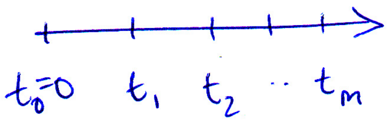
Lecture 7: Initial value problems (IVP)

$$\begin{cases} \dot{u}(x,t) + A(u(x,t)) = f(x,t) & \forall (x,t) \in \Omega \times [0,T] \\ u(x,t) = w(x,t) & \forall (x,t) \in \Gamma \times [0,T] \\ u(x,0) = u_0(x) & \end{cases}$$

Time stepping: Discretize in time $[0,T]$



solution at $t = t_m$ is given by solution & data at earlier time steps t_n, $n < m$.

Stability: growth/decay of perturbations of a solution with time.

(Time & space) Discretization introduces perturbations.

In general: error accumulate (grow with time)

Parabolic problem: $(Av, v) \geq 0$, $(Aw, v) = (w, Av)$ for all v, w.

Parabolic problems are dissipative: errors do not accumulate!

(**Elliptic problems**: $(Av, v) = a(v,v) \geq \alpha_1 \|v\|_V^2$ $\forall v \in V$)

Ex: Heat equation

$$\begin{cases} \dot{u} - \Delta u = f & (x,t) \in \Omega \times (0,T] \\ u = 0 & x \in \partial\Omega \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

$$(Av,v) = (-\Delta v, v) = (\nabla v, \nabla v) = \|\nabla v\|^2 \geq 0$$

$\Rightarrow$ Heat equation is parabolic!

Stability/Energy estimates: mult. by function & integrate

1. Mult. by u & integrate $\Rightarrow \int_\Omega (\dot{u} - \Delta u) u \, dx = (\dot{u} - \Delta u, u) = 0$
(assume that $f = 0$)

$$\Rightarrow (\dot{u}, u) - (\Delta u, u) = 0 \iff \frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 = 0$$

$$\left(\frac{d}{dt}\|u\|^2 = \frac{d}{dt}\int_\Omega |u|^2 dx = \frac{d}{dt}\int_\Omega u_1^2 + u_2^2 + u_3^2 \, dx = \int_\Omega 2u_1\dot{u}_1 + \dots + 2u_3\dot{u}_3 = 2(\dot{u}, u)\right)$$

Integrate in time $\Rightarrow$ $\boxed{\|u(T)\|^2 + 2\int_0^T \|\nabla u\|^2 dt = \|u_0\|^2}$ (1)

$2\int_0^t \|\nabla u\|^2 dt$ increase with time t

$\Rightarrow$ $\|u(t)\|^2$ decrease with time t

(dissipation of energy $\|u(t)\|^2$)

2. Mult. by $-t\Delta u$ & integrate

$$\Rightarrow (\dot{u}, -t\Delta u) - (\Delta u, -t\Delta u) = 0$$

$$\left(\frac{1}{2}\frac{d}{dt}\left(t\|\nabla u\|^2\right) = \frac{1}{2}\|\nabla u\|^2 + t(\nabla\dot{u}, \nabla u) = \frac{1}{2}\|\nabla u\|^2 + (\dot{u}, -t\Delta u)\right)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(t\|\nabla u\|^2\right) - \frac{1}{2}\|\nabla u\|^2 + t\|\Delta u\|^2 = 0$$

Integrate in time from 0 to T $\Rightarrow$

$$\frac{T}{2}\|\nabla u(T)\|^2 + \int_0^T t\|\Delta u\|^2 dt = \frac{1}{2}\int_0^T \|\nabla u\|^2 dt \underset{(1)}{\leq} \frac{1}{4}\|u_0\|^2$$

$$\Rightarrow \boxed{\frac{T}{2}\|\nabla u(T)\|^2 + \int_0^T t\|\Delta u\|^2 dt \leq \frac{1}{4}\|u_0\|^2} \quad (2)$$

3. Mult. by $t^2\Delta^2 u$ & integrate $\Rightarrow$

$$(\dot{u}, t^2\Delta^2 u) - (\Delta u, t^2\Delta^2 u) = 0$$

$$\left(\frac{1}{2}\frac{d}{dt}(t^2\|\Delta u\|^2) = t\|\Delta u\|^2 + t^2(\Delta\dot{u}, \Delta u) = t\|\Delta u\| + (\dot{u}, t^2\Delta^2 u)\right)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}(t^2\|\Delta u\|^2) - t\|\Delta u\|^2 - (\Delta u, t^2\Delta^2 u) = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}(t^2\|\Delta u\|^2) \leq t\|\Delta u\|^2 \quad \left((\Delta u, t^2\Delta^2 u) = t^2\|\nabla(\Delta u)\|^2 \geq 0\right)$$

$$\Rightarrow \frac{T^2}{2}\|\Delta u(T)\|^2 \leq \int_0^T t\|\Delta u\|^2 dt \leq \frac{1}{4}\|u_0\|^2$$

$$\Rightarrow \boxed{\|\Delta u(T)\| \leq \frac{1}{\sqrt{2}\,T}\|u_0\|} \quad (3)$$

<u>Poincaré-Friedrich Ineq</u>: There exist $c>0$
s.t. $\|\nabla v\|^2 \geq c\|v\|^2 \quad \forall v: v=0$ on Γ
(and $v\in H^1(\Omega)$)

<u>$f\neq 0$</u>: mult by u & integrate:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 = (f,u)$$

<u>P.F.</u> $\Rightarrow \frac{1}{2}\frac{d}{dt}\|u\|^2 + c\|u\|^2 \leq (f,u) \leq \|f\|\,\|u\| \leq \frac{1}{2c}\|f\|^2 + \frac{c}{2}\|u\|^2$

$$\left(\begin{array}{l} ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon}{2}b^2 \quad \forall a,b,\ \varepsilon>0 \\ \underline{\text{Proof}}: \ 0 \leq (a-\varepsilon b)^2 = a^2 - 2\varepsilon ab + \varepsilon^2 b^2 \end{array} \right)$$

$$\Rightarrow \quad \frac{d}{dt}\|u\|^2 + c\|u\|^2 \leq \frac{1}{c}\|f\|^2$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{ct}\|u\|^2\right) \leq \frac{e^{ct}}{c}\|f\|^2$$

$$\Rightarrow \|u(T)\|^2 \leq e^{-cT}\|u_0\|^2 + \frac{1}{c}\int_0^T e^{-c(T-t)}\|f\|^2 dt$$

<u>With $f=0$: energy dissipates exponentially!</u>

Time discretization with the θ-method

Mult. Heat eqn. with test function v

& integrate in space over Ω:

$$(\dot{u}, v) + a(u, v) = 0 \qquad \text{with } a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

Discretize in time using the θ-method:

$$\xrightarrow[\;t_{m-1}\quad t_m \quad t_{m+1}\;]{} t$$

Find u_h^{m+1} ~~[illegible]~~ in V_h s.t. (for each time step)

$$\left(\frac{u_h^{m+1} - u_h^m}{k}, v\right) + a\left(u_h^{m+\theta}, v\right) = 0 \qquad \forall v \in V_h$$

$$\begin{cases} \theta = 0 \Rightarrow \text{Forward Euler method in time} \\ \theta = 1 \Rightarrow \text{Backward Euler} \quad \text{—— " ——} \\ \theta = 1/2 \Rightarrow \text{Crank-Nicolson} \end{cases}$$

Investigate the stability of the θ-method:

Set $v = u_h^{m+\theta} \Rightarrow \left(\frac{u_h^{m+1} - u_h^m}{k}, u_h^{m+\theta}\right) + a\left(u_h^{m+\theta}, u_h^{m+\theta}\right) = 0$

$$u_h^{m+\theta} = \theta u_h^{m+1} + (1-\theta) u_h^m = k\left(\theta - \tfrac{1}{2}\right) \frac{u_h^{m+1} - u_h^m}{k} + \frac{u_h^{m+1} + u_h^m}{2}$$

$$\Rightarrow k\left(\theta - \tfrac{1}{2}\right) \left\| \frac{u_h^{m+1} - u_h^m}{k} \right\|^2 + \frac{\|u_h^{m+1}\|^2 - \|u_h^m\|^2}{2k} + \|\nabla u_h^{m+\theta}\|^2 = 0 \quad (*)$$

Assume $\theta \in [1/2, 1] \Rightarrow \theta - 1/2 \geq 0$

$$\Rightarrow \frac{\|u_h^{m+1}\|^2 - \|u_h^m\|^2}{2k} + \|\nabla u_h^{m+\theta}\|^2 \leq 0$$

$$\Rightarrow \boxed{\|u_h^{m+1}\|^2 \leq \|u_h^m\|^2} \qquad (k \leq M)$$

$\Rightarrow$ θ-method unconditionally stable for $\theta \in [1/2, 1]$
(including backward Euler, C-N)

Forward Euler ($\theta = 0$) only <u>conditionally stable</u>:

stable for $k \leq C h^2$

$\Rightarrow$ very small time steps (stiff problem)

$\Rightarrow$ choose implicit method ($\theta > 0$).

"<u>Proof</u>": $(*) \Rightarrow$

$$\|u_h^{m+1}\|^2 - \|u_h^m\|^2 = \underbrace{-\left(2k\|\nabla u_h^{m+\theta}\|^2 + 2k^2(\theta - \tfrac{1}{2})\left\|\frac{u_h^{m+1} - u_h^m}{k}\right\|^2\right)}_{D}$$

<u>$D \leq 0 \Rightarrow$ stability</u>: $\|u_h^{m+1}\|^2 \leq \|u_h^m\|^2$

$$D \leq 0 \Rightarrow -\left(2k\|\nabla u_h^{m+\theta}\|^2 + 2k^2(\theta - \tfrac{1}{2})\left\|\frac{u_h^{m+1} - u_h^m}{k}\right\|^2\right) \leq 0$$

$$\Rightarrow 2k\|\nabla u_h^{m+\theta}\|^2 + 2k^2(\theta - \tfrac{1}{2})\left\|\frac{u_h^{m+1} - u_h^m}{k}\right\|^2 \geq 0$$

$$\theta = 0 \Rightarrow 2\cancel{k}\|\nabla u_h^m\|^2 \geq k^{\cancel{2}}\left\|\frac{u_h^{m+1} - u_h^m}{k}\right\|^2 \sim k\|\Delta u_h^m\|^2$$

$$\nabla u_h^m \sim \frac{u_h^m}{h}, \; \Delta u_h^m \sim \frac{u_h^m}{h^2} \Rightarrow \left(\frac{u_h^m}{h}\right)^2 \gtrsim k\left(\frac{u_h^m}{h^2}\right)^2 \Rightarrow \boxed{k \lesssim h^2}$$

Wave equation

$$(1)\quad \begin{cases} \ddot{u} - \Delta u = f & \text{in } \Omega \times (0,T] \\ u = 0 & \text{on } \Gamma \times (0,T] \\ u(x,0) = u_0(x) & \text{in } \Omega \\ \dot{u}(x,0) = \dot{u}_0(x) & \text{in } \Omega \end{cases}$$

Write as first order system: $u_1 = \dot{u}$, $u_2 = u$

$$(2)\quad \begin{cases} \dot{u}_1 - \Delta u_2 = 0 & \text{in } \Omega \times (0,T] \\ -\Delta \dot{u}_2 + \Delta u_1 = 0 & \text{in } \Omega \times (0,T] \\ u_1(x,0) = \dot{u}_0(x) & \text{in } \Omega \\ u_2(x,0) = u_0(x) & \text{in } \Omega \\ u_2 = 0 & \text{on } \Gamma \times (0,T] \end{cases}$$

Mult. (1) by $\dot{u}$ & integrate over Ω: (set $f=0$)

$$(\ddot{u} - \Delta u, \dot{u}) = (\ddot{u}, \dot{u}) - (\Delta u, \dot{u}) = (\ddot{u}, \dot{u}) + (\nabla u, \nabla \dot{u}) = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(\|\dot{u}\|^2 + \|\nabla u\|^2\right) = 0 \Rightarrow \boxed{\frac{d}{dt}\left(\|\dot{u}\|^2 + \|\nabla u\|^2\right) = 0}$$

Energy conservation!

Similar by mult. (2) by (u_1, u_2) & integrate $\Rightarrow$

$$(\dot{u}_1, u_1) - (\Delta u_2, u_1) + (-\Delta \dot{u}_2, u_2) + (\Delta u_1, u_2) = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u_1\|^2 + \underbrace{\cancel{(\nabla u_2, \nabla u_1)}}_{=0} + (\nabla \dot{u}_2, \nabla u_2) - \underbrace{\cancel{(\nabla u_1, \nabla u_2)}}_{=0} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\|u_1\|^2 + \|\nabla u_2\|^2\right) = 0$$

Discretize Wave eqn. (2) by Crank-Nicolson ($\theta = \frac{1}{2}$)

Find $(u_1^{m+1}, u_2^{m+1}) \in V_h \times V_h$ s.t.

$$\Rightarrow \left(\frac{u_1^{m+1} - u_1^m}{k}, v_1\right) + a\left(u_2^{m+\theta}, v_1\right)$$

$$+ a\left(\frac{u_2^{m+1} - u_2^m}{k}, v_2\right) - a\left(u_1^{m+\theta}, v_2\right) = 0 \quad \forall (v_1, v_2) \in V_h \times V_h$$

Investigate energy conservation for C-N:

Set $v_1 = u_1^{m+\theta}$, $v_2 = u_2^{m+\theta}$

$$\Rightarrow k\left(\theta - \frac{1}{2}\right) \left\|\frac{u_1^{m+1} - u_1^m}{k}\right\|^2 + \frac{\|u_1^{m+1}\|^2 - \|u_1^m\|^2}{2k} + \cancel{a\left(u_2^{m+\theta}, u_1^{m+\theta}\right)}^{0}$$

$$+ k\left(\theta - \frac{1}{2}\right) \left\|\frac{\nabla(u_2^{m+1} - u_2^m)}{k}\right\|^2 + \frac{\|\nabla u_2^{m+1}\|^2 - \|\nabla u_2^m\|^2}{2k} - \cancel{a\left(u_1^{m+\theta}, u_2^{m+\theta}\right)}^{0} = 0$$

C-N: $\theta = \frac{1}{2} \Rightarrow$

$$\boxed{\|u_1^{m+1}\|^2 + \|\nabla u_2^{m+1}\|^2 = \|u_1^m\|^2 + \|\nabla u_2^m\|^2}$$

C-N energy conservative!

Puffin:

In Puffin the backward Euler method is typically used in the examples.

ODE:
$$\begin{cases} \dot{u}(t) = f(u(t)) \\ u(0) = u_0 \end{cases}$$

FEM for ODE: cG(q) & dG(q)

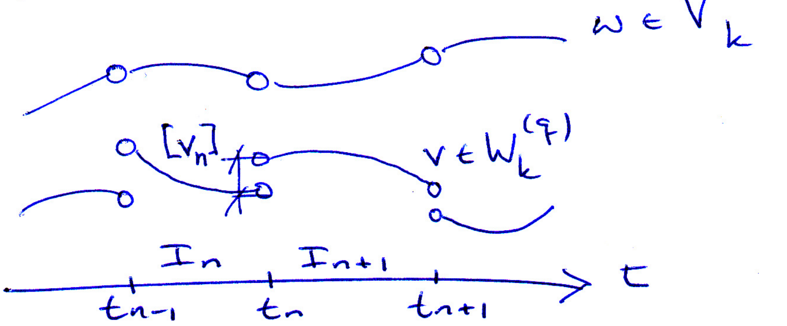
cG(q): Find $U \in V_k^{(q)}$ with $U(0) = u_0$ s.t.
$$\int_0^T \dot{U} v \, dt = \int_0^T f(U) v \, dt \quad \forall v \in W_k^{(q-1)}$$

dG(q): Find $U \in W_k^{(q)}$ with $U_0^- = u_0$ s.t.
$$\int_0^T \dot{U} v \, dt + \sum_{n=1}^{N} [U_{n-1}] v_{n-1}^+ = \int_0^T f(U) v \, dt \quad \forall v \in W_k^{(q)}$$

$W_k^{(q)} = \{ v : v|_{I_i} \in \mathcal{P}^{(q)}(I_i) \}$

$V_k^{(q)} = \{ v : v \in W_k^{(q)} \text{ \& } v \text{ continuous} \}$

$[v_n] = v_n^+ - v_n^-$



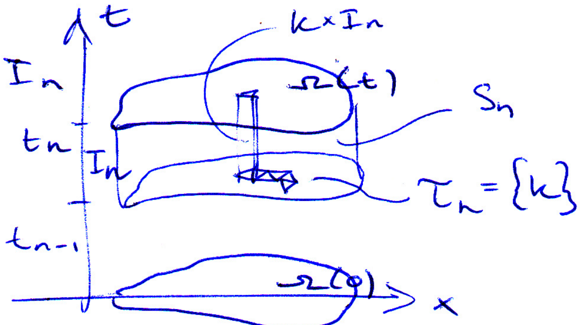
$k_n = t_n - t_{n-1}$

Semidiscretization: e.g. θ-method (time) + FEM (space)

Space-time FEM: Discretize space-time simultaneously: basis functions $v = v(x,t)$

Space-time slab $S_n = \Omega \times I_n$

Space-time prism $K \times I_n$



$V_n = V_{h_n}$ space of hat functions on space mesh $\mathcal{T}_n = \{K\}$

$$W_{kn}^{(r)} = \left\{ v(x,t) : v(x,t) = \sum_{j=0}^{r} t^j \varphi_j(x),\ \varphi_j \in V_n,\ (x,t) \in S_n \right\}$$

$$W_k^{(r)} = \left\{ v : v|_{S_n} \in W_{kn}^{(r)},\ n = 1, 2, \ldots, N \right\}$$

$v \in W_k^{(r)}$ typically discont. across t_n: $[v_n] = v_n^+ - v_n^- \neq 0$

cG(1) dG(0) for Heat equation:

For $n = 1, \ldots, N$ find $U \in W_k^{(0)}$:

$$\int_{I_n} \left((\dot{U}, v) + (\nabla U, \nabla v) \right) dt + \left([U_{n-1}], v_{n-1}^+ \right) = \int_{I_n} (f, v)\, dt \qquad \forall v \in W_{kn}^{(0)}$$

$$\left(\dot{U} = 0,\ U|_{S_n} = U^n(x) \in V_h,\ [U_{n-1}] = U_n - U_{n-1} \right)$$

$$\Rightarrow \int_{I_n} (\nabla U, \nabla v)\, dt + (U_n, v_n) = (U_{n-1}, v_n) + \int_{I_n} (f, v)\, dt$$

Further reading: 9.1–9.2 (ODE), 16, 17 (Heat & Wave eqns)