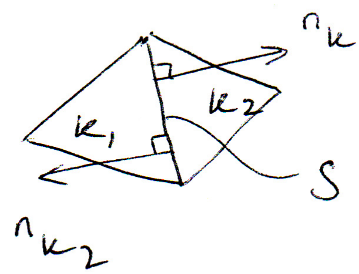
Divide integrals over elements K & Green's formula

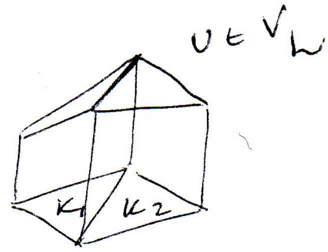
$$\|\nabla e\|^2 = \sum_K \int_K (f + \Delta U)(e - \tilde{\pi}_h e)\, dx - \sum_K \int_{\partial K} \frac{\partial U}{\partial n_K}(e - \tilde{\pi}_h e)\, ds$$

$\frac{\partial U}{\partial n_K}$ with respect to outward normal n_K of element boundary ∂K



Each internal edge $S \in \mathcal{S}_h$ occurs twice, with opposite signs of outward normal:





Let $\partial_S v$ be the derivative of v in one of the directions n_{K_1} or n_{K_2}: $\partial_S v$ will in general be discontinuous over S.

Let $[\partial_S v]$ be the jump (difference) in $\partial_S v$ from the two triangles K_1 & K_2.

Rewrite the surface integrals over $\partial\Omega$ as integrals over S instead:

$$\sum_K \int_{\partial K} \frac{\partial U}{\partial n_K}(e - \tilde{\pi}_h e)\, ds = \sum_{S \in \mathcal{S}_h} \int_S [\partial_S U](e - \tilde{\pi}_h e)\, ds$$

$$\Rightarrow \|\nabla e\|^2 = \sum_K \int_K (f + \Delta U)(e - \tilde{\pi}_h e)\, dx + \sum_{S \in \mathcal{S}_h} \int_S [\partial_S U](e - \tilde{\pi}_h e)\, ds$$

Then distribute the jump to each of the two triangles to get

$$\|\nabla e\|^2 = \sum_K \int_K (f + \Delta U)(e - \tilde{\pi}_h e)\, dx + \frac{1}{2}\int_{\partial K} [\partial_S U](e - \tilde{\pi}_h e)\, ds$$

<u>First term</u>: $\left| \sum_K \int_K h_K (f+\Delta U) h_K^{-1} (e - \tilde{\pi}_h e)\, dx \right|$

$\leq \| h R_1(U) \| \, \| h^{-1}(e - \tilde{\pi}_h e) \| \leq C_i \| h R_1(U) \| \, \| \nabla e \|$

with $R_1(U) = |f + \Delta U|$ on each $K \in \mathcal{T}_h$.

<u>Second term</u>: $\left| \sum_K \frac{1}{2} \int_{\partial K} h_K^{-1} [\partial_s U] (e - \tilde{\pi}_h e) h_K \, ds \right|$

$\leq \ldots$

$\Rightarrow$ <u>A posteriori error estimate</u> (Thm 15.3)

There exist constant C_i depending only on $\mathcal{T}_h$ s.t.

$$\| \nabla u - \nabla U \| \leq C_i \| h R(U) \|$$

with $R(U) = R_1(U) + R_2(U)$

$R_1(U) = |f + \Delta U|$ on $K \in \mathcal{T}_h$

$R_2(U) = \frac{1}{2} \max_{S \subset \partial K} h_K^{-1} |[\partial_s U]|$ on $K \in \mathcal{T}_h$

<u>Further reading</u>:

CDE 5.14.2 (interpolation)

CDE 8.2–8.6, 15.2–15.3 (error estimation)

Non-homogeneous Dirichlet b.c.

Poisson 1D : $\begin{cases} -u'' = f & \text{on } (0,1) \\ u(0) = g, \; u(1) = 0 \end{cases}$

Variational formulation (mult. by test function & integrate over $(0,1)$) :

$$\int_0^1 -u'' v \, dx = \int_0^1 f v \, dx$$

Partial integration $\Rightarrow \int_0^1 u'v' dx - [u'v]_0^1 = \int_0^1 fv \, dx$

Choose $v \in V_0 = \{ v : \|v\|^2 + \|v'\|^2 < \infty, \; v(0) = v(1) = 0 \}$

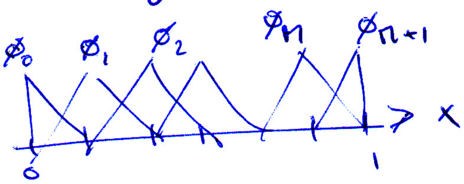
$\Rightarrow [u'v]_0^1 = 0$

Search for $u \in V_g = \{ v : \|v\|^2 + \|v'\|^2 < \infty, \; v(0) = g, \; v(1) = 0 \}$.

Find $u \in V_g$: $\int_0^1 u'v' dx = \int_0^1 f v \, dx \quad \forall v \in V_0$

FEM : Find $U \in V_h^g \subset V_g$: $\int_0^1 U'v' dx = \int_0^1 fv \, dx \quad \forall v \in V_h^0 \subset V_0$

$$U = \sum_{j=0}^{M+1} \xi_j \phi_j(x)$$



$U(1) = \xi_{M+1} = 0$; $U(0) = \xi_0 = g$

$$\Rightarrow U = \sum_{j=1}^{M} \xi_j \phi_j(x) + g \phi_0(x)$$

FEM : $\int_0^1 U'v' dx = \sum_{j=1}^{M} \xi_j \int_0^1 \phi_j' v' dx + g \int_0^1 \phi_0' v' dx = \int_0^1 fv dx \quad \forall v \in V_h^0$

$$\Rightarrow \underbrace{\sum_{j=1}^{M} \xi_j \int_0^1 \phi_j \phi_i \, dx}_{Ax} = \underbrace{\int_0^1 f \phi_i \, dx - g \int_0^1 \phi_0' \phi_i \, dx}_{b} \quad i = 1, \dots, M$$