# Lecture 6 (Chapter 21)

## Abstract framework

(i) Hilbert space $V$; with norm $\|\cdot\|_V$ and scaler product.

(ii) Bilinear form $a: V \times V \rightarrow \mathbb{R}$; determined by underlying differential equation.

(iii) Linear form $L: V \rightarrow \mathbb{R}$ determined by data.

We will formulate our differential equation using a bilinear and a linear form, and ~~using~~ search for a solution in the Hilbert space $V$.

A bilinear form $a(\cdot,\cdot)$ is a function taking values in $V \times V$ into $\mathbb{R}$. That is, $a(v,w) \in \mathbb{R}$ for all $v, w \in V$, such that $a(v,w)$ is linear in each argument:

$$a(\alpha_1 v_1 + \alpha_2 v_2, w_1) = \alpha_1 a(v_1, w_1) + \alpha_2 a(v_2, w_1)$$

and $a(v_1, \alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 a(v_1, w_1) + \alpha_2 a(v_1, w_2)$

for all $\alpha_i \in \mathbb{R}$, $v_i, w_i \in V$.

A linear form $L(\cdot)$ is a function on $V$ such that $L(v) \in \mathbb{R}$ for all $v \in V$, and linear in $v$: $L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$

The abstract problem is: Find $u \in V$ s.t. (2)

(*) $\quad a(u,v) = L(v)$ for all $v \in V$.

Problem: Do such solutions $u \in V$ exist?

Problem: If so: Is such a solution unique?

"Existence" & "Uniqueness"

Depending on $a(\cdot,\cdot)$, $L(\cdot)$, and $V$, we may prove existence & uniqueness of solutions.

- Assume that $a(\cdot,\cdot)$ is V-elliptic or coercive:

  There exist constant $\varkappa_1 > 0$ : $a(v,v) \geq \varkappa_1 \|v\|_V^2 \quad \forall v \in V$

- $a(\cdot,\cdot)$ is continuous: there exist constant $\varkappa_2$ such that $|a(v,w)| \leq \varkappa_2 \|v\|_V \|w\|_V \quad \forall v,v \in V$

- $L(\cdot)$ is continuous: there exist constant $\varkappa_3$ such that $|L(v)| \leq \varkappa_3 \|v\|_V \quad \forall v \in V$

Lax-Milgram Theorem (Th 21.1)

Suppose $a(\cdot,\cdot)$ is a continuous, V-elliptic bilinear form on $V$, and $L(\cdot)$ is a linear form on $V$. Then there exist unique $u \in V$ satisfying (*), and $\|u\|_V \leq \frac{\varkappa_3}{\varkappa_1}$

$L$ is "continuous" since by linearity:

$$|L(v)-L(w)| = |L(v-w)| \le \varkappa_3 \|v-w\|_V$$

so that $L(v) \to L(w)$ if $\|v-w\|_V \to 0$ ($v \to w$ in $V$).

Similar with $a(\cdot,\cdot)$.

<u>Energy norm</u>: $\|v\|_a = \sqrt{a(v,v)}$

- $\|v\|_a \ge 0$ since $\|v\|_a^2 = a(v,v) \ge \varkappa_1 \|v\|_V^2 \ge 0$
- $\|\cdot\|_a$ & $\|\cdot\|_V$ are <u>equivalent norms</u>:
  $$\varkappa_1 \|v\|_V^2 \le \|v\|_a^2 \le \varkappa_2 \|v\|_V^2$$

If we choose the norm in $V$ to be $\|\cdot\|_a$ we get $\varkappa_1 = \varkappa_2 = 1$.

<u>Abstract Galerkin method</u>: Find $U \in V_h$ s.t.

$$a(U,v) = L(v) \text{ for all } v \in V_h$$

where $V_h \subset V$ finite dimensional subspace.

<u>Galerkin orthogonality</u>: $a(u-U,v) = 0 \quad \forall v \in V_h$

A priori error estimate (Thm 21.3):

$$\|u-U\|_V \leq \frac{\varkappa_2}{\varkappa_1}\|u-v\|_V \quad \forall v \in V_h$$

If $\|\cdot\|_V = \|\cdot\|_a$ then $\|u-U\|_a \leq \|u-v\|_a$

(Galerkin solution $U$ optimal in energy norm)

Proof: For all $v \in V_h$:

$$\varkappa_1 \|u-U\|_V^2 \leq a(u-U, u-U) = a(u-U, u-U) + a(u-U, U-v)$$

$$= a(u-U, u-v) \leq \varkappa_2 \|u-U\|_V \|u-v\|_V \qquad \square$$

Sobolev spaces $H^1(\Omega)$ & $H^1_0(\Omega)$

$$H^1(\Omega) = \left\{ v : \int_\Omega (|\nabla v|^2 + v^2)\, dx < \infty \right\}$$

$$(v,w)_{H^1(\Omega)} = \int_\Omega (\nabla u \cdot \nabla w + vw)\, dx$$

$$\|v\|_{H^1(\Omega)} = \sqrt{(v,w)_{H^1(\Omega)}} = \left( \int_\Omega (|\nabla v|^2 + v^2)\, dx \right)^{1/2}$$

$H^1_0(\Omega)$ is a subspace of $H^1(\Omega)$ with the same norm and scaler product such that

$$H^1_0(\Omega) = \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma \right\}$$

Poincaré's-Friedrichs inequality (Thm 21.4):

There exist a constant $C$ depending on $\Omega$, such that for all $v \in H^1(\Omega)$:

$$\|v\|^2_{L_2(\Omega)} \leq C\left( \|v\|^2_{L_2(\Gamma)} + \|\nabla v\|^2_{L_2(\Omega)}\right)$$

Proof:

Trace theorem (Thm 21.5):

If $\Omega$ is a bounded domain with boundary $\Gamma$, then there exist a constant $C$ such that for all $v \in H^1(\Omega)$:

$$\|v\|_{L_2(\Gamma)} \leq C \|v\|_{H^1(\Omega)}$$

Proof:

Ex: $\begin{cases} -\Delta u + u = f & \text{in } \Omega \subset \mathbb{R}^d \\ \partial_n u = 0 & \text{on } \Gamma \end{cases}$ (6)

Find variational formulation: mult. by $v$ & integrate

$$\int_\Omega (-\Delta u + u) v \, dx = \int_\Omega (\nabla u \cdot \nabla v + uv) dx - \int_\Gamma \cancel{\partial_n u}^{\,0} v \, ds$$
$$= \int_\Omega f v \, dx$$

Find $u \in H^1(\Omega)$ s.t. $a(u,v) = L(v) \quad \forall v \in H^1(\Omega)$

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv) \, dx, \quad L(v) = \int_\Omega f v \, dx$$

Check if L-M applies: $V = H^1(\Omega)$.

(i) $a(\cdot,\cdot)$ V-elliptic?

$$a(v,v) = \int_\Omega (|\nabla v|^2 + v^2) \, dx = \|v\|^2_{H^1(\Omega)} = \|v\|^2_V$$

$\Rightarrow a(\cdot,\cdot)$ V-elliptic with $\alpha_1 = 1$

(ii) $a(\cdot,\cdot)$ continuous?

$$|a(v,w)| = \left|\int_\Omega (\nabla v \cdot \nabla w + vw) \, dx\right| \le \|\nabla v\| \, \|\nabla w\| + \|v\| \, \|w\|$$
$$= (\|\nabla v\| \;\; \|v\|) \cdot (\|\nabla w\| \;\; \|w\|)$$
$$\le (\|\nabla v\|^2 + \|v\|^2)^{1/2} (\|\nabla w\|^2 + \|w\|^2)^{1/2} = \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}$$
$$= \|v\|_V \|w\|_V \Rightarrow a(\cdot,\cdot) \text{ cont. with } \alpha_2 = 1$$

(iii) $L(\cdot)$ continuous?

$$|L(v)| = \left|\int_\Omega f v \, dx\right| \le \|f\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} \le \|f\|_{L_2(\Omega)} \|v\|_{H^1(\Omega)}$$
$$= \|f\|_{L_2(\Omega)} \|v\|_V \Rightarrow L(\cdot) \text{ cont. with } \alpha_3 = \|f\|_{L_2(\Omega)}$$

(i), (ii), (iii) $\Rightarrow$ L-M applies!

Ex: $\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \Gamma \end{cases}$ (Poisson eqn. Dirichlet b.c.)

Find $u \in V$: $a(u,v) = L(v) \quad \forall v \in V$

$V = H_0^1(\Omega)$

$a(v,w) = \int_\Omega \nabla v \cdot \nabla w \, dx \qquad L(v) = \int_\Omega f v \, dx$

$\|v\|^2_{H^1(\Omega)} = \|\nabla v\|^2_{L_2(\Omega)} + \|v\|^2_{L_2(\Omega)} \leq (1+C)\|\nabla v\|^2_{L_2(\Omega)} = (1+C)a(v,v)$

Poincaré-Friedrich ineq.

$\Rightarrow a(\cdot,\cdot)$ V-elliptic with $\varkappa_1 = (1+C)^{-1} > 0$

$a(v,w) = (\nabla v, \nabla w) \leq \|\nabla v\| \, \|\nabla w\| \leq \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} = \|v\|_V \|w\|_V$

$\Rightarrow a(\cdot,\cdot)$ continuous with $\varkappa_2 = 1$

$L(\cdot)$ continuous as before. $(\varkappa_3 = \|f\|_{L_2(\Omega)})$

$\Rightarrow$ L-M applies $\Rightarrow$ existence & uniqueness!

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Ex: $\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \partial_n u = g & \text{on } \Gamma \end{cases}$

$V = H^1(\Omega)$, $a(u,v)$ as with $g = 0$

$L(v) = \int_\Omega f v \, dx + \int_\Gamma g v \, ds$

$|L(v)| \leq \|f\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma)} \|v\|_{L_2(\Gamma)}$

$\leq \|f\|_{L_2(\Omega)} \|v\|_{H^1(\Omega)} + \|g\|_{L_2(\Gamma)} \|v\|_{H^1(\Omega)}$

$\Rightarrow L(\cdot)$ cont. with $\varkappa_3 = (\|f\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma)})$