# Principles of Wireless Sensor Networks

https://www.kth.se/social/course/EL2745/

Lecture 8

## Static Distributed Estimation

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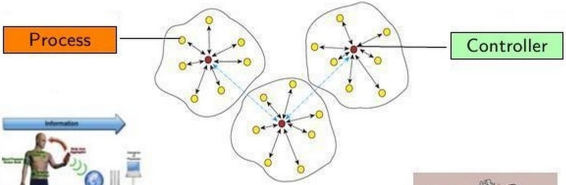
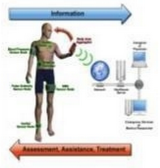
# Course content

- Part 1
  - Lec 1: Introduction to WSNs
  - Lec 2: Introduction to Programming WSNs
- Part 2
  - Lec 3: Wireless Channel
  - Lec 4: Physical Layer
  - Lec 5: Medium Access Control Layer
  - Lec 6: Routing
- Part 3
  - Lec 7: Distributed Detection
  - Lec 8: Static Distributed Estimation
  - Lec 9: Dynamic Distributed Estimation
  - Lec 10: Positioning and Localization
  - Lec 11: Time Synchronization
- Part 4
  - Lec 12: Wireless Sensor Network Control Systems 1
  - Lec 13: Wireless Sensor Network Control Systems 2
  - Lec 14: Summary and Project Presentations

# Today's lecture




- Today we study how to perform static estimation from noisy measurements of the sensors

- "Static" means that the estimation is of a variable (constant or random) that does not evolve over time

# Motivation for Static Estimation

- Plays a central role in many WSNs applications
- Accurately predicts the **parameters** of a **phenomenon**
- **Communication:** Position, navigation
- **Monitoring:** Pollution, Process Conearthquake magnitude
- **Surveillance:** Crowd density, intruders, attitude

# Today's learning goals

- What are the **fundamental aspects** of distributed estimation?
- Estimation over a Star and a General topology?
- What is the LMMSE estimator?
- How to make a static sensor fusion?

# Outline

- Star and general topologies
- Estimation from one sensor
- Distributed estimation in a star topology
- Distributed estimation in a general topology

# Outline

- Star and general topology
- Estimation from one sensor
  - Model of the measurements for one sensor
  - Model of the estimator
  - Mean Squared Error (MSE)
  - LMMSE estimate
- Distributed estimation from many sensors
  - Star topology
  - General topology

# Topology 1: Star topology

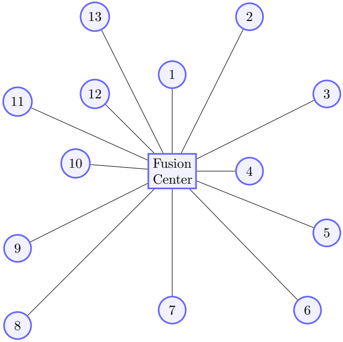



Figure: Network with a star topology: Solid lines indicating that there is message communication between nodes. The fusion center receives information from all other nodes.

- The **phenomenon** is observed by a number of sensors organized as a star
- Multiple sensors make measurements
- Measurements are transmitted to a fusion center

# Topology 2: General topology

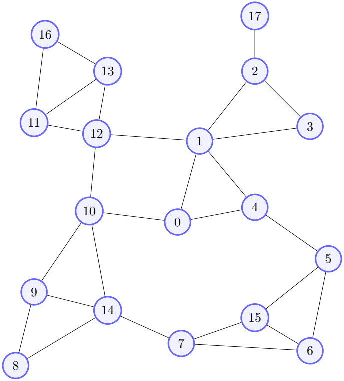



Figure: Network with an Arbitrary Topology: Solid lines indicating that there is communication between nodes. There is no node acting as fusion center.

- The **phenomenon** is observed by a number of sensors organized arbitrarily
- Multiple sensors make measurements
- Measurements are not transmitted to a fusion center
  - Indeed, no fusion center. Every node is a sort of local fusion center

# Outline

- Star and General topology
- Estimation from one sensor
  - Model of the measurements for one sensor
  - Model of the estimator
  - Mean Squared Error (MSE)
  - LMMSE estimate
- Distributed estimation from many sensors
  - Star topology
  - General topology

# Model of the measurements for one sensor

- Let's consider only one sensor
- **Linear** measurements (i.e., measurements and the parameters are related linearly) with **noise** or **measurement errors**

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v} \tag{1}$$

- $\mathbf{y}$: sensor measurement(s)
- $\mathbf{H}$: a known matrix
- $\mathbf{x}$: what we want to estimate
- $\mathbf{v}$: unknown noise or measurement error

# Model of the measurements for one sensor

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- $\mathbf{y}$: sensor measurement(s)
- $\mathbf{H}$: a known matrix
- $\mathbf{x}$: what we want to estimate
- $\mathbf{v}$: unknown noise or measurement error
- Goal: How to estimate **x** out of the measurement **y**?

# Model of the estimator

**Linear** estimator, i.e., the estimator and the measurements are assumed to be linearly related

$$\hat{\mathbf{x}}(\mathbf{L}) = \mathbf{L}\mathbf{y}$$

- $\mathbf{y}$: sensor measurement(s)
- $\hat{\mathbf{x}}(\mathbf{L})$: estimator of $\mathbf{x}$, dependent on $\mathbf{L}$
- We need to compute a good estimate $\hat{\mathbf{x}}(\cdot) \Rightarrow$ what matrix $\mathbf{L}$ to be used?
- **Performance criterion** for computing $\mathbf{L}$?

# Mean Squared Error (MSE)

A good estimate $\hat{\mathbf{x}}(\cdot)$ is found by considering the **MSE**, which is given by the trace of **error covariance matrix** $\mathbf{C}$ of the estimator error

- In particular, for fixed $\mathbf{L}$, MSE is defined as

$$\begin{aligned} \mathrm{MSE}(\mathbf{L}) &= \mathsf{Tr}\left\{\mathbf{C}(\mathbf{L})\right\} \\ &= \mathsf{Tr}\left\{\mathsf{E}\left\{\left(\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x}\right)\left(\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x}\right)^{\mathsf{T}}\right\}\right\} \\ &= \textstyle\sum_{i=1}^{N} \mathsf{E}(\hat{x}_i(\mathbf{L}) - x_i)^2 \end{aligned}$$

- Let $\mathbf{L}^{\star} = \arg\min_{\mathbf{L}} \mathrm{MSE}(\mathbf{L})$

- Then, $\hat{\mathbf{x}} = \mathbf{L}^{\star}\mathbf{y}$ is called the **linear minimum MSE (LMMSE)** estimate of $\mathbf{x}$

# LMMSE estimate

## Proposition 1

*Consider a random variable* $\mathbf{x}$ *being observed by a sensor that generates measurements of the form* $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$. *Then LMMSE estimator of* $\mathbf{x}$ *given* $\mathbf{y}$ *is*

$$\hat{\mathbf{x}} = \underbrace{\mathbf{P}\mathbf{H}^T\mathbf{R}_{\mathbf{v}}^{-1}}_{\mathbf{L}^\star}\mathbf{y} \ ,$$

*where*

$$\mathbf{P} = \left(\mathbf{R}_{\mathbf{x}}^{-1} + \mathbf{H}^T\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\right)^{-1} \ ,$$

$\mathbf{R}_{\mathbf{x}}$ is the covariance matrix of $\mathbf{x}$, and $\mathbf{R}_{\mathbf{v}}$ is the noise covariance matrix.

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- We need to show that $\mathbf{L}^\star = \mathbf{P}\mathbf{H}^\mathsf{T}\mathbf{R}_{\mathbf{v}}^{-1}$

# LMMSE estimate proof

Advanced topic, not requested for the exam

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**Proof:**

Preliminaries:

$$
\begin{aligned}
&(1) \quad \mathbf{A} + \mathbf{B} \succeq \mathbf{B} \text{ when } \mathbf{A} \succeq \mathbf{0} \\
&(2) \quad \mathbf{A} \succeq \mathbf{B} \Rightarrow \mathsf{Tr}(\mathbf{A}) \geq \mathsf{Tr}(\mathbf{B}) \\
&(3) \quad (\mathbf{A} + \mathbf{BC})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{I} + \mathbf{CA}^{-1}\mathbf{B}\right)^{-1}\mathbf{CA}^{-1}
\end{aligned}
$$

# LMMSE estimate proof

**Proof:**

$$\mathbf{C}(\mathbf{L}) = \mathsf{E}\left\{(\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x})(\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x})^\mathsf{T}\right\} = \mathsf{E}\left\{(\mathbf{L}\mathbf{y} - \mathbf{x})(\mathbf{L}\mathbf{y} - \mathbf{x})^\mathsf{T}\right\}$$

# LMMSE estimate proof

**Proof:**

$$\mathbf{C}(\mathbf{L}) = \mathsf{E}\left\{\left(\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x}\right)\left(\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x}\right)^{\mathsf{T}}\right\} = \mathsf{E}\left\{\left(\mathbf{L}\mathbf{y} - \mathbf{x}\right)\left(\mathbf{L}\mathbf{y} - \mathbf{x}\right)^{\mathsf{T}}\right\}$$
$$= \mathsf{E}\left\{\left(\mathbf{L}\mathbf{H} - \mathbf{I}\right)\mathbf{x}\mathbf{x}^{\mathsf{T}}\left(\mathbf{L}\mathbf{H} - \mathbf{I}\right)^{\mathsf{T}} + \mathbf{L}\mathbf{v}\mathbf{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}}\right\} = \left(\mathbf{L}\mathbf{H} - \mathbf{I}\right)\mathbf{R}_{\mathbf{x}}\left(\mathbf{L}\mathbf{H} - \mathbf{I}\right)^{\mathsf{T}} + \mathbf{L}\mathbf{R}_{\mathbf{v}}\mathbf{L}^{\mathsf{T}}$$

# LMMSE estimate proof

**Proof:**

$$
\begin{aligned}
\mathbf{C}(\mathbf{L}) &= \mathsf{E}\left\{(\hat{\mathbf{x}}(\mathbf{L})-\mathbf{x})(\hat{\mathbf{x}}(\mathbf{L})-\mathbf{x})^{\mathsf{T}}\right\} = \mathsf{E}\left\{(\mathbf{Ly}-\mathbf{x})(\mathbf{Ly}-\mathbf{x})^{\mathsf{T}}\right\} \\
&= \mathsf{E}\left\{(\mathbf{LH}-\mathbf{I})\mathbf{x}\mathbf{x}^{\mathsf{T}}(\mathbf{LH}-\mathbf{I})^{\mathsf{T}} + \mathbf{L}\mathbf{v}\mathbf{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}}\right\} = (\mathbf{LH}-\mathbf{I})\mathbf{R_x}(\mathbf{LH}-\mathbf{I})^{\mathsf{T}} + \mathbf{L}\mathbf{R_v}\mathbf{L}^{\mathsf{T}} \\
&= \mathbf{L}\left(\mathbf{H}\mathbf{R_x}\mathbf{H}^{\mathsf{T}} + \mathbf{R_v}\right)\mathbf{L}^{\mathsf{T}} - \mathbf{L}\mathbf{H}\mathbf{R_x} - \mathbf{R_x}\mathbf{H}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} + \mathbf{R_x} \\
&= \left(\mathbf{L}-\mathbf{R_x}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R_x}\mathbf{H}^{\mathsf{T}}+\mathbf{R_v}\right)^{-1}\right)\left(\mathbf{H}\mathbf{R_x}\mathbf{H}^{\mathsf{T}}+\mathbf{R_v}\right)\left(\mathbf{L}-\mathbf{R_x}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R_x}\mathbf{H}^{\mathsf{T}}+\mathbf{R_v}\right)^{-1}\right) \\
&\qquad + \mathbf{R_x} - \mathbf{R_x}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R_x}\mathbf{H}^{\mathsf{T}}+\mathbf{R_v}\right)^{-1}\mathbf{H}\mathbf{R_x}
\end{aligned}
$$

# LMMSE estimate proof

**Proof:**

$$
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\mathbf{C}(\mathbf{L}) &= \mathsf{E}\left\{\left(\hat{\mathbf{x}}(\mathbf{L})-\mathbf{x}\right)\left(\hat{\mathbf{x}}(\mathbf{L})-\mathbf{x}\right)^{\mathsf{T}}\right\} = \mathsf{E}\left\{\left(\mathbf{L}\mathbf{y}-\mathbf{x}\right)\left(\mathbf{L}\mathbf{y}-\mathbf{x}\right)^{\mathsf{T}}\right\} \\
&= \mathsf{E}\left\{\left(\mathbf{L}\mathbf{H}-\mathbf{I}\right)\mathbf{x}\mathbf{x}^{\mathsf{T}}\left(\mathbf{L}\mathbf{H}-\mathbf{I}\right)^{\mathsf{T}} + \mathbf{L}\mathbf{v}\mathbf{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}}\right\} = \left(\mathbf{L}\mathbf{H}-\mathbf{I}\right)\mathbf{R}_{\mathbf{x}}\left(\mathbf{L}\mathbf{H}-\mathbf{I}\right)^{\mathsf{T}} + \mathbf{L}\mathbf{R}_{\mathbf{v}}\mathbf{L}^{\mathsf{T}} \\
&= \mathbf{L}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}+\mathbf{R}_{\mathbf{v}}\right)\mathbf{L}^{\mathsf{T}} - \mathbf{L}\mathbf{H}\mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} + \mathbf{R}_{\mathbf{x}} \\
&= \left(\mathbf{L}-\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}+\mathbf{R}_{\mathbf{v}}\right)^{-1}\right)\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}+\mathbf{R}_{\mathbf{v}}\right)\left(\mathbf{L}-\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}+\mathbf{R}_{\mathbf{v}}\right)^{-1}\right) \\
&\qquad\qquad + \mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}+\mathbf{R}_{\mathbf{v}}\right)^{-1}\mathbf{H}\mathbf{R}_{\mathbf{x}} \\
&\succeq \mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}+\mathbf{R}_{\mathbf{v}}\right)^{-1}\mathbf{H}\mathbf{R}_{\mathbf{x}}
\end{aligned}
$$

# LMMSE estimate proof

The lower bound is achieved when

$$
\begin{aligned}
\mathbf{L} &= \mathbf{R_x}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R_x}\mathbf{H}^{\mathsf{T}}+\mathbf{R_v}\right)^{-1} \\
&= \mathbf{R_x}\mathbf{H}^{\mathsf{T}}\left(\mathbf{R_v}^{-1} - \mathbf{R_v}^{-1}\mathbf{H}\left(\mathbf{I}+\mathbf{R_x}\mathbf{H}^{\mathsf{T}}\mathbf{R_v}^{-1}\mathbf{H}\right)^{-1}\mathbf{R_x}\mathbf{H}^{\mathsf{T}}\mathbf{R_v}^{-1}\right) \\
&= \left(\mathbf{I} - \mathbf{R_x}\mathbf{H}^{\mathsf{T}}\mathbf{R_v}^{-1}\mathbf{H}\left(\mathbf{I}+\mathbf{R_x}\mathbf{H}^{\mathsf{T}}\mathbf{R_v}^{-1}\mathbf{H}\right)^{-1}\right)\mathbf{R_x}\mathbf{H}^{\mathsf{T}}\mathbf{R_v}^{-1} \\
&= \left(\mathbf{I}+\mathbf{R_x}\mathbf{H}^{\mathsf{T}}\mathbf{R_v}^{-1}\mathbf{H}\right)^{-1}\mathbf{R_x}\mathbf{H}^{\mathsf{T}}\mathbf{R_v}^{-1} = \left(\mathbf{R_x}^{-1}+\mathbf{H}^{\mathsf{T}}\mathbf{R_v}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathsf{T}}\mathbf{R_v}^{-1} = \mathbf{P}\mathbf{H}^{\mathsf{T}}\mathbf{R_v}^{-1} \ \square
\end{aligned}
$$

# LMMSE estimate

**Recap:**

Consider the linear system of measurements given in (1), i.e., $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$. Let $\hat{\mathbf{x}}$ denote the LMMSE estimator of $\mathbf{x}$ given $\mathbf{y}$. Then we have

$$\mathbf{P}^{-1}\hat{\mathbf{x}} = \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{y} \ , \tag{2}$$

where

$$\mathbf{P} = \left(\mathbf{R}_{\mathbf{x}}^{-1} + \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\right)^{-1} = \text{error covariance of } \hat{\mathbf{x}}.$$

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- Several measurements from **one sensor** can be seen in case of **multiple sensors**

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- **No** need to send all the measurements to a **central data processing**

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Consider the linear system of measurements given in (1), i.e., $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$. Let $\hat{\mathbf{x}}$ denote the LMMSE estimator of $\mathbf{x}$ given $\mathbf{y}$. Then we have

$$\mathbf{P}^{-1}\hat{\mathbf{x}} = \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{y} \ , \qquad (2)$$

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- Relation (2) has been derived for **the case of one sensor**
- In **the case of multiple sensors**, relation (2) suggests the possibility of **combining local estimates directly**
- Several measurements from **one sensor** can be seen in case of **multiple sensors**
- **No** need to send all the measurements to a **central data processing**
- This is called **static sensor fusion**

# Considering $\mathbf{x}$ as a deterministic value

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- If the information included has a non zero mean the estimate need to be corrected in the following way

$$\mathbf{P}^{-1}(\hat{\mathbf{x}} - \bar{\mathbf{x}}) = \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}(\mathbf{y} - H\bar{\mathbf{x}})$$

- We assume $\bar{\mathbf{x}} = 0$ for readability reasons

# Outline

- Star and General topology
- Estimation from one sensor
  - Model of the measurements for one sensor
  - Model of the estimator
  - Mean Squared Error (MSE)
  - LMMSE estimate
- Distributed estimation from many sensors
  - Star topology
  - General topology

# Static sensor fusion, star topology

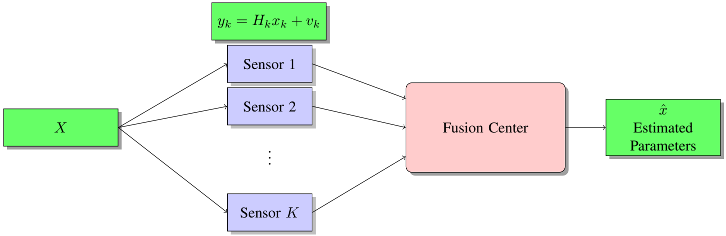


Figure: Illustration of how the process in static sensor fusion is preformed.

- Now we move to a case of **many sensors in a star topology**

# Static sensor fusion, star topology

## Proposition 2

*Consider a random variable* $\mathbf{x}$ *being observed by* $K$ *sensors that generate measurements of the form*

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \mathbf{v}_k, \quad k = 1, \ldots, K$$

- $\mathbf{y}_k =$ $k$*th sensor measurement(s)*
- $\mathbf{H}_k =$ *a matrix known to the* $k$*th sensor*
- $\mathbf{x} =$ *what we want to estimate*
- $\mathbf{v}_k =$ *noise or measurement error at* $k$*th sensor,* $\mathbf{v}_k$ *and* $\mathbf{v}_j$ $(j \neq k)$ *are uncorelated*

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*Let* $\hat{\mathbf{x}}$ *denote the LMMSE estimator of* $\mathbf{x}$ *given* $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_K)$*, as obtained at the fusion center. Then*

$$\mathbf{P}^{-1}\hat{\mathbf{x}} = \sum_{k=1}^{K} \mathbf{P}_k^{-1}\hat{\mathbf{x}}_k \ ,$$

*where* $\mathbf{P}$ *is the estimate error covariance corresponding to* $\hat{\mathbf{x}}$ *and* $\mathbf{P}_k$ *is the error covariance corresponding to* $\hat{\mathbf{x}}_k$*.*

# Static sensor fusion, star topology

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*where* $\mathbf{P}$ *is the estimate error covariance corresponding to* $\hat{\mathbf{x}}$ *and* $\mathbf{P}_k$ *is the error covariance corresponding to* $\hat{\mathbf{x}}_k$*. Furthermore,*

$$\mathbf{P}^{-1} = -(K-1)\mathbf{R}_{\mathbf{x}}^{-1} + \sum_{k=1}^{K} \mathbf{P}_k^{-1} \ ,$$

# Proof of proposition 2

**Proof:** Note that overall linear system is given by

$$\underbrace{\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_K \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{H}_1 \\ \vdots \\ \mathbf{H}_K \end{bmatrix}}_{\mathbf{H}} \mathbf{x} + \underbrace{\begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_K \end{bmatrix}}_{\mathbf{v}}$$

Now use Proposition 1

$$\mathbf{P}^{-1}\hat{\mathbf{x}} = \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{y} = \begin{bmatrix} \mathbf{H}_1^{\mathsf{T}} \cdots \mathbf{H}_K^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{\mathbf{v}_1}^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{R}_{\mathbf{v}_2}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{R}_{\mathbf{v}_K}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_K \end{bmatrix}$$

$$= \sum_{k=1}^{K} \mathbf{H}_k^{\mathsf{T}} \mathbf{R}_{\mathbf{v}_k}^{-1} \mathbf{y}_k$$

$$= \sum_{k=1}^{K} \mathbf{P}_k^{-1} \hat{\mathbf{x}}_k$$

# Proof of proposition 2

Moreover, from Proposition 1

$$
\begin{aligned}
\mathbf{P}^{-1} &= \mathbf{R}_{\mathbf{x}}^{-1} + \underbrace{\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}} \\
&= \mathbf{R}_{\mathbf{x}}^{-1} + \textstyle\sum_{k=1}^{K} \underbrace{\mathbf{H}_k^{\mathsf{T}}\mathbf{R}_{\mathbf{v}_k}^{-1}\mathbf{H}_k} \\
&= \mathbf{R}_{\mathbf{x}}^{-1} + \textstyle\sum_{k=1}^{K} \left(\mathbf{P}_k^{-1} - \mathbf{R}_{\mathbf{x}}^{-1}\right) = -(K-1)\mathbf{R}_{\mathbf{x}}^{-1} + \textstyle\sum_{k=1}^{K} \mathbf{P}_k^{-1} \ ,
\end{aligned}
$$

# Static sensor fusion from multiple sensors

- By Proposition 2, **complexity** of the **fusion center goes down** considerably
- **Some computational load is delegated** to the distributed **sensors**
- **Each estimate is weighted** by the inverse of the error covariance matrix
- The **higher the confidence** we have in a particular sensor, the **higher the trust** we place in its measurement

# Outline

- Star and General topology
- Estimation from one sensor
  - Model of the measurements for one sensor
  - Model of the estimator
  - Mean Squared Error (MSE)
  - LMMSE estimate
- Distributed estimation from many sensors
  - Star topology
  - General topology

# Network with arbitrary topology

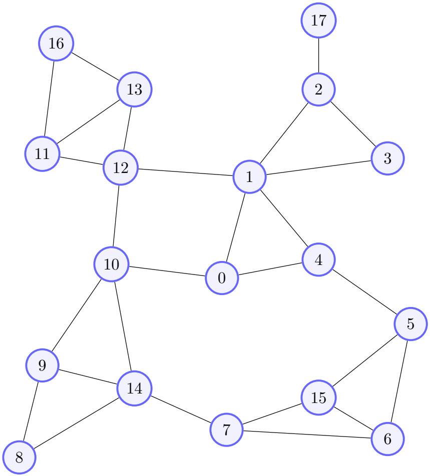


Figure: Network with a Arbitrary Topology: Solid lines indicating that there is message communication between nodes. There is no node acting as fusion center.

# Network with arbitrary topology

- Star topology: essentially a **two step procedure**
  - All the nodes transmit local estimates to a central node (**called fusion center**)
  - Central node calculates and transmits the weighted sum of the local estimates back
- Final outcome is a **weighted average**
- $\Rightarrow$ Generalize the approach to an **arbitrary graph**
- This approaches are along the lines of **average consensus algorithms**
- **No fusion center**

# Static sensor fusion with limited communication range

**Example scenario:**

- $K$ nodes, each measure a **scalar** value $x$, measurements are noisy
- Nodes are connected according to an arbitrary graph
- Each node wants to calculate the average of all the scalars

$$y_k = x + v_k \ , \ k = 1, \ldots, K$$

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**Important:** Provided the noise is iid Gaussian then the maximum likelihood **(ML)** estimate $\hat{x}$ of $x$ is given by the average of all $y_k$ values, i.e.,

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**Question:** How to obtain $\hat{x}$ just by coordinating with **adjacent neighbors** (no central fusion center)?

# Static sensor fusion with limited communication range

**One way:**

- Iterative method, iterations $n = 0, 1, 2, \dots$
- Each sensor $k$, during iteration $0$, set $x_{0,k} = y_k$
- Each sensor $k$ implements the dynamical system

$$x_{n+1,k} = x_{n,k} + h \sum_{j \in \mathcal{N}_k} (x_{n,j} - x_{n,k}) \ ,$$

  where $\mathcal{N}_k$ is the adjacent sensors of sensor $k$
- Just **local communications**

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- Just **local communications**
- **Compact form**

$$\mathbf{x}_{n+1} = (\mathbf{I} - h\mathbf{L})\mathbf{x}_n \ , \ n = 0, 1, 2, \ldots \ ,$$

  where $\mathbf{L}$ is the **Graph Laplacian matrix**

# Static sensor fusion with limited communication range

**Question:** When $n \to \infty$ do we get $(\mathbf{x}_{n+1})_k = \hat{x}$ for all $k = 1, \ldots, K$

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**Answer:** YES

if and only if

$$(\mathbf{I} - h\mathbf{L})\mathbf{1} = \mathbf{1}, \quad \mathbf{1}^\mathsf{T}(\mathbf{I} - h\mathbf{L}) = \mathbf{1}^\mathsf{T}, \quad \rho\left((\mathbf{I} - h\mathbf{L}) - \mathbf{1}\mathbf{1}^\mathsf{T}\right) < 1$$

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- Condition 2 is true: Each column sum of $\mathbf{L}$ is $0$
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the idea extends in a straightforward manner to more general models such as

$$x_{n+1,k} = x_{n,k} + h\mathbf{W}_k^{-1} \sum_{j \in \mathcal{N}_k} (x_{n,j} - x_{n,k})$$

# Summary

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# Next lecture

- Dynamic distributed estimation