EL1820 Modeling of Dynamical Systems

Lecture 10 - System identification as a model building tool

- Experiment design
- Examination and prefiltering of data
- Model structure selection
- Model validation
Today’s goal

You should be able to

- use system identification as a systematic model-building tool
- do a careful experiment design/data collection to enable good model estimation
- select the appropriate model structure and model order
- validate that the estimated model is able to reproduce the observed data
System identification procedure

User choices

- *Experimental condition* input, sampling freq, open/closed loop, ...

- *Model structure* linear/nonlinear, parametric/nonparametric, ...

- *Identification method* least squares, PEM, ...
Experiments and data collection

Often good to use a two-stage approach

1. Preliminary experiments
   - step/impulse response tests to get basic understanding of system dynamics
   - linearity, stationary gains, time delays, time constants, sampling interval

2. Data collection for model estimation
   - carefully designed experiment to enable good model fit
   - operating point, input signal type, number of data points to collect, etc.
Preliminary experiments

• The input signal in an identification experiment can have a significant influence on the resulting model

• Typical choices:
  – (Approximate) impulse functions
  – Step functions
  – Pseudorandom binary sequences (PRBS)
  – Sinusoids
Preliminary experiments: step response

Easy to apply and useful for obtaining basic information about system

- dead-times, static gain, time constants, resonances and aids sampling time selection (rule-of-thumb: 4-10 samples / rise time)
- More power at low freq’s than at high freq’s
- Very sensitive to noise (needs large amplitude to get good results)
- User choices: Amplitude $A$, duration $T$
Preliminary experiments: sinusoids

\[ u(t) = \alpha \sin(\omega t + \phi) \]

- Easy to apply
- Useful for directly obtaining freq. domain information
- Focuses on freq’s of interest
- Amplitude can be traded off for the duration of the experiment, i.e. for \( \alpha \) larger the experiment can be shortened
- Only provides information at one freq, hence many experiments needed to obtain an adequate model
Preliminary experiments: multi-sines

\[ u(t) = \sum_{k=1}^{m} \alpha_k \sin(\omega_k t + \phi_k) \]

- If \( \phi_k \)'s are chosen equal to 0, the amplitude can be large.
- Schreoder method: Choose \( \phi_k = \frac{k^2 \pi}{m} \) (if \( \alpha_k \)'s are equal).
- \( \omega_k \)'s should be in the freq. region of interest, and multiples of \( h \).
Tests for verifying linearity

For linear systems, response is independent of operating point

- test linearity by a sequence of step response tests for different operating points
Tests for detecting friction

Friction can be detected by using small step increases in input

Input moves every two or three steps
Designing experiment for model estimation

Input signal should excite all relevant frequencies

- estimated model accurate in freq’s where input has much energy
- good choice: binary signal with random ”hold times” (e.g., PRBS)

![Graph showing a binary signal with random hold times](image)

Trade-off in selection of signal amplitude

- constraints on the input (economic, safety, actuator limits, ...)
- large amplitude gives high signal-to-noise ratio, low variance
- most systems are nonlinear for large input amplitudes

Sampling freq? Typically $5 - 10 \times$ time constant of the process

Many pitfalls if estimating a model of a system under closed-loop!
Identifiability

**Definition** Model parametrization is *identifiable* if
\[ \hat{y}[k|\theta^*] = \hat{y}[k|\theta] \Rightarrow \theta^* = \theta \]

Many reasons for loss of identifiability

1. Over-parametrization of model \((e.g., y[k] = \alpha \beta u[k])\)
2. Input signal not persistently exciting
3. Closed-loop control

**Example** Is the model
\[ y[k + 1] = b_0 u[k] + b_1 u[k - 1] \]
identifiable? what if \(u[k]\) is white noise? or if \(u[k]\) is constant?
Identifiability: persistent excitation

Recall from Lecture 9: If the underlying system is given by

\[ y[k] = \theta_1 u[k - 1] + \cdots + \theta_n u[k - n] + v[k] = \theta^T \varphi[k] + v[k] \]

then the parameter \( \hat{\theta} \) that makes the model

\[ y[k] = \hat{\theta}^T \varphi[k] \]

best fit given measured data sequences \( \{u[k]\} \) and \( \{y[k]\} \) is

\[ \hat{\theta} = (\varphi_N^T \varphi_N)^{-1} \varphi_N^T y_N \]

An input signal is *persistently exciting* (p.e.) of order \( n \) if

\[ R = \lim_{N \to \infty} \frac{1}{N} \varphi_N^T \varphi_N \in \mathbb{R}^{n \times n} \]

is invertible when \( u[k] \) and \( v[k] \) are independent (compare exercises!)
**Identifiability: persistent excitation cont’d**

**Fact:** Characterization of p.e. in freq. domain

$u[k]$ is p.e. of order $n_\theta$ if $\Phi_u(\omega) \neq 0$ for at least $n_\theta$ distinct freq’s in $[-\pi, \pi]$

**Examples**

- $u[k] = \text{const}$ (or a step) is p.e. of order $\leq 1$
- $u[k] = \sin[\omega k]$ is p.e. of order $\leq 2$
- $u[k]$ with continuous $\Phi_u(\omega)$ is p.e. of every order

**Fact** For rational structures, a necessary condition for identifiability is that $u[k]$ is p.e. of sufficiently high order:

- ARX / ARMAX: p.e. of order $n_b$
- OE / BJ: p.e. of order $n_a + n_b$
Identification in closed-loop

Many pitfalls when trying to identify dynamics of system in closed-loop

Feedback control

- reduces signal variations $\Rightarrow$ potential problems with p.e.
- makes input dependent of noise $\Rightarrow$ estimates can be biased

Example Identifying linear system under P-control

\[
\begin{align*}
    y[k] + ay[k - 1] &= bu[k - 1] + e[k] \\
    u[k] &= -fy[k]
\end{align*}
\]

$\Rightarrow$ $y[k] + \hat{a}y[k - 1] = \hat{b}u[k - 1] + e[k]$

Cannot identify $a$ and $b$ separately (even if we know feedback gain $f$!)
Identification in closed-loop cont’d

Closed-loop system with a reference term (\(i.e., \ u[k] = f(r[k] - y[k])\)):

\[ y[k] = -(a + bf)y[k - 1] + bfr[k] + e[k] \]

Can identify \(a, b\) if we know \(f\)!

ARX (with ref, dotted) works fine, while spectral analysis (without ref, dashed) fails completely
Data collection

Sampling time selection and anti-alias filtering central!
System identification – an iterative procedure

- Experiment design
- Data collection
- Data prefiltering
- Model structure selection
- Parameter estimation
- Model validation

Model OK?

Yes

No
Prefiltering of data

Remove

- transient needed to reach desired operating point
- mean values of input and output signals, i.e., work with

$$\Delta u[k] = u[k] - \frac{1}{T} \sum_{i=1}^{T} u[i]$$
$$\Delta y[k] = y[k] - \frac{1}{T} \sum_{i=1}^{T} y[i]$$

- trends (use `detrend` in Matlab)
- outliers (“obviously erroneous data points”)

Lecture 10
Prefiltering of data cont’d

Prefiltering: useful tool for emphasizing freq. ranges in estimation

If input and output are filtered through the same filter

\[ y_F[k] = L(q)y[k] \quad u_F[k] = L(q)u[k] \]

then the frequency domain interpretation of PEM becomes

\[
\lim_{N \to \infty} \hat{\theta}_N = \arg \min_{\theta} \int_{-\pi}^{\pi} \left| G_0(e^{i\omega}) - G(e^{i\omega}; \theta) \right|^2 \frac{\Phi_u(\omega)|L(e^{i\omega})|^2}{|H_*(e^{i\omega})|^2} \, d\omega
\]

Good fit where \( \Phi_u \) and \( |L(e^{i\omega})| \) are large, \( |H_*(e^{i\omega})| \) small

**Example** ARX: \( H_*(q) = A^{-1}(q) \), focus on high freq’s unless we prefilter!
Choice of model structure

1. Start with non-parametric estimates (correlation/spectral analysis)
   – give information about model order and important freq. regions

2. Prefilter input/output data to emphasize important freq. ranges

3. Begin with ARX models

4. Select model orders via
   – cross-validation (simulate model and compare with new data)
   – Akaike’s Information Criterion, i.e., pick the model that minimizes

\[
\left(1 + 2 \frac{d}{N}\right) \sum_{k=1}^{N} \varepsilon^2[k; \theta]
\]

(where \(d\) is the number of estimated parameters in the model)
System identification – an iterative procedure

Experiment design
Data collection
Data prefiltering
Model structure selection
Parameter estimation
Model validation
Model OK?
Yes
No
Model validation

A critical evaluation: “is model good enough”?  
- typically depends on the purpose of the model

Example

\[ G(s) = \frac{1}{(s + 1)(s + a)} \]

Open- and closed-loop responses for \( a = -0.01, 0, 0.01 \):

Insufficient for open-loop prediction, good enough for closed-loop control
Model validation cont’d

Bode diagrams reveal why model is good enough for closed-loop control

Different low-freq. behavior, similar responses around cross-over freq.
Principles of model validation

Try to verify that observations behave according to modelling assumptions

1. Compare model simulation/prediction with real data

2. Compare estimated model’s frequency response and spectral analysis estimate

3. Perform statistical tests on prediction errors
Validation: simulation and prediction

Split data into two parts: one for estimation and one for validation

Apply input signal in validation data set to estimated model

Compare simulated output with output stored in validation data set

![Measured and simulated output](image)
Statistical model validation

If we fit the parameters of the model

$$y[k] = G(q; \theta)u[k] + H(q; \theta)e[k]$$

to data, the residuals

$$\varepsilon[k] = H(q; \theta)^{-1}\{y[k] - G(q; \theta)u[k]\}$$

represent a disturbance that explains mismatch between model and observed data

If the model is correct, the residuals should be

– white, and

– uncorrelated with $u$
Statistical model validation

To test if the residuals $\varepsilon[k]$ are white, we estimate their autocovariance

$$\hat{R}_\varepsilon(\tau) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon[k] \varepsilon[k-\tau]$$

and verify that its components lie in a 95% confidence region around zero

- large components indicate unmodelled dynamics

Example Since $N \sum_{\tau=1}^{k} \frac{\hat{R}_\varepsilon^2(\tau)}{\hat{R}_\varepsilon^2(0)} \xrightarrow{d} \chi^2(k) \approx \mathcal{N}(k, 2k)$ for large $k$, a test of significance level 0.05 is:

The residuals $\varepsilon[k]$ are not white if

$$\sum_{\tau=1}^{k} \frac{\hat{R}_\varepsilon^2(\tau)}{\hat{R}_\varepsilon^2(0)} > \frac{k + 1.65\sqrt{2k}}{N}$$
Statistical model validation cont’d

Independence tested by verifying that cross-covariance function

\[ \hat{R}_{\varepsilon u}(\tau) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon[k]u[k - \tau] \]

lies within a 95% confidence region around zero

- large components indicate unmodeled dynamics
- \( \hat{R}_{\varepsilon u}(\tau) \neq 0 \) for \( \tau < 0 \) (non-causality) \( \Rightarrow \) presence of feedback

Example A test of significance level 0.05 is:

\[ \varepsilon[k] \text{ and } u[k] \text{ are not uncorrelated if } \sum_{\tau=1}^{k} \frac{\hat{R}^2_{\varepsilon u}(\tau)}{\hat{R}_\varepsilon(0)\hat{R}_u(0)} > \frac{k + 1.65\sqrt{2k}}{N} \]
Statistical model validation cont’d

Autocorrelation of residuals

Crosscorrelation between input and output

Samples
Summary

System identification – an iterative procedure in several steps

- Experiment design
  - preliminary experiments detect basic system behavior
  - carefully designed experiment enables good model estimation
    (choice of sampling interval, anti-alias filters, input signal)

- Examination and prefiltering of data
  - remove outliers and trends

- Model structure selection

- Model validation
  - cross-validation and residual tests
Next lecture

Identification of nonlinear models