Lecture 5

$$\begin{cases} -(au')' + cu = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

with $a(x) > 0$ & $c(x) \geq 0$

Variational formulation: Find $u \in V$ s.t.

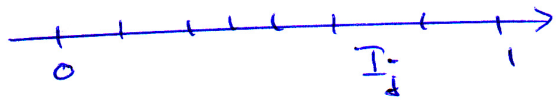
$$\int_0^1 (au'v' + cuv)\,dx = \int_0^1 fv\,dx \quad \forall v \in V$$

$$V = \{ v : \|v\|^2 + \|v'\|^2 < \infty,\ v(0) = v(1) = 0 \}$$

Galerkin FEM (cG(1)): Find $U \in V_h$ s.t.

$$\int_0^1 (aU'v' + cUv)\,dx = \int_0^1 fv\,dx \quad \forall v \in V_h$$

$V_h = \{ v : v$ cont. p.w. linear on I_j, $v(0) = v(1) = 0$



$I_j = (x_{j-1}, x_j)$

We want to estimate the L_2-norm of the error $e = u - U$. To do this we introduce the dual problem:

$$\begin{cases} -(a\varphi')' + c\varphi = e & \text{in } (0,1) \\ \varphi(0) = \varphi(1) = 0 \end{cases}$$

A posteriori error estimate of L_2 error

(2)

$$\|e\|^2 = \int_0^1 e(-(a\varphi')' + c\varphi)\,dx = \int_0^1 (a e'\varphi' + c e\varphi)\,dx$$

$$= \int_0^1 (a u'\varphi' + c u\varphi)\,dx - \int_0^1 (a U'\varphi' + c U\varphi)\,dx$$

$$= \int_0^1 f\varphi\,dx - \int_0^1 (a U'\varphi' + c U\varphi)\,dx$$

$$= \int_0^1 f(\varphi - \pi_h\varphi) - \sum_{j=1}^{M+1} \int_{I_j} (a U'(\varphi - \pi_h\varphi)' + cU(\varphi - \pi_h\varphi))\,dx$$

Integrate by parts, using that boundary terms vanish since $(\varphi - \pi_h\varphi)(x_j) = 0$ for all nodes x_j.

$$\|e\|^2 = \int_0^1 (f + (aU')' - cU)(\varphi - \pi_h\varphi)\,dx$$

$$= \int_0^1 h^2 R(U)\, h^{-2}(\varphi - \pi_h\varphi)\,dx$$

$$\leq \|h^2 R(U)\| \|h^{-2}(\varphi - \pi_h\varphi)\|$$

with residual $R(U) = f + (aU')' - cU$

Interpolation error estimate $\|h^{-2}(\varphi - \pi_h\varphi)\| \leq C_i \|\varphi''\|$

$$\Rightarrow \|e\|^2 \leq \|h^2 R(U)\|\, C_i \|\varphi''\| \frac{\|e\|}{\|e\|}$$

Thm. 15.4

$$\|e\| \leq C_i \|h^2 R(U)\| \frac{\|\varphi''\|}{\|e\|} \leq \boxed{S\, C_i \|h^2 R(U)\|}$$

with stability factor $S = \max_{\xi \in L_2(I)} \frac{\|\varphi''\|}{\|\xi\|}$

where φ is the dual solution with e replaced by ξ as data.

A priori error estimate for L_2-norm

Assume $c=0$ & constant mesh size h.

Theorem 15.5: $\|u-U\| \le C_i S_a \|h(u-U)'\|_a \le C_i^2 S_a \|h^2 u''\|_a$

with $S_a = \max\limits_{g \neq 0} \frac{\|\varphi''\|_a}{\|g\|}$ with φ dual sol. with g as data.

Proof: $\|e\|^2 = \int_0^1 a e' \varphi' \, dx = \int_0^1 a e' (\varphi - \pi_h \varphi)' \, dx$

$\le \|h e'\|_a \|h^{-1}(\varphi - \pi_h \varphi)'\|_a \le C_i \|h e'\|_a \|\varphi''\|_a$

And note that $\|h e'\|_a \le C_i \|h^2 u''\|_a$ $\square$

2D error estimates:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

Variational formulation: Find $u \in V$ s.t.

$(\nabla u, \nabla v) = (f, v) \quad \forall v \in V$

cG(1): Find $U \in V_h$ s.t.

$(\nabla U, \nabla v) = (f, v) \quad \forall v \in V_h$

Dual problem:

$$\begin{cases} -\Delta \varphi = e & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma \end{cases}$$

Thm 15.6 (Strong stability):

If Ω convex with polygonal boundary, or if Ω is general with smooth boundary, then there exist a constant S independent of f such that

$$\|D^2 u\| \leq S\|\Delta u\| = S\|f\|$$

(If Ω convex $\Rightarrow S = 1$)

A priori error estimate:

$$\|e\|^2 = (u-U, u-U) = (\nabla(u-U), \nabla\varphi)$$

$$= (\nabla(u-U), \nabla(\varphi - \pi_h\varphi)) \leq C_i \|h\nabla(u-U)\| \, \|D^2\varphi\|$$

$$\leq S C_i \|h\nabla(u-U)\| \cdot \|e\|$$

$$\Rightarrow \|e\| \leq S C_i \|h\nabla(u-U)\| \leq S C_i^2 \|h^2 D^2 u\|$$

A posteriori error estimate:

$$\|e\|^2 = (\nabla(u-U), \nabla\varphi) = (f, \varphi) - (\nabla U, \nabla\varphi)$$

$$= (f, \varphi - \pi_h\varphi) - (\nabla U, \nabla(\varphi - \pi_h\varphi))$$

Partial integration, interpolation error est., strong stability $\Rightarrow$ $\|u-U\| \leq S C_i \|h^2 R(U)\|$

$$R(U) = R_1(U) + R_2(U)$$

$$R_1(U) = |f + \Delta U| \quad \text{on } K \in \mathcal{T}_h$$

$$R_2(U) = \frac{1}{2}\max_{S\subset\partial K} h_K^{-1} |[\partial_S U]| \quad \text{on } K \in \mathcal{T}_h$$

Adaptive error control

A posteriori error estimates only involves the computed solution, and can thus be used in an adaptive algorithm.

Ex. $\|\nabla u - \nabla U\| \le C_i \|h R(U)\|$

$\|u - U\| \le S C_i \|h^2 R(U)\|$

To guarantee that $\|\nabla u - \nabla U\| < TOL$, choose $h = h(x)$ such that

$C_i \|h R(U)\| < TOL$.

Adaptive alg.

1. Choose initial coarse mesh $\mathcal{T}_h^{(0)}$
2. Compute FEM solution $U \in V_h$
3. Compute residual $R(U)$ & evaluate $C_i \|h R(U)\|$
   If smaller than TOL stop, else
4. Refine the mesh where $\int_K (h_K R(U))^2 dx$ is large.

Further reading:

CDE 15.3, 15.5